Efficiency, envy-freeness, and Bayesian incentive compatibility in economies with one indivisible good and money*

Yuji Fujinaka†

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Abstract

We consider the problem of allocating an indivisible object when monetary transfers are possible. We assume that each agent privately knows his own preference and has incomplete information about the preferences of other agents. We formulate a class of allocation rules and establish that each allocation rule in the class satisfies *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*. This yields a positive result when compared with the impossibility results in the literature on *strategy-proofness*. Further, we formulate an auxiliary property—the *anonymous and additive transfer property*—that is satisfied by many of the important and well-analyzed allocation rules, such as the first-price and second-price auction rules, the Shapley value allocation rule (Shapley, 1953; and Littlechild and Owen, 1973), and the equal welfare rule (Tadenuma and Thomson, 1993). We also establish a characterization theorem: an allocation rule satisfies *efficiency*, *envy-freeness*, *Bayesian incentive compatibility*, and the *anonymous and additive transfer property* if and only if it is an allocation rule in the class defined above.

Keywords: Bayesian Nash implementation, Fair allocation, Indivisible good, Mechanism design.

JEL codes: C72, C78, D61, D63, D71.

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†Graduate School of Economics, Kobe University, 2-1, Rokkodai, Nada, Kobe 657-8501, JAPAN; fujinaka@stu.kobe-u.ac.jp ; http://www.geocities.jp/yuji_fujinaka/

1 Introduction

We study the problem of allocating one indivisible object when monetary transfers are possible. This model applies to the problem of locating a public facility or the assigning of a certain right. For example, Kunreuther, Kleindorfer, Knez, and Yaksick (1987), Kleindorfer and Serter (1994), and Sakai (2005a,b) interpret the object as a noxious facility; they examine where the facility can be sited as well as how monetary compensations can be made for the region in which it is sited. In the model, an *allocation* is a pair of vectors: an *assignment vector* that determines who receives the object and a *monetary transfer vector* that determines how to make monetary compensations. An *allocation rule* is a function that associates each preference profile with an allocation. We assume that each agent has quasi-linear preferences. The agent's preference is thus characterized by his valuation of the indivisible object. We also suppose that each agent privately knows his own valuation of the object and is only aware that the agents' valuations are made independently from the identical distribution; in other words, we consider the problem under incomplete information with independent private valuations.

Since preferences or valuations are privately known, agents may have an incentive to manipulate the allocation rule by strategically misrepresenting preferences. In order to prevent such strategic behavior by agents, we should impose an incentive compatible constraint on allocation rules. In the model with incomplete information, one of the most attractive constraints is *strategy-proofness*.[1] *Strategy-proofness* implies that the truthful revelation of preference is a weakly dominant strategy for each agent. However, in our model, it is well known that any *efficient* allocation rule is not strategy-proof (Holmström, 1979; Ohseto, 2000; and Schummer, 2000). Therefore, by the revelation principle, we conclude that *efficiency* cannot be achieved under the notion of weakly dominant strategy equilibrium.

We consider an incentive compatible condition weaker than *strategy-proofness*: *Bayesian incentive compatibility*. *Bayesian incentive compatibility* implies that for each agent, a truthful revelation of his preference maximizes his expected payoff given that all other agents also report truthfully. In contrast with the results on *strategy-proofness* in many studies, results in which the incentive condition is consistent with *efficiency* are established (for example, Arrow, 1979 and d'Aspremont and Gérard-Varet, 1979). However, these studies seldom focus on the fairness of allocation rules; this is a significant property in allocation problems (such as the problem of locating a public facility), where each agent has identical rights on the indivisible object. In this paper, we study one of the most commonly used notions, namely, *envy-freeness* (Foley, 1967), which implies that every agent weakly prefers his own consumption to that of any other agent.[2] We examine whether or not there

[1]It is clear that the concept of *strategy-proofness* does not require us to make strong assumptions about information structure.

[2]Note that in contrast with the results on *efficiency*, those of *strategy-proofness* and *envy-freeness* are consistent in our model. Ohseto (2006) characterizes the set of allocation rules that satisfies the two properties. Sakai (2006) presents a study of these allocation rules on a domain with non-quasi-linear preferences.

exists an allocation rule that satisfies not only *efficiency* and *Bayesian incentive compatibility* but also *envy-freeness*.

We establish that the answer to the above question is in the affirmative. We formulate a class of allocation rules, denoted by Φ^*. We establish that each allocation rule in Φ^* satisfies *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*. Our result implies that in the direct revelation game, the *efficient* and *envy-free* allocation rule can be implemented through a Bayesian Nash equilibrium where the strategies require truth-telling.[3] Therefore, we can achieve *efficiency* and *envy-freeness* simultaneously under the weak equilibrium notion.[4] This is a very positive result when compared with the impossibility results in the literature on *strategy-proofness*. Note that our result is analogous to that of Morgan (2004). In the common valuations model in which agents' valuations of the object are common, Morgan examines the same topic as ours and provides a positive result. Our result implies that a positive theorem that is identical to that of Morgan can be established in the private valuations model.

Next, we consider the question of whether or not there exists an allocation rule that is not part of Φ^* but that satisfies the three properties of *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*. Since it is difficult for us to solve the question completely, we would like to restrict our attention to allocation rules that satisfy a certain anonymous and additive monetary transfer property and provide a partial answer to the question. Thus, we formulate the *anonymous and additive transfer property* as an additional property. Although the property is restrictive, it is satisfied by many of the important and well-studied allocation rules in the literature on the subject. Prominent examples are the first-price and second price auction rules, the Shapley value allocation rule (Shapley, 1953; and Littlechild and Owen, 1973), and the equal welfare rule (Tadenuma and Thomson, 1993). We establish the following theorem: an allocation rule satisfies *efficiency*, *envy-freeness*, *Bayesian incentive compatibility*, and the *anonymous and additive transfer property* if and only if it is an allocation rule in Φ^*. This theorem implies that if we pay attention to only the class of allocation rules that satisfy the *anonymous and additive transfer property*, the answer to the second question is in the negative.

The paper is organized as follows: Section 2 defines the basic notion, the properties of allocation rules, and our incentive compatible constraint. Sections 3 and 4 provide the two main theorems: the existence theorem and the characterization theorem. Section 5 concludes the discussion.

[3]We employ the notion of *partial* implementation, which is weaker than that of *full* implementation. In general environments, our *Bayesian incentive compatibility* composes a necessary and sufficient condition for full Bayesian Nash implementability. See Jackson (1991) for details.

[4]In our model with complete information, Fujinaka and Sakai (2007a) study the relationship between *full* Nash implementability and fairness. Their result implies that any *envy-free* allocation rule is not full Nash implementable.

2 Model

2.1 Basic notion

Let $I \equiv \{1, 2, \ldots, n\}$ be a finite set of the *agents*. There is an indivisible object to be assigned to one of agents. We assume that monetary transfers are possible.

Each agent's preference is characterized by a value of V_i that is his valuation of the indivisible object. Each V_i takes on values in $\mathcal{V}_i \equiv [\underline{v}, \overline{v}] \subset \mathbb{R}$ with $\underline{v} < \overline{v}$. The agents' valuations are independently and identically distributed on $\mathcal{V}_i$ according to the distribution function $F : \mathcal{V}_i \to [0, 1]$. Suppose that F admits the density function $f \equiv F'$ that satisfies $f(v_i) > 0$ for each $v_i \in \mathcal{V}_i$. Let

$$\mathcal{V} \equiv \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_n,$$

and $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathcal{V}$ be a profile of valuations. Let

$$\mathcal{V}_{-i} \equiv \mathcal{V}_1 \times \cdots \times \mathcal{V}_{i-1} \times \mathcal{V}_{i+1} \times \cdots \times \mathcal{V}_n,$$

and $\mathbf{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \in \mathcal{V}_{-i}$. Define $f(\mathbf{v})$ to be the joint density of $\mathbf{v} \in \mathcal{V}$. Since the valuations are independently distributed, $f(\mathbf{v}) = f(v_1) \times \cdots \times f(v_n)$ for each $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathcal{V}$. Similarly, define $f_{-i}(\mathbf{v}_{-i})$ to be the joint density of $\mathbf{v}_{-i}$. Each agent privately knows the realization of his own valuation and only knows that the other agents' valuations are independently distributed according to F. We assume that all components of the model other than the realized valuations are commonly known to all agents.

Agent i with valuation v_i has the quasi-linear utility function $u_i(\,\cdot\,; v_i) : \{0, 1\} \times \mathbb{R} \to \mathbb{R}$ such that

$$u_i(x_i, t_i; v_i) \equiv v_i x_i + t_i.$$

Given $i \in I$, $x_i = 1$ (resp. $x_i = 0$) represents that agent i receives (resp. does not receive) the object. And $t_i \geq 0$ (resp. $t_i < 0$) is the amount of money he is paid (resp. pays).

An *assignment vector* $x = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^I$ such that $\sum_{i \in I} x_i = 1$ represents who receives the object. The set of assignment vectors is denoted by X, i.e.,

$$X \equiv \{x \in \{0, 1\}^I : \sum_{i \in I} x_i = 1\}.$$

A *monetary transfer vector* $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^I$ such that $\sum_{i \in I} t_i \leq 0$ represents how to make monetary transfers. The set of monetary transfer vectors is denoted by T, i.e.,

$$T \equiv \{t \in \mathbb{R}^I : \sum_{i \in I} t_i \leq 0\}.$$

An *allocation* (x, t) is a pair of vectors: an assignment vector $x \in X$ and a monetary transfer vector $t \in T$. Let $A \equiv X \times T$ be the set of allocations. Let $(x, t) = (x_i, t_i)_{i \in I} \in A$. Each (x_i, t_i) denotes agent i's consumption bundle.

An *allocation rule* ϕ is a function $\phi : \mathcal{V} \to A$ that associates each profile of valuations $\mathbf{v} \in \mathcal{V}$ with an allocation $\phi(\mathbf{v}) = (\phi_i(\mathbf{v}))_{i \in I} = (x_i(\mathbf{v}), t_i(\mathbf{v}))_{i \in I} \in A$.

2.2 Properties of allocation rules

We introduce efficiency and fairness properties of allocation rules that we consider in the present paper, *efficiency* and *envy-freeness*.[5]

We first introduce an efficiency condition. *Efficiency* states that no unanimous welfare improvement is possible. An allocation $(x,t) \in A$ is *efficient* at $\mathbf{v} \in \mathcal{V}$ if there does not exist $(x',t') \in A$ such that for each $i \in I$, $u_i(x'_i, t'_i; v_i) \geq u_i(x_i, t_i; v_i)$, and for some $j \in I$, $u_j(x'_j, t'_j; v_j) > u_j(x_j, t_j; v_j)$.

Efficiency: An allocation rule ϕ is *efficient* if for each $\mathbf{v} \in \mathcal{V}$, $\phi(\mathbf{v})$ is *efficient* at $\mathbf{v}$.

We next introduce a fairness requirement. *Envy-freeness* states that every agents weakly prefers his own consumption to that of any other agent (Foley, 1967). Formally, an allocation $(x,t) \in A$ is *envy-free* at $\mathbf{v} \in \mathcal{V}$ if for each $i, j \in I$, $u_i(x_i, t_i; v_i) \geq u_i(x_j, t_j; v_i)$.

Envy-freeness: An allocation rule ϕ is *envy-free* if for each $\mathbf{v} \in \mathcal{V}$, $\phi(\mathbf{v})$ is *envy-free* at $\mathbf{v}$.

The next proposition characterizes the set of *efficient* and *envy-free* allocations in our model.[6]

Proposition 1. For each $\mathbf{v} \in \mathcal{V}$, an allocation $(x,t) \in A$ is *efficient* and *envy-free* at $\mathbf{v}$ if and only if, letting $x_j = 1$,

$$v_j = \max_{i \in I} v_i, \tag{1}$$

$$\sum_{i \in I} t_i = 0, \tag{2}$$

$$\frac{\max_{h \neq j} v_h}{n} \leq t_i = t_k \leq \frac{v_j}{n} \quad \text{for each } i, k \in I \setminus \{j\}. \tag{3}$$

Proof. Let $(x,t) \in A$ is *efficient* and *envy-free* at $\mathbf{v} \in \mathcal{V}$. By *efficiency*, (1) and (2) are obvious. By *envy-freeness*, $t_i = t_k$ for each $i, k \in I \setminus \{j\}$. Consider agent $i \in I \setminus \{j\}$ such that $v_i = \max_{h \neq j} v_h$. (2) implies that $t_j = -(n-1)t_i$. Since j and i do not envy each other, we have that $v_j + t_j \geq t_i$, and $t_i \geq v_i + t_j$. Therefore, we have that

$$\frac{v_i}{n} \leq t_i \leq \frac{v_j}{n}.$$

Conversely, let $(x,t) \in A$ satisfies (1)-(3). Obviously, it is *efficient* at $\mathbf{v}$ by (1) and (2). It is sufficient to show that for each $i \in I \setminus \{j\}$, i and j do not envy each other. Notice that $t_j = -(n-1)t_i$ by (2). Since for agent j,

$$v_j + t_j = v_j - (n-1)t_i \geq v_j - \frac{n-1}{n} v_j \geq t_i,$$

[5]Note that our notions of efficiency and fairness are *ex post*. Since we will establish certain possibility theorems in the following sections, this assumption strengthens the results.

[6]General versions of Proposition 1 can be found in Tadenuma and Thomson (1995) and Bochet and Sakai (2007).

agent j does not envy agent i. Similarly, since for agent i,

$$t_i \geq \frac{\max_{h \neq j} v_h}{n} \geq \frac{v_i}{n} = v_i - \frac{n-1}{n} v_i \geq v_i - (n-1)t_i = v_i + t_j,$$

agent i does not envy agent j. Therefore, we complete the proof. □

Next, we introduce incentive compatible conditions on allocation rules. First one is *strategy-proofness* that implies that no one can gain by misrepresentation of his own preference.

Strategy-proofness: An allocation rule is *strategy-proof* if for each $\mathbf{v} \in \mathcal{V}$, each $i \in I$, and each $v'_i \in \mathcal{V}_i$, $u_i(\phi_i(\mathbf{v}); v_i) \geq u_i(\phi_i(v'_i, \mathbf{v}_{-i}); v_i)$.

A general result obtained by Holmström (1979) leads us to the following proposition[7]:

Proposition 2. There exists no *efficient* and *strategy-proof* allocation rule.

Based on Proposition 2, we can conclude that *efficiency* cannot be achieved under the notion of *strategy-proofness* or weakly dominant strategy equilibrium. Next, we consider an incentive compatible condition that is weaker than strategy-proofness, *Bayesian incentive compatibility. Bayesian incentive compatibility* implies that for each agent, a truthful revelation of his preference maximizes his expected payoff given that all other agents report their own preferences truthfully.

Bayesian incentive compatibility: An allocation rule ϕ is *Bayesian incentive compatible* if for each $i \in I$ and $v_i \in \mathcal{V}_i$,

$$\int_{\mathcal{V}_{-i}} u_i(\phi_i(v_i, \mathbf{v}_{-i}); v_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \geq \int_{\mathcal{V}_{-i}} u_i(\phi_i(v'_i, \mathbf{v}_{-i}); v_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i},$$

for all $v'_i \in \mathcal{V}_i$.

Since *Bayesian incentive compatibility* is weaker than *strategy-proofness*, it is consistent with *efficiency* in contrast with Proposition 2. Arrow (1979) and d'Aspremont and Gérard-Varet (1979) originally established this fact by showing an example of an allocation rule that satisfied the two properties. Such an allocation rule is called an "expected externality" mechanism. For each $\mathbf{v} \in \mathcal{V}$ and each $k = 1, 2, \ldots, n$, let $\mathbf{v}^k$ be the k-th highest valuation among $\mathbf{v} = (v_1, v_2 \ldots, v_n)$ respectively.[8] Let $\phi^E(\mathbf{v}) = (x^E_i(\mathbf{v}), t^E_i(\mathbf{v}))_{i \in I}$ be an allocation rule that satisfies the following: for each $\mathbf{v} \in \mathcal{V}$, letting $x^E_j(\mathbf{v}) = 1$, and $W_{-i}(\mathbf{v}) \equiv \sum_{h \neq i} v_h x^E_h(\mathbf{v})$ for each $i \in I$,

$$v_j = \mathbf{v}^1$$

$$t^E_i(\mathbf{v}) = E_{\mathbf{v}_{-i}}\left[W_{-i}(v_i, \mathbf{v}_{-i})\right] - \frac{1}{n-1} \sum_{h \neq i} E_{\mathbf{v}_{-h}}\left[W_{-h}(v_h, \mathbf{v}_{-h})\right] \quad \text{for each } i \in I.$$

[7] Ohseto (2000) and Schummer (2000) discuss Holmström's result in our environment.

[8] For example, if $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\mathbf{v}^k = v_k$ for each $k = 1, 2, \ldots, n$.

Note that given the assignment vector function $x^E(\cdot)$, for each $i \in I$ and each $\mathbf{v} \in \mathcal{V}$, $W_{-i}(\mathbf{v})$ denotes the aggregate benefit of agents other than agent i when the profile of valuations is $\mathbf{v}$; further, for each $i \in I$ and each $v_i \in \mathcal{V}_i$, $E_{\mathbf{v}_{-i}}[W_{-i}(v_i, \mathbf{v}_{-i})]$ denotes the "expected" aggregate benefit of agents other than agent i when his valuation is v_i. The allocation rule ϕ^E is *efficient* and *Bayesian incentive compatible*. However, it does not satisfy *envy-freeness*.[9] Further, the succeeding results in the literature on the subject seldom discuss the fairness properties of allocation rules. Therefore, we do not know whether or not there exists an allocation rule that satisfies the three desirable requirements of *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*. This is the topic that we examine in this paper.

3 Existence Theorem

In this section, we show that there exists an allocation rule that satisfies *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*. Let $\phi^*(\mathbf{v}) = (x_i^*(\mathbf{v}), t_i^*(\mathbf{v}))_{i \in I}$ be an allocation rule that satisfies the following: for each $\mathbf{v} \in \mathcal{V}$, letting $x_j^*(\mathbf{v}) = 1$,

$$v_j = \mathbf{v}^1 \tag{4}$$

$$t_i^*(\mathbf{v}) = \begin{cases} -\dfrac{n-1}{n}\left(\mathbf{v}^1 - \displaystyle\int_{\mathbf{v}^2}^{\mathbf{v}^1} F(v)dv\right) & \text{if } i = j \\ \\ \dfrac{1}{n}\left(\mathbf{v}^1 - \displaystyle\int_{\mathbf{v}^2}^{\mathbf{v}^1} F(v)dv\right) & \text{if } i \in I \setminus \{j\}. \end{cases} \tag{5}$$

The allocation rule ϕ^* stipulates that an agent with the highest valuation accepts the indivisible object and then pays each of the other agents $\frac{1}{n}\left(\mathbf{v}^1 - \int_{\mathbf{v}^2}^{\mathbf{v}^1} F(v)dv\right)$. Note that for each $\mathbf{v} \in \mathcal{V}$, monetary transfers $(t_i^*(\mathbf{v}))_{i \in I}$ depend only on the highest and second highest valuations among $\mathbf{v}$. Let Φ^* be the set of such allocation rules. We establish the following theorem.

Theorem 1. Each allocation rule ϕ^* in Φ^* satisfies *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*.

Proof. Let $\phi^*(\mathbf{v}) = (x_i^*(\mathbf{v}), t_i^*(\mathbf{v}))_{i \in I}$ be an allocation rule that satisfies (4) and (5). For each $\mathbf{v} \in \mathcal{V}$, by (5), it is obvious that $\sum_{i \in I} t_i^*(\mathbf{v}) = 0$. Note that since $\mathbf{v}^1 \geq \mathbf{v}^2$, it holds that $0 \leq \displaystyle\int_{\mathbf{v}^2}^{\mathbf{v}^1} F(v)dv \leq \mathbf{v}^1 - \mathbf{v}^2$. This implies that for each $i \in I \setminus \{j\}$,

$$\frac{1}{n}\mathbf{v}^2 = \frac{1}{n}\mathbf{v}^1 - \frac{1}{n}(\mathbf{v}^1 - \mathbf{v}^2) \leq \frac{1}{n}\mathbf{v}^1 - \frac{1}{n}\int_{\mathbf{v}^2}^{\mathbf{v}^1} F(v)dv = t_i^*(\mathbf{v}) \leq \frac{1}{n}\mathbf{v}^1.$$

[9] Note that the monetary transfer function t^E is additively separable with respect to $v_1, v_2, \ldots, v_n$. It is simple to check that allocation rules with additively separable monetary transfer do not satisfy *envy-freeness*.

Therefore, by Proposition 1, we have that ϕ^* satisfies *efficiency* and *envy-freeness*.

Let $G(v) \equiv F(v)^{n-1}$ for each $v \in \mathcal{V}_i$, and $g \equiv G'$, i.e., $g(v) = (n-1)f(v)F(v)^{n-2}$. Given $i \in I$, let Y_1 and Y_2 denote the highest and second highest valuations among the $n-1$ remaining agents respectively. In other words, Y_1 and Y_2 is the highest and second highest order statistics of $V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_n$. Then, the joint density of the first and second order statistics is

$$f(y_1, y_2) = (n-1)(n-2)f(y_1)f(y_2)F(y_2)^{n-3}$$

if $y_1 \geq y_2$ and 0 otherwise.[10]

Fix $i \in I$, and $\hat{v}_i \in \mathcal{V}_i$. We would like to show that for agent i with valuation $\hat{v}_i$, truth revelation of his own valuation maximizes his expected payoff given that all other agents reveal their own valuations truthfully, i.e.,

$$\hat{v}_i \in \arg\max_{v_i \in \mathcal{V}_i} \int_{\mathcal{V}_{-i}} u_i(\phi_i^*(v_i, \mathbf{v}_{-i}); \hat{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}.$$

When agent i with true valuation $\hat{v}_i$ reports his valuation as v_i, then his expected utility is

$$\begin{aligned}
&\int_{\mathcal{V}_{-i}} u_i(\phi_i^*(v_i, \mathbf{v}_{-i}); \hat{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \\
&\quad = \int_{\underline{v}}^{v_i} \int_{\underline{v}}^{y_1} \left(\hat{v}_i - \frac{n-1}{n}(v_i - \int_{y_1}^{v_i} F(v)dv) \right) f(y_1, y_2) dy_2 dy_1 \\
&\quad + \int_{v_i}^{\overline{v}} \int_{\underline{v}}^{v_i} \frac{1}{n}(y_1 - \int_{v_i}^{y_1} F(v)dv) f(y_1, y_2) dy_2 dy_1 \\
&\quad + \int_{v_i}^{\overline{v}} \int_{v_i}^{y_1} \frac{1}{n}(y_1 - \int_{y_2}^{y_1} F(v)dv) f(y_1, y_2) dy_2 dy_1. \qquad (6)
\end{aligned}$$

Differentiating (6) with respect to v_i, we obtain that

$$\frac{d}{dv_i} \int_{\mathcal{V}_{-i}} u_i(\phi_i^*(v_i, \mathbf{v}_{-i}); \hat{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} = \hat{v}_i g(v_i) - v_i g(v_i). \qquad (7)$$

By (7), when $v_i = \hat{v}_i$, the first-order and second-order conditions are satisfied because

$$\begin{aligned}
\frac{d}{dv_i} \int_{\mathcal{V}_{-i}} u_i(\phi_i^*(\hat{v}_i, \mathbf{v}_{-i}); \hat{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} &= \hat{v}_i g(\hat{v}_i) - \hat{v}_i g(\hat{v}_i) = 0 \\
\frac{d^2}{dv_i^2} \int_{\mathcal{V}_{-i}} u_i(\phi_i^*(\hat{v}_i, \mathbf{v}_{-i}); \hat{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} &= \hat{v}_i g'(\hat{v}_i) - g(\hat{v}_i) - \hat{v}_i g'(\hat{v}_i) \\
&= -g(\hat{v}_i) < 0.
\end{aligned}$$

Therefore, we can conclude that

$$\hat{v}_i \in \arg\max_{v_i \in \mathcal{V}_i} \int_{\mathcal{V}_{-i}} u_i(\phi_i^*(v_i, \mathbf{v}_{-i}); \hat{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}.$$

□

[10] See Krisha (2002, Appendix C pp.265-268)

It follows from this theorem that truth-telling is a Bayesian Nash equilibrium in the direct revelation game associated with an allocation rule ϕ^* in Φ^*. Therefore, we can conclude that *efficiency* and *envy-freeness* are simultaneously realized under the notion of Bayesian Nash equilibrium. This is a very positive result as compared with the impossibility theorems under the notion of weakly dominant strategy equilibrium. It is noteworthy that most of the existing literature studies *individual rationality* (for example, Myerson and Satterthwaite, 1983; and Cramton, Gibbons, and Klemperer, 1987)—and not fairness—as an additional property of allocation rules. Only Morgan (2004) focuses on fairness in the model where each agent has a common valuation for the indivisible object. Our theorem implies that a same positive result identical to that of Morgan can be established in the model where each agent has a private valuation for the object.

4 Characterization Theorem

In the previous section, we established the existence of an allocation rule that satisfies *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*. However, this result does not clarify the class of all allocation rules that satisfies the three requirements. We still leave unanswered the question of whether or not there exists another allocation rule satisfying the three requirements.

In this section, we provide a partial answer to that question. We restrict our attention to a class of allocation rules that is familiar and has been well-studied in the related literature. We formulate the following property: monetary transfers are (i) anonymous, i.e., they depend only on the levels of valuations, and are (ii) additive with respect to the levels. It can be recalled that for each $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathcal{V}$ and for each $k = 1, 2, \ldots, n$, let $\mathbf{v}^k$ be the k-th highest valuation among $\mathbf{v}$. Further, let $k(\mathbf{v}) \in I$ be an agent with the k-th highest valuation among $\mathbf{v}$, i.e., $v_{k(\mathbf{v})} = \mathbf{v}^k$.

Anonymous and additive transfer property: An allocation rule $\phi(\cdot) = (x(\cdot), t(\cdot))$ satisfies the *anonymous and additive transfer property* if there exist differentiable functions $\tau_\ell^k : [\underline{v}, \overline{v}] \to \mathbb{R}$, $k, \ell = 1, 2, \ldots, n$ such that for each $\mathbf{v} \in \mathcal{V}$ and for each $\ell = 1, 2, \ldots, n$, $t_{\ell(\mathbf{v})}(\mathbf{v}) = \tau_\ell^1(\mathbf{v}^1) + \tau_\ell^2(\mathbf{v}^2) + \cdots + \tau_\ell^n(\mathbf{v}^n)$.

Although the property concerning monetary transfers is restrictive, the class of allocation rules that satisfy the property includes those that are important and have been well-studied in the related literature.

The first examples of allocation rules that satisfy the property are the first-price and second-price auction rules. Those are the most well-studied allocation rules in auction theory. Let $\phi^I(\mathbf{v}) = (x_i^I(\mathbf{v}), t_i^I(\mathbf{v}))_{i \in I}$ and $\phi^{II}(\mathbf{v}) = (x_i^{II}(\mathbf{v}), t_i^{II}(\mathbf{v}))_{i \in I}$ be the first-price and second-price auction rules respectively, i.e., ϕ^I is an allocation rule such that for each $\mathbf{v} \in \mathcal{V}$, letting $x^I_j(\mathbf{v}) = 1$,

$$v_j = \mathbf{v}^1$$
$$t^I_i(\mathbf{v}) = \begin{cases} -\mathbf{v}^1 & \text{if } i = j \\ \\ 0 & \text{if } i \in I \setminus \{j\}, \end{cases}$$

and ϕ^{II} is an allocation rule such that for each $\mathbf{v} \in \mathcal{V}$, letting $x^{II}_j(\mathbf{v}) = 1$,

$$v_j = \mathbf{v}^1$$
$$t^{II}_i(\mathbf{v}) = \begin{cases} -\mathbf{v}^2 & \text{if } i = j \\ \\ 0 & \text{if } i \in I \setminus \{j\}. \end{cases}$$

The first-price and second-price auction rules are rules in which an agent with the highest valuations receives the indivisible object, and then pays the auctioneer the highest and second-highest valuations respectively. These auction rules obviously satisfy the *anonymous and additive transfer property.*

The next allocation rule based on the Shapley value (Shapley, 1953) also satisfies the monetary transfer property. Let $\phi^{sh}(\mathbf{v}) = (x^{sh}_i(\mathbf{v}), t^{sh}_i(\mathbf{v}))_{i \in I}$ be an allocation rule such that for each $\mathbf{v} \in \mathcal{V}$, letting $x^{sh}_j(\mathbf{v}) = 1$,

$$v_j = \mathbf{v}^1$$
$$t^{sh}_i(\mathbf{v}) = \begin{cases} -\mathbf{v}^1 + \dfrac{\mathbf{v}^n}{n} + \dfrac{\mathbf{v}^{n-1} - \mathbf{v}^n}{n-1} + \cdots + \dfrac{\mathbf{v}^1 - \mathbf{v}^2}{1} & \text{if } i = j \\ \\ \dfrac{\mathbf{v}^n}{n} + \dfrac{\mathbf{v}^{n-1} - \mathbf{v}^n}{n-1} + \cdots + \dfrac{\mathbf{v}^k - \mathbf{v}^{k+1}}{k} & \text{if } i \neq j \text{ with } i = k(\mathbf{v}). \end{cases}$$

At this allocation rule ϕ^{sh}, for each $\mathbf{v} \in \mathcal{V}$ and each $k = 1, 2, \ldots, n$, agent $k(\mathbf{v})$ enjoys the utility level,

$$\frac{\mathbf{v}^n}{n} + \frac{\mathbf{v}^{n-1} - \mathbf{v}^n}{n-1} + \cdots + \frac{\mathbf{v}^k - \mathbf{v}^{k+1}}{k}.$$

This value is the Shapley value (Shapley, 1953) when the worth of each coalition $S \subseteq I$ is defined by $\max_{i \in S} v_i$.[11]

The next class of allocation rules is one of the most important and well-analyzed class in the context of fair allocation problem. For each $\alpha \in [0, 1]$, let $\phi^\alpha(\mathbf{v}) = (x^\alpha_i(\mathbf{v}), t^\alpha_i(\mathbf{v}))_{i \in I}$ be an allocation rule such that for each $\mathbf{v} \in \mathcal{V}$, letting $x^\alpha_j(\mathbf{v}) = 1$,

$$v_j = \mathbf{v}^1$$
$$t^\alpha_i(\mathbf{v}) = \begin{cases} -\dfrac{n-1}{n}((1-\alpha)\mathbf{v}^1 + \alpha \mathbf{v}^2) & \text{if } i = j \\ \\ \dfrac{1}{n}((1-\alpha)\mathbf{v}^1 + \alpha \mathbf{v}^2) & \text{if } i \in I \setminus \{j\}. \end{cases}$$

[11] This argument is based on the simple expression of the Shapley value by Littlechild and Owen (1973)

Each allocation rule ϕ^α is a rule in which an agent with the highest valuation receives the indivisible object, and then pays each of the other agents $\frac{(1-\alpha)\mathbf{v}^1+\alpha\mathbf{v}^2}{n}$. For $\alpha = 0$, ϕ^α is an equal welfare rule that is defined by Tadenuma and Thomson (1993) in which each agent enjoys the "equal utility" $\frac{\mathbf{v}^1}{n}$. For $\alpha = \frac{1}{2}$, ϕ^α is the split-the-difference rule described by Samuelson (1985). It is obvious that each allocation rule ϕ^α satisfies the *anonymous and additive transfer property.* Note that although, by Proposition 1, each allocation rule also satisfies *efficiency* and *envy-freeness*, it does not satisfy *Bayesian incentive compatibility* (Güth and van Damme, 1986; and Cramton, Gibbons, and Klemperer, 1987).[12]

We provide a characterization of *efficient*, *envy-free*, and *Bayesian incentive compatible* allocation rules in the class of allocation rules that satisfy the *anonymous and additive transfer property.*

Theorem 2. An allocation rule ϕ satisfies *efficiency*, *envy-freeness*, *Bayesian incentive compatibility*, and the *anonymous and additive transfer property* if and only if it is an allocation rule in Φ^*.

Proof. By Theorem 1 and the definition of Φ^*, the "if" part of the theorem is obvious. Therefore, we only show the "only if" part.

Let $\phi(\cdot) = (x(\cdot), t(\cdot))$ be an allocation rule that satisfies the four requirements. By Proposition 1, for each $\mathbf{v} \in \mathcal{V}$, $t_{k(\mathbf{v})}(\mathbf{v}) = t_{\ell(\mathbf{v})}(\mathbf{v})$ for each $k, \ell = 2, 3, \dots, n$. For each $\mathbf{v} \in \mathcal{V}$, let $t_\alpha(\mathbf{v})$ and $t_\nu(\mathbf{v})$ be the amount of money that the accepter of the object receives and that the non-accepters do respectively. By Proposition 1, $t_\alpha(\mathbf{v}) = t_{1(\mathbf{v})}(\mathbf{v})$ and $t_\nu(\mathbf{v}) = t_{\ell(\mathbf{v})}(\mathbf{v})$ for some $\ell \neq 1$. Also, by Proposition 1, $t_\alpha(\mathbf{v}) = -(n-1)t_\nu(\mathbf{v})$ and $\frac{\mathbf{v}^2}{n} \leq t_\nu(\mathbf{v}) \leq \frac{\mathbf{v}^1}{n}$ for each $\mathbf{v} \in \mathcal{V}$.

By the *anonymous and additive transfer property*, there exist differentiable functions $\tau^k : [\underline{v}, \overline{v}] \to \mathbb{R}$, $k = 1, 2, \dots, n$ such that for each $\mathbf{v} \in \mathcal{V}$, $t_\nu(\mathbf{v}) = \tau^1(\mathbf{v}^1) + \tau^2(\mathbf{v}^2) + \cdots + \tau^n(\mathbf{v}^n)$. Thus, for each $\mathbf{v} \in \mathcal{V}$,

$$\frac{\mathbf{v}^2}{n} \leq t_\nu(\mathbf{v}) = \tau^1(\mathbf{v}^1) + \tau^2(\mathbf{v}^2) + \cdots + \tau^n(\mathbf{v}^n) \leq \frac{\mathbf{v}^1}{n}. \tag{8}$$

For each $v, v' \in [\underline{v}, \overline{v}]$, let us consider profiles of valuations $\mathbf{v} = (\overline{v}, \overline{v}, \dots, \overline{v}, v)$ and $\mathbf{v}' = (\overline{v}, \overline{v}, \dots, \overline{v}, v')$. By (8), we have that

$$t_\nu(\mathbf{v}) = \tau^1(\overline{v}) + \tau^2(\overline{v}) + \cdots + \tau^{n-1}(\overline{v}) + \tau^n(v) = \frac{\overline{v}}{n}$$
$$t_\nu(\mathbf{v}') = \tau^1(\overline{v}) + \tau^2(\overline{v}) + \cdots + \tau^{n-1}(\overline{v}) + \tau^n(v') = \frac{\overline{v}}{n}.$$

These equations imply that $\tau^n(v) = \tau^n(v')$ for each $v, v' \in [\underline{v}, \overline{v}]$. Therefore, τ^n is constant, i.e., $\tau^n(v) = c_n \in \mathbb{R}$ for each $v \in [\underline{v}, \overline{v}]$. Similar arguments lead us to the

[12]The results of Güth and van Damme (1986) and Cramton, Gibbons, and Klemperer (1987) imply that allocations realized through a Bayesian Nash equilibrium in the direct revelation game associated with ϕ^α is not *envy-free* at a true valuation. This fact contrasts greatly with the results in the model with complete information (Tadenuma and Thomson, 1995; Fujinaka and Sakai, 2006; and Fujinaka and Sakai, 2007b).

result that for each $k = 3, 4, \ldots, n-1$, τ^k is constant, i.e., $\tau^k(v) = c_k \in \mathbb{R}$ for each $v \in [\underline{v}, \overline{v}]$. Thus, we have that for each $\mathbf{v} \in \mathcal{V}$, $t_\nu(\mathbf{v}) = \tau^1(\mathbf{v}^1) + \tau^2(\mathbf{v}^2) + \sum_{k=3}^n c_k$. Obviously, we may rewrite this as

$$t_\nu(\mathbf{v}) = \tau^1(\mathbf{v}^1) + \tau^2(\mathbf{v}^2). \tag{9}$$

For each $v \in [\underline{v}, \overline{v}]$, let us consider a profile of valuations $\mathbf{v}$ such that $\mathbf{v}^1 = \mathbf{v}^2 = v$. By (8) and (9), for each $v \in [\underline{v}, \overline{v}]$,

$$\tau^1(v) + \tau^2(v) = \frac{v}{n}. \tag{10}$$

Since τ^1 and τ^2 is differentiable, by differentiating this, we can obtain that for each $v \in [\underline{v}, \overline{v}]$,

$$\frac{d\tau^1(v)}{dv} + \frac{d\tau^2(v)}{dv} = \frac{1}{n}. \tag{11}$$

For each agent $i \in I$ and each true valuation $\hat{v}_i \in \mathcal{V}_i$, when he reports his valuation as v_i, then his expected utility is, recalling $f(y_1, y_2)$ is the joint density of the first and second order statistics of $V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_n$,

$$\begin{aligned}
&\int_{\mathcal{V}_{-i}} u_i(\phi_i(v_i, \mathbf{v}_{-i}); \hat{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \\
&= \int_{\underline{v}}^{v_i} \int_{\underline{v}}^{y_1} [\hat{v}_i - (n-1)(\tau^1(v_i) + \tau^2(y_1))] f(y_1, y_2) dy_2 dy_1 \\
&\quad + \int_{v_i}^{\overline{v}} \int_{\underline{v}}^{v_i} (\tau^1(y_1) + \tau^2(v_i)) f(y_1, y_2) dy_2 dy_1 \\
&\quad + \int_{v_i}^{\overline{v}} \int_{v_i}^{y_1} (\tau^1(y_1) + \tau^2(y_2)) f(y_1, y_2) dy_2 dy_1.
\end{aligned}$$

By *Bayesian incentive compatibility*,

$$\begin{aligned}
&\frac{d}{dv_i} \int_{\mathcal{V}_{-i}} u_i(\phi_i(\hat{v}_i, \mathbf{v}_{-i}); \hat{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \\
&= -(n-1) \frac{d\tau^1(\hat{v}_i)}{dv} F(\hat{v}_i)^{n-1} + (n-1) \frac{d\tau^2(\hat{v}_i)}{dv} (1 - F(\hat{v}_i)) F(\hat{v}_i)^{n-2} = 0.
\end{aligned}$$

Therefore, we have that

$$\frac{d\tau^1(\hat{v}_i)}{dv} F(\hat{v}_i) - \frac{d\tau^2(\hat{v}_i)}{dv} (1 - F(\hat{v}_i)) = 0.$$

Substituting (11) for this , we can obtain that

$$\frac{d\tau^1(\hat{v}_i)}{dv} = \frac{1}{n}(1 - F(\hat{v}_i)).$$

It follows from the fundamental theorem of calculus that for each $v \in [\underline{v}, \overline{v}]$,

$$\begin{aligned}\tau^1(v) &= \tau^1(\underline{v}) + \int_{\underline{v}}^{v} \frac{1}{n}(1 - F(t))dt \\ &= \tau^1(\underline{v}) - \frac{\underline{v}}{n} + \frac{1}{n}\left(v - \int_{\underline{v}}^{v} F(t)dt\right).\end{aligned}$$

Substituting (10) for this, we have that for each $v \in [\underline{v}, \overline{v}]$,

$$\tau^2(v) = -\tau^1(\underline{v}) + \frac{\underline{v}}{n} + \frac{1}{n}\int_{\underline{v}}^{v} F(t)dt.$$

Therefore, by (9), we can conclude that for each $\mathbf{v} \in \mathcal{V}$,

$$t_\nu(\mathbf{v}) = \frac{1}{n}\left(\mathbf{v}^1 - \int_{\mathbf{v}^2}^{\mathbf{v}^1} F(v)dv\right).$$

Therefore, we can conclude that the allocation rule ϕ is in Φ^*. □

This theorem gives us a partial answer to the question posed above. It follows from this theorem that in the class of allocation rules that satisfy the *anonymous and additive transfer property*, only allocation rules in Φ^* satisfy the three desirable properties of *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*. It is, however, necessary to note that we have not answered the above question completely. There may exist an allocation rule satisfying the three desirable properties and that does not satisfy the *anonymous and additive transfer property*. Unfortunately, since the *anonymous and additive transfer property* is restrictive, it is difficult for us to infer the complete answer from Theorem 2. Therefore, we must leave the problem unsolved.

5 Concluding remarks

We establish the result that there exists an allocation rule satisfying *efficiency*, *envy-freeness*, and *Bayesian incentive compatibility*. Further, in the important class of allocation rules that satisfies a certain anonymous and additive monetary transfer property, there does not exist any allocation rule that satisfies the three properties except any allocation rule in Φ^*. In the direct revelation game associated with an allocation rule ϕ^* in Φ^*, truth-telling is a Bayesian Nash equilibrium, and *efficiency* and *envy-freeness* are achieved simultaneously. However, the direct revelation game is not "simple" in the same sense as that of Cramton, Gibbons, and Klemperer (1987) and McAfee (1992). "Simple" games (or mechanisms) are those in which the rules of the games do not depend on the distribution of the valuation of agents, F. We have not addressed the issue of designing the "simple" mechanism that implements the allocation rule ϕ^*. This issue is left to be addressed in the future research.

We establish a positive result in the private valuations model. Morgan (2004) also produces a positive result in the common valuations model. These two models are regarded as special cases of the interdependent valuations model in which each agent only has partial information about his valuation. It would also be interesting if future research were to formulate a framework for unifying the two positive results in the interdependent valuations model.

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