Spatial Economics and Potential Games

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Outline

- Potential Games with a Continuum of Players
  Sandholm (2001)
- Myopic Evolutionary Dynamics in Potential Games
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- Perfect Foresight Dynamics in Potential Games
  Hofbauer and Sorger (1999)
- Core-Periphery Model as a Potential Game
  Oyama (2006)

Potential Games with a Continuum of Players

A continuum of homogeneous and anonymous players with mass 1;
$A = \{1, \dots, n\}$: (common) set of actions.

$\Delta = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$: set of action distributions.
($\bar{\Delta} \subset \mathbb{R}^n$: a neighborhood of Δ; $e_i \in \Delta$: ith vertex.)

A population game is a profile $(u_i)_{i=1}^n$ of C^1 functions $u_i \colon \Delta \to \mathbb{R}$ ($u_i(x)$: the payoff to action i when the action distribution is $x \in \Delta$).

Ex.:
Random matching play of a symmetric two-player normal form game:

$$u(x) = Ux \quad (x \in \Delta), \qquad (u(x) = (u_1(x), \dots, u_n(x))')$$

$U \in \mathbb{R}^n \times \mathbb{R}^n$.

Ex. (cont.):
Traffic networks: $A =$ set of routes.

Spatial economics: $A =$ set of locations (regions, countries...)
(e.g., Krugman (1991): presence of trade costs).

Economic development: $A =$ set of sectors
(e.g., Matsuyama (1991, 1992)).

Potential Functions for Population Games (Sandholm (2001))

Definition 1. A function $v\colon \bar{\Delta} \to \mathbb{R}$ is said to be a *potential function* of $(u_i)_{i=1}^n$ if for all $i, j = 1, \dots, n$,

$$\frac{\partial v}{\partial x_i}(x) - \frac{\partial v}{\partial x_j}(x) = u_i(x) - u_j(x) \quad \text{for all } x \in \Delta. \tag{1}$$

$(u_i)_{i=1}^n$ is said to be a *potential game* if it admits a potential function.

$x^* \in \Delta$ is a Nash equilibrium of $(u_i)_{i=1}^n$
$\Longleftrightarrow$ it satisfies the Kuhn-Tucker first-order conditions for

Maximize $v(x)$ subject to $x \in \Delta$.

Local (in particular, global) maximizers $\Rightarrow$ Nash equilibria.

Examples

(1) Two-action ($n = 2$) case: a potential function trivially exists:

$$v(x_1, x_2) = \int_0^{x_1} u_1(x_1, 1 - x_1)\, dx_1 + \int_0^{x_2} u_2(1 - x_2, x_2)\, dx_2.$$

(2) Random-matching of a symmetric $n \times n$ game: $u(x) = Ux$:
If $U = V + W$ where

$$V\text{: symmetric}, \qquad W = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \\ w_1 & w_2 & \cdots & w_n \\ \vdots & \vdots & & \vdots \\ w_1 & w_2 & \cdots & w_n \end{pmatrix},$$

then this game is a potential game with

$$v(x) = \frac{1}{2} x' V x.$$

A Characterization of Potential Games

(Hofbauer (1985), Hofbauer and Sigmund (1998, Theorem 19.5.4))

Suppose $(u_i)_{i=1}^n$ is defined on a neighborhood of Δ.

$(u_i)_{i=1}^n$ admits a potential function
$\Longleftrightarrow$ it satisfies *triangular integrability*:

$$\frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_k}(x) + \frac{\partial u_k}{\partial x_i}(x) = \frac{\partial u_i}{\partial x_k}(x) + \frac{\partial u_k}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x),$$

or equivalently, *symmetric externalities*:

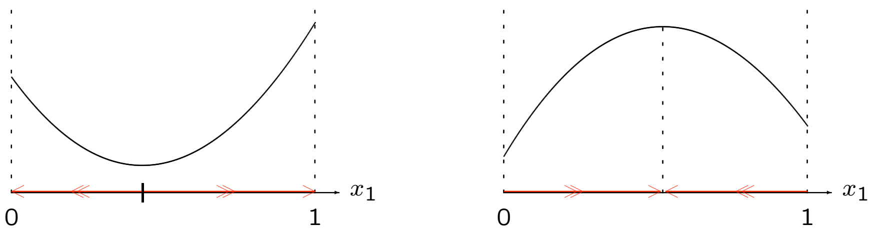
$$\frac{\partial (u_i - u_k)}{\partial (e_j - e_k)}(x) = \frac{\partial (u_j - u_k)}{\partial (e_i - e_k)}(x).$$

$\left(= \frac{\partial (u_i - u_k)}{\partial x_j}(x) - \frac{\partial (u_i - u_k)}{\partial x_k}(x)\right)$

Myopic Evolutionary Dynamics in Potential Games

A potential function works as a global Lyapunov function for many reasonable myopic dynamics.

Ex. ($n = 2$)



Myopic Evolutionary Dynamics in Potential Games

A potential function works as a global Lyapunov function for many reasonable myopic dynamics.

- Best response dynamics
  Gilboa and Matsui (1991), Matsui (1992), Hofbauer (1995)
- Perturbed best response dynamics
  Hofbauer (2000), Hofbauer and Sandholm (2002)
- Replicator dynamics
  Taylor and Jonker (1978), Hofbauer and Sigmund (1998)

Best Response Dynamics (1/2)

$$\dot{x} \in B(x) - x, \tag{BRD}$$

where
$B(x) = \{\alpha \in \Delta \mid \alpha_i > 0 \Rightarrow u_i(x) \geq u_j(x) \text{ for all } j\} = \arg\max_{\alpha\in\Delta} \alpha' u(x)$.

Interpretation:
During $[t, t+dt)$, fraction $\lambda \cdot dt$ of players can revise actions ($\lambda \equiv 1$):

$$x(t+dt) = \alpha(t)\lambda dt + x(t)(1-\lambda dt) \qquad \alpha(t) \in \Delta,$$

where revising players play *myopic* best responses to $x(t)$:

$$\alpha_i(t) > 0 \Rightarrow u_i(x(t)) \geq u_j(x(t)) \quad \forall\, j.$$

Best Response Dynamics (2/2)

$$\dot{x} \in B(x) - x, \tag{BRD}$$

where
$B(x) = \{\alpha \in \Delta \mid \alpha_i > 0 \Rightarrow u_i(x) \geq u_j(x) \text{ for all } j\} = \arg\max_{\alpha\in\Delta} \alpha' u(x)$.

Observation. If $(u_i)_{i=1}^n$ has a potential function v, then

$$\begin{aligned} \frac{d}{dt}v(x) &= \nabla v(x)'\dot{x} \\ &= \alpha'\nabla v(x) - x'\nabla v(x) \geq 0, \qquad (\alpha \in B(x)) \end{aligned}$$

with equality only at Nash equilibria.
(Hofbauer (2000))

Perturbed Best Response Dynamics (1/3)

$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$: random vector with a strictly positive density on $\mathbb{R}^n$.
$\tilde{u}_j = u_j + \varepsilon_j$: perturbed payoff.
$C_i(u) = \Pr(\arg\max_j \tilde{u}_j = i)$: choice probability function ($u = (u_1, \ldots, u_n)'$).

$$\dot{x}_i = \Pr(\arg\max_j \tilde{u}_j(x) = i) - x_i. \tag{PBRD}$$

Ex.:
If ε_i: i.i.d. with extreme value distribution ($F(x) = \exp[-\exp[-\eta^{-1}x - \gamma]]$),
then this is the logit best response dynamics:

$$\Pr(\arg\max_j \tilde{u}_j = i) = \frac{\exp(\eta^{-1}u_i)}{\sum_j \exp(\eta^{-1}u_j)} \qquad (\eta \in (0,\infty)\text{: "noise level"}).$$

Perturbed Best Response Dynamics (2/3)

$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$: random vector with a strictly positive density on $\mathbb{R}^n$.
$\tilde{u}_j = u_j + \varepsilon_j$: perturbed payoff.
$C_i(u) = \Pr(\arg\max_j \tilde{u}_j = i)$: choice probability function ($u = (u_1, \ldots, u_n)'$).

$$\dot{x}_i = \Pr(\arg\max_j \tilde{u}_j(x) = i) - x_i. \qquad \text{(PBRD)}$$

Interpretation:
- Heterogeneity in players' preferences;

- Idiosyncratic preference shocks at individual level (+ "no aggregate uncertainty").

Perturbed Best Response Dynamics (3/3)

$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$: random vector with a strictly positive density on $\mathbb{R}^n$.
$\tilde{u}_j = u_j + \varepsilon_j$: perturbed payoff.
$C_i(u) = \Pr(\arg\max_j \tilde{u}_j = i)$: choice probability function ($u = (u_1, \ldots, u_n)'$).

$$\dot{x}_i = \Pr(\arg\max_j \tilde{u}_j(x) = i) - x_i. \qquad \text{(PBRD)}$$

Theorem. There exists $V : \mathrm{int}\Delta \to \mathbb{R}$ (convex, steep near boundary):

$$C(u) = \arg\max_{z \in \mathrm{int}\Delta} z'u - V(z). \qquad (C(u) = (C_1(u), \ldots, C_n(u))')$$

(ex.: Logit $\Rightarrow V(z) = \eta \sum_j z_j \log z_j$: "entropy function".)

Theorem. (Hofbauer (2000)) If $(u_i)_{i=1}^n$ has a potential function v, then $v(x) - V(x)$ is a Lyapunov function of (PBRD).

Replicator Dynamics

$$\dot{x}_i = x_i(u_i(x) - \bar{u}(x)), \qquad \text{(RD)}$$

$\bar{u}(x) = x'u(x)$: average payoff.

Satisfies "Positive Correlation" (Sandholm (2001)):

$$u(x)'\dot{x} = \sum_i x_i(u_i(x))^2 - (\bar{u}(x))^2 \geq 0$$

with "=" if $u_i(x) = \bar{u}(x)$ for all i with $x_i > 0$.

Observation. If $(u_i)_{i=1}^n$ has a potential function v, then

$$\frac{d}{dt}v(x) = \nabla v(x)'\dot{x} = u(x)'\dot{x} \geq 0.$$

Application: Evolutionary Implementation

Negative externality:
Sandholm (2002, 2005).

Positive externality + stochastic evolution:
Sandholm (2007a).

Perfect Foresight Dynamics in Potential Games

Players choose current actions based on *expectations* about the future behavior pattern of the society.

We consider equilibrium paths, or *perfect foresight paths*, of the dynamics model.

Expected path $\phi^e \colon [0, \infty) \to \Delta$

$\Downarrow$

path $\phi' \colon [0, \infty) \to \Delta$ resulting from best responses to ϕ^e.

Equilibrium paths (from a given initial state)

$=$ Fixed points of the correspondence $\phi^e \mapsto \phi'$.

Perfect Foresight Dynamics

Poisson action revisions with param $\lambda \equiv 1$ (independent across players).

$\longrightarrow$ Fraction $\lambda \cdot dt$ of players can revise their actions.

Expected discounted utility (during a lock-in):

$$V_i(\phi^e)(t) = (\lambda + \theta) \int_t^\infty e^{-(\lambda+\theta)(s-t)} u_i(\phi^e(s))\, ds.$$

$\theta > 0$: discount rate. (θ/λ: *degree of friction*).

$\phi^e \colon [0, \infty) \to \Delta$: anticipated feasible path.

A perfect foresight path (PF-path) is a path of distributions along which revising players choose an action that maximizes the expected discounted utility.

Perfect Foresight Paths

Definition.

$\phi \colon [0, \infty) \to \Delta$ is a PF-path if

$$\dot\phi(t) = \lambda\big(\alpha(t) - \phi(t)\big) \quad \alpha(t) \in \Delta$$

$$(\phi(t + dt) = \alpha(t)\lambda dt + \phi(t)(1 - \lambda dt).)$$

such that

$$\alpha_i(t) > 0 \Rightarrow V_i(\phi)(t) \geq V_j(\phi)(t) \quad \forall j.$$

$V_i(\phi)(t)$: expected discounted utility from action i at time t.

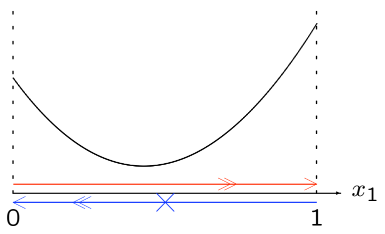
Observation. (Oyama, Takahashi, and Hofbauer (2003))

For each initial state, a PF-path exists.

Ex.: $n = 2$ with a convex potential

Suppose the discount rate $\theta > 0$ is sufficiently small

(i.e., future payoffs are important):



$\longrightarrow$ is a PF-path $\Rightarrow (x_1, x_2) = (1, 0)$ is *globally accessible*.

$\longleftarrow$ is NOT a PF-path $\Rightarrow (x_1, x_2) = (1, 0)$ is *absorbing*.

Stability under PFD

Definition.

A distribution $x^* \in \Delta$ is *absorbing* if
$\exists$ neighborhood of x^*, $\forall$ PF-path converges to x^*.

A state $x^* \in \Delta$ is *globally accessible* if
$\forall$ initial distribution, $\exists$ PF-path that converges to x^*.

Theorem. (Hofbauer and Sorger (JET 1999))
Suppose $(u_i)_{i=1}^n$ admits a potential function v, and assume $\{x^*\} = \arg\max_{x\in\Delta} v(x)$.

Then, x^* is the unique state that is absorbing and globally accessible for any sufficiently small discount rate.

Sketch of Proof.

Consider the maximization problem (v: potential function):

$$\text{Maximize} \quad J(\phi) = \int_0^\infty \theta e^{-\theta t} v(\phi(t))\, dt$$
$$\text{subject to} \quad \phi : \text{feassible path.}$$

ϕ^*: solution $\Rightarrow$ ϕ^*: perfect foresight path.

Turnpike property:
There exists $\bar{\theta} > 0$ such that if $\theta \in (0, \bar{\theta})$,
then for any optimal solution ϕ,
$\phi(t)$ gets close to the potential maximizer for some t.

$+$ Neighborhood turnpike property (= absorption)

$\Rightarrow$ Global accessibility.

An Open Problem

Suppose $x^* \in \Delta$ is a unique global potential maximizer.

For each initial condition x^0 in a neighborhood of x^*,
is a PF-path from x^0 unique?

(Note: absorption does not require uniqueness of PF-paths.)

Core-Periphery Model as a Potential Game

"New Economic Geography" (Krugman (1991) and others)

General equilibrium models with
- scale economies (Dixit-Stiglitz type),
- trade costs ("iceberg"),
- production factor mobility (with two locations).

Interested in distributions of manufacturing firms
that are stable under *myopic* evolutionary dynamics.

Standard result:
Trade costs: Large $\Rightarrow$ Dispersion;
Small $\Rightarrow$ Agglomeration (a "core" and a "periphery").
Only "history" matters.

History versus Expectations

Relevance of forward-looking expectations in relocation decisions.
$\leftarrow$ standard response: "Difficult to consider expectations".

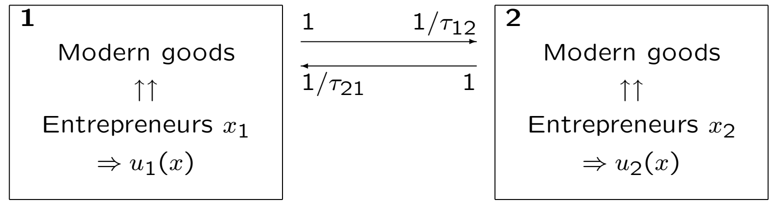
Oyama (2006) considers a version of CP model with n countries which admits a potential function.

$\Rightarrow$ Apply the results by Hofbauer and Sorger (1999):
a (unique) global potential maximizer is uniquely stable under PFD
(for small discount rates).

$\Rightarrow$ Just characterize the shape of the potential function
for various values of the key parameters (such as trade costs).

Model

Ex.: $n = 2$



$\tau_{ji} > 1$: trade cost from country j to country i,
$u_i(x)$: indirect utility from locating in country i.

$\Rightarrow (u_i)_{i=1}^n$: reduced population game.

Key Assumption

Trade cost depends only on the trade barrier of the destination:

$$\tau_{ji} = \tau_{ki} \ (=: \tau_i) \quad \text{for all } j, k \neq i.$$

$\tau_i > 1$: *trade barrier* of country i.

(+ Quasi-linear preference.)

Observation.

The population game $(u_i)_{i=1}^n$ admits a potential function v.

Trade barriers:
Low $\Rightarrow$ v: convex;
Hight $\Rightarrow$ v: concave.

Proposition.

Assume trade barriers are sufficiently low.
If (trade barrier of country i^*) $>$ (trade barrier of country j) $\forall j \neq i$,
then agglomeration in i^* is the absorbing and globally accessible state
for a sufficiently small discount rate.

(Expectations matter.)

Proposition.

If trade barriers are sufficiently high,
then there exists a unique potential maximizer $x^* \in \mathrm{int}(\Delta)$,
and any PF-path converges to x^* for any discount rate.

(Expectations as well as history play no role.)

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Notes

1 Potential Games with a Continuum of Players

Sandholm (2001). See also Sandholm (2007b).

2 Myopic Evolutionary Dynamics in Potential Games

Hofbauer (2000), Sandholm (2001), Hofbauer and Sandholm (2002).

Textbooks: Fudenberg and Levine (1998), Hofbauer and Sigmund (1998), Weibull (1997).

Application of perturbed best response dynamics to New Economic Geography: Murata (2003), Tabuchi and Thisse (2002).

3 Perfect Foresight Dynamics in Potential Games

Industrialization: Matsuyama (1991, 1992).

Random matching: Matsui and Matsuyama (1995), Hofbauer and Sorger (1999), Oyama, Takahashi, and Hofbauer (2003).

See also Hofbauer and Sorger (2002), Oyama (2002), Matsui and Oyama (2006), Oyama and Tercieux (2004), Takahashi (2005).

4 Core-Periphery Model as a Potential Game

Krugman (1991).

Textbooks: Baldwin *et al.* (2003), Fujita, Krugman, and Venables (1999).