7. Choquet Integral

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Ellsberg Paradox

- An urn contains 30 Red balls and 60 Green and Blue balls.

  $\Omega = \{R, G, B\}$
- Bets (or acts)
  - $f_R(R) = 10$, $f_R(G) = 0$, $f_R(B) = 0$
  - $f_G(R) = 0$, $f_G(G) = 10$, $f_G(B) = 0$
  - $f_{RB}(R) = 10$, $f_{RB}(G) = 0$, $f_{RB}(B) = 10$
  - $f_{GB}(R) = 0$, $f_{GB}(G) = 10$, $f_{GB}(B) = 10$
- Typical preferences:

  $f_R \succ f_G$ and $f_{RB} \prec f_{GB}$

- Expected utility theory fails to explain these preferences.
- p: probability measure on Ω

  $u\colon \mathbb{R} \to \mathbb{R}$: utility function $\qquad (0 = u(0) \neq u(10))$
- $\int u(f_R)dp = p(\{R\})u(10)$, $\int u(f_G)dp = p(\{G\})u(10)$

  $\int u(f_{RB})dp = p(\{R, B\})u(10)$,
  $\int u(f_{GB})dp = p(\{G, B\})u(10)$
- $\int u(f_R)dp > \int u(f_G)dp \iff p(\{R\}) > p(\{G\})$

  $\int u(f_{RB})dp < \int u(f_{GB}) \iff p(\{R\}) < p(\{G\})$

  $\cdots$ contradiction!

Capacities

- $\Omega = \{1, \ldots, n\}$: (Finite) state space

Definition 7.1
$v\colon 2^{\Omega} \to [0, 1]$ is called a *capacity* if it satisfies the following:

1. $v(\emptyset) = 0$,
2. $v(\Omega) = 1$, and
3. $v(E) \leq v(F)$ whenever $E \subset F$.

- A capacity v is *additive* if
  $v(E) + v(F) = v(E \cap F)$ for all $E, F \subset \Omega$ with $E \cap F = \emptyset$.
  - *Probability measures* are a special case of capacities.
- A capacity v is *convex* if it is supermodular with respect to $\subset$:
  $v(E) + v(F) \leq v(E \cup F) + v(E \cap F)$ for all $E, F \subset \Omega$.

Choquet Integral

- For $X : \Omega \to \mathbb{R}$, order the states so that $X(\omega_1) \leq \cdots \leq X(\omega_n)$.
- We write $v(X \geq t) = v(\{\omega \in \Omega \mid X(\omega) \geq t\})$.

Definition 7.2

For $X : \Omega \to \mathbb{R}$, the *Choquet integral* of X with respect to capacity v is defined by

$$\begin{aligned}\int X dv &= \int_0^{\infty} v(X \geq t)dt + \int_{-\infty}^{0} (v(X \geq t) - 1)dt \\ &= X(\omega_1) + \sum_{k=2}^{n} (X(\omega_k) - X(\omega_{k-1}))v(\{\omega_k, \ldots, \omega_n\}) \\ &= \sum_{k=1}^{n} X(\omega_k)(v(\{\omega_k, \ldots, \omega_n\}) - v(\{\omega_{k+1}, \ldots, \omega_n\})).\end{aligned}$$

Properties

1. $\int \mathbf{1}_E dv = v(E)$ for any $E \subset \Omega$.
2. [Monotonicity] If $X \leq Y$, then $\int X dv \leq \int Y dv$.
3. [Homogeneity] $\int tX dv = t \int X dv$ for any $t \geq 0$.
4. [Comonotonic additivity] If X and Y are *comonotonic*, i.e., $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$ for all $\omega, \omega' \in \Omega$, then $\int (X+Y) dv = \int X dv + \int Y dv$.
5. If v is additive, then $\int X dv = \sum_{\omega \in \Omega} X(\omega) v(\{\omega\})$.

Choquet Expected Utility Theory

- A preference relation $\precsim$ on the set of functions from Ω to $\mathbb{R}$ admits a Choquet expected utility representation if there exist a capacity v and a utility function u such that

$$f \precsim g \iff \int u(f)dv \leq \int u(g)dv.$$

Ellsberg Paradox "Resolved"

- $\Omega = \{R, G, B\}$
- Bets
  - $f_R(R) = 10$, $f_R(G) = 0$, $f_R(B) = 0$
  - $f_G(R) = 0$, $f_G(G) = 10$, $f_G(B) = 0$
  - $f_{RB}(R) = 10$, $f_{RB}(G) = 0$, $f_{RB}(B) = 10$
  - $f_{GB}(R) = 0$, $f_{GB}(G) = 10$, $f_{GB}(B) = 10$
- $f_R \succ f_G$ and $f_{RB} \prec f_{GB}$

- Assume $0 = u(0) < u(10)$.
- For a capacity v,

  $\int u(f_R)dv = v(\{R\})u(10)$, $\int u(f_G)dv = v(\{G\})u(10)$

  $\int u(f_{RB})dv = v(\{R,B\})u(10)$,
  $\int u(f_{GB})dv = v(\{G,B\})u(10)$
- If, for example,

$$v(\{R\}) = v(\{R,G\}) = v(\{R,B\}) = \tfrac{1}{3},$$
$$v(\{G\}) = v(\{B\}) = 0,\ v(\{G,B\}) = \tfrac{2}{3},$$

  then

  $\int u(f_R)dv = \frac{1}{3}u(10) > \int u(f_G)dv = 0$,

  $\int u(f_{RB})dv = \frac{1}{3}u(10) < \int u(f_{GB})dv = \frac{2}{3}u(10)$.

- This capacity v is not additive and is convex.
- In particular,

$$v(\{R, G\}) + v(\{R, B\}) = \frac{1}{3} + \frac{1}{3} < v(\{R\}) + v(\Omega) = \frac{1}{3} + 1.$$

- Perception of uncertainty/ambiguity +
  aversion to uncertainty/ambiguity

Maxmin Expected Utility Theory

- Denote by Δ the set of probability measures on Ω:

$$\Delta = \{p \in \mathbb{R}^n \mid p_i \geq 0,\ \sum_{i=1}^n p_i = 1\}.$$

- A preference relation $\precsim$ on the set of functions from Ω to $\mathbb{R}$ admits a maxmin expected utility representation if there exist a convex closed set $C \subset \Delta$ and a utility function u such that

$$f \precsim g \iff \min_{p \in C} \int u(f) dp \leq \min_{p \in C} \int u(g) dp.$$

- Non-singleton $C \cdots$ perception of uncertainty/ambiguity

  $\min_{p \in C} \cdots$ aversion to uncertainty/ambiguity

Ellsberg Paradox "Resolved"

- $\Omega = \{R, G, B\}$
- Bets
  - $f_R(R) = 10$, $f_R(G) = 0$, $f_R(B) = 0$
  - $f_G(R) = 0$, $f_G(G) = 10$, $f_G(B) = 0$
  - $f_{RB}(R) = 10$, $f_{RB}(G) = 0$, $f_{RB}(B) = 10$
  - $f_{GB}(R) = 0$, $f_{GB}(G) = 10$, $f_{GB}(B) = 10$
- $f_R \succ f_G$ and $f_{RB} \prec f_{GB}$

- Assume $0 = u(0) < u(10)$.
- Let, for example,

$$C = \{p \in \Delta \mid p(\{R\}) = \tfrac{1}{3}\}.$$

- Then

$\min_{p\in C} \int u(f_R)dp = \frac{1}{3}u(10) > \min_{p\in C} \int u(f_G)dp = 0$,
$\min_{p\in C} \int u(f_{RB})dp = \frac{1}{3}u(10) < \min_{p\in C} \int u(f_{GB})dp = \frac{2}{3}u(10)$.

Relationship

- The *core* of capacity v:

$$C(v) = \{p \in \Delta \mid p(E) \geq v(E) \text{ for all } E \subset \Omega\}.$$

Proposition 7.1

For a capacity v, define $I(X) = \int X dv$ ($X : \Omega \to \mathbb{R}$).
The following statements are equivalent:

1. *v is convex.*
2. *$I(X) = \min_{p \in C(v)} \int X dp$ for all $X : \Omega \to \mathbb{R}$.*
3. *$I(X + Y) \geq I(X) + I(Y)$ for all $X, Y : \Omega \to \mathbb{R}$.*

- Under homogeneity, 3 is equivalent to concavity of I.

- Recall the capacity v given by

$$v(\{R\}) = v(\{R,G\}) = v(\{R,B\}) = \frac{1}{3},$$
$$v(\{G\}) = v(\{B\}) = 0,\ v(\{G,B\}) = \frac{2}{3},$$

- $C(v) = \{p \in \Delta \mid p(\{R\}) = \frac{1}{3}\}$.
- Bets
  - $f_R(R) = 10$, $f_R(G) = 0$, $f_R(B) = 0$
  - $f_G(R) = 0$, $f_G(G) = 10$, $f_G(B) = 0$
  - $f_B(R) = 0$, $f_B(G) = 0$, $f_B(B) = 10$.

  $f_{RB} = f_R + f_B$, $f_{GB} = f_G + f_B$
- $I(f_R + f_B) > I(f_R) + I(f_B)$, $I(f_G + f_B) > I(f_G) + I(f_B)$

  Compatible with $I(f_R) > I(f_G)$, $I(f_R + f_B) < I(f_G + f_B)$.

Proof of Proposition 7.1

- $1 \Rightarrow 2$: By Propositions 5.2 and 6.9.
- Indeed, fix any $X : \Omega \to \mathbb{R}$, and suppose that $X(\omega_1) \leq \cdots \leq X(\omega_n)$.
- Let $\alpha^\sigma \in \Delta$ be the marginal contribution vector associated with permutation $\sigma = (\omega_n, \ldots, \omega_1)$.

  Then by the definition of the Choquet integral, we have $\int X dv = \int X d\alpha^\sigma$.
- Also, by the definition of the Choquet integral, we have $\int X dv \leq \int X dp$ for any $p \in C(v)$.
- Since $\alpha^\sigma \in C(v)$ by the convexity of v (Proposition 5.2), we therefore have $\int X dv = \min_{p \in C(v)} \int X dp$.

- $2 \Rightarrow 3$:

  Let $p \in C(v)$ be such that $I(X+Y) = \int (X+Y) dp$.
- Then $I(X+Y) = \int X dp + \int Y dp \geq I(X) + I(Y)$.
- $3 \Rightarrow 1$:

  Fix any $E, F \subset \Omega$.
- Since $\mathbf{1}_{E \cup F}$ and $\mathbf{1}_{E \cap F}$ are comonotonic, we have

$$
\begin{aligned}
& v(E \cup F) + v(E \cap F) \\
& = I(\mathbf{1}_{E \cup F}) + I(\mathbf{1}_{E \cap F}) \\
& = I(\mathbf{1}_{E \cup F} + \mathbf{1}_{E \cap F}) \qquad \text{(by comonotonic additivity)} \\
& = I(\mathbf{1}_E + \mathbf{1}_F) \\
& \geq I(\mathbf{1}_E) + I(\mathbf{1}_F) = v(E) + v(F).
\end{aligned}
$$