Producer Theory

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Production Sets

- $Y \subset \mathbb{R}^L$: production set
- For $(y_1, \ldots, y_L) \in Y$
  - $y_\ell > 0 \Rightarrow \ell$: output
  - $y_\ell < 0 \Rightarrow \ell$: input
- Example: Suppose $L = 5$ and $y = (-5, 2, -6, 3, 0) \in Y$.
  - revenue $= p_2 \times y_2 + p_4 \times y_4$
  - cost $= p_1 \times (-y_1) + p_3 \times (-y_3)$
  - profit $= [p_2 \times y_2 + p_4 \times y_4] - [p_1 \times (-y_1) + p_3 \times (-y_3)] = p \cdot y$
- If a production function f is given where L is the output, then

$$Y = \{(-z_1, \ldots, -z_{L-1}, q) \mid q \leq f(z_1, \ldots, z_{L-1}),\ z_\ell \geq 0\}.$$

Properties of Production Sets

1. Y is nonempty.
2. Y is closed.
3. No free lunch: $Y \cap \mathbb{R}^L_+ \subset \{0\}$
4. Possibility of inaction: $0 \in Y$
5. Free disposal: If $y \in Y$ and $y' \leq y$, then $y' \in Y$, or equivalently, $Y - \mathbb{R}^L_+ \subset Y$.

   ($A - B = \{c \mid c = a - b$ for some $a \in A$ and $b \in B\}$)
6. Irreversibility: If $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

Properties of Production Sets

7. Nonincreasing returns to scale:
   If $y \in Y$, then $\alpha y \in Y$ for all $\alpha \in [0, 1]$.

8. Nondecreasing returns to scale:
   If $y \in Y$, then $\alpha y \in Y$ for all $\alpha \geq 1$.

9. Constant returns to scale:
   If $y \in Y$, then $\alpha y \in Y$ for all $\alpha \geq 0$.

   (I.e., Y is a *cone*.)

10. Additivity: $Y + Y \subset Y$.

11. Convexity:
    If $y, y' \in Y$, then $\alpha y + (1 - \alpha)y' \in Y$ for all $\alpha \in [0, 1]$.

12. Y is a *convex cone*:
    If $y, y' \in Y$, then $\alpha y + \beta y' \in Y$ for all $\alpha \geq 0$ and $\beta \geq 0$.

Convexity

Proposition 1

Y is additive and exhibits nonincreasing returns to scale if and only if it is a convex cone.

Constant Returns to Scale

Proposition 2

If $Y \subset \mathbb{R}^L$ *is convex and* $0 \in Y$*, then there is a constant returns, convex production set* $Y' \subset \mathbb{R}^{L+1}$ *such that*
$Y = \{y \in \mathbb{R}^L \mid (y, -1) \in Y'\}$.

- Decreasing returns reflect the scarcity of some underlying, unlisted input ("entrepreneurial factor").

Proof

- Let
  $Y' = \{y' \in \mathbb{R}^{L+1} \mid y' = \alpha(y, -1) \text{ for some } y \in Y \text{ and } \alpha \geq 0\}$.

Profit Maximization

$$\max_{y} \; p \cdot y$$
$$\text{s. t.} \quad y \in Y$$

- Supply correspondence:

$$\begin{aligned} y(p) &= \arg\max_{y \in Y} \; p \cdot y \\ &= \{ y \in \mathbb{R}^L \mid y \in Y \text{ and } p \cdot y \geq p \cdot y' \text{ for all } y' \in Y \} \end{aligned}$$

- Profit function:

$$\pi(p) = \max_{y \in Y} \; p \cdot y$$

- Analogous to expenditure minimization!

Properties of π and y

Proposition 3

Suppose Y is nonempty and closed.

1. *π is homogeneous of degree one.*
2. *π is convex.*
3. *If Y is convex and satisfies free disposal, then $Y = \{y \in \mathbb{R}^L \mid p \cdot y \leq \pi(p) \text{ for all } p \geq 0\}$.*
4. *y is homogeneous of degree zero.*
5. *If Y is convex, then $y(p)$ is a convex set for all p. If Y is strictly convex, then $y(p)$ is single-valued (if nonempty).*
6. *[Hotelling's lemma] If $y(p)$ is a singleton, then $\nabla\pi(p) = y(p)$.*
7. *If y is a continuously differentiable function, then $Dy(p)$ is symmetric and positive semi-definite, and $Dy(p)p = 0$.*

Cost Minimization

f: Production function

$$\min_{z \geq 0} \quad w \cdot z$$
$$\text{s. t.} \quad f(z) \geq q$$

- Conditional factor demand correspondence:

$$z(w, q) = \arg\min\{w \cdot z \mid f(z) \geq q\}$$

- Cost function:

$$c(w, z) = \min\{w \cdot z \mid f(z) \geq q\}$$

- Analogous to expenditure minimization!

Properties of c and z
Proposition 4

1. *c is homogeneous of degree one in w and nondecreasing in q.*
2. *c is concave in w.*
3. *If f is nondecreasing and quasi-concave, then $Y = \{(-z, q) \mid z \geq 0$ and $w \cdot z \leq c(w, q)$ for all $w \gg 0\}$.*
4. *z is homogeneous of degree zero in w.*
5. *If f is quasi-concave, then $z(w, q)$ is a convex set. If f is strictly quasi-concave, then $z(w, q)$ is single-valued.*
6. *[Shepard's lemma] If $z(w, q)$ is a singleton, then $\nabla_w c(w, q) = z(w, q)$.*
7. *If z is a continuously differentiable function, then $D_w z(w, q)$ is symmetric and negative semi-definite, and $D_w z(w, q) w = 0$.*

8. If f is homogeneous of degree one (i.e., exhibits constant returns to scale), then c and z are homogeneous of degree one in q.
9. If f is concave, then c is convex in q.