Advanced Mathematics for Economic Theory
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Notes on Baire Functions

Let C be the set of continuous functions on $\mathbb{R}^\ell$, and B the smallest set of functions that contains C and is closed under pointwise convergence.

Proposition 1. *B is closed under linear combination, i.e., if $f, g \in B$, then $af + bg \in B$ for all $a, b \in \mathbb{R}$.*

Proof. Let

$$C^* = \{g \in B \mid af + bg \in B \text{ for all } f \in C \text{ and all } a, b \in \mathbb{R}\}.$$

Since C is closed under linear combination, we have $C \subset C^*$. If $\{g_n\} \subset C^*$, meaning that, for all $f \in C$, $\{af + bg_n\} \subset B$, and $\lim g_n = g$ (pointwise), then $\lim(af + bg_n) = af + b \lim g_n = af + bg \in B$ since B is closed under pointwise convergence, and therefore $g \in B$. Hence $B = C^*$, which implies that for all $f \in C$ and all $g \in B$, $af + bg \in B$ for all $a, b \in \mathbb{R}$. Therefore, if we let

$$C^{**} = \{f \in B \mid af + bg \in B \text{ for all } g \in B \text{ and all } a, b \in \mathbb{R}\},$$

then we have $C \subset C^{**}$. Furthermore, if $\{f_n\} \subset C^{**}$ and $\lim f_n = f$, then, for all $g \in B$, $\lim(af_n + bg) = a \lim f_n + bg = af + bg \in B$, and therefore $f \in B$. Hence $B = C^{**}$. ∎

As is clear from the proof, B is closed under any operation under which C is closed and such that this operation and taking pointwise limits are interchangeable.