1. Farkas' Lemma

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Things to know from linear algebra

- For $S = \{x^1, x^2, \ldots\} \subset \mathbb{R}^n$,
  we also use S to denote the index set, i.e., $S = \{1, 2, \ldots\}$.
- For a finite set $S = \{x^1, x^2, \ldots, x^J\} \subset \mathbb{R}^n$, the vector

  $$\sum_{j \in S} \lambda_j x^j$$

  with $\lambda_1, \ldots, \lambda_J \in \mathbb{R}$ is called a *linear combination* of S.
- For $S \subset \mathbb{R}^n$, the set of all linear combinations of finite subsets of S is called the *span* of S and denoted by $\operatorname{span}(S)$.
- A finite set $S \subset \mathbb{R}^n$ is *linearly independent* (LI) if

  $$\sum_{j \in S} \lambda_j x^j = 0 \Rightarrow \lambda_j = 0 \text{ for all } j \in S.$$

- A finite set $S \subset \mathbb{R}^n$ is *linearly dependent* (LD) if it is not linearly independent.

- For $S \subset \mathbb{R}^n$, a finite $B \subset S$ is a *maximal LI subset* of S if $B \cup \{x\}$ is LD for any $x \in S \setminus B$.

  A maximal LI subset of S is called a *basis* of S.

- Write e^i for the ith standard unit vector of $\mathbb{R}^n$, i.e., $e^i = (0, \ldots, \underset{i}{1}, \ldots, 0)$.

  $\{e^1, \ldots, e^n\}$ is a basis of $\mathbb{R}^n$.

Proposition 1.1

If B is a basis of $S \subset \mathbb{R}^n$, then $\mathrm{span}(S) = \mathrm{span}(B)$.

Proposition 1.2

1. *If a finite $S \subset \mathbb{R}^n$ is LI, then $|S| \leq n$.*
2. *Any $S \subset \mathbb{R}^n$, $S \neq \emptyset$, has a basis.*
3. *Suppose that B is a basis of $S \subset \mathbb{R}^n$ and B' is a finite LI subset of S.*
   - $|B'| \leq |B|$.
   - *B' is a basis of S if and only if $|B'| = |B|$.*

- The *rank* of $S \subset \mathbb{R}^n$, denoted $\mathrm{rank}(S)$, is the cardinality $|B|$ of any basis B of S.
- The *dimension* of $\mathrm{span}(S)$, denoted $\dim[\mathrm{span}(S)]$, is the rank of S, $\mathrm{rank}(S)$.

- $\mathbb{R}^{m\times n}$: Set of $m \times n$ matrices
- For $A \in \mathbb{R}^{m\times n}$, we write a^j for the jth column vector of A.
- We often identify $A \in \mathbb{R}^{m\times n}$ with the set of its column vectors, i.e., $A = \{a^1, \dots, a^n\} \subset \mathbb{R}^m$.
- For $A \in \mathbb{R}^{m\times n}$, $A^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ denotes the transpose of A.
- For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we write $xy = \sum_{j=1}^n x_j y_j$ (instead of $x \cdot y$ or $x^{\mathrm{T}} y$).
- For $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:
  - $Ax = \sum_{j=1}^n x_j a^j \in \mathbb{R}^m$
    $\cdots$ linear combination of column vectors $a^1, \dots, a^n$ with coefficients $x_1, \dots, x_n$
  - $yA = (ya^1, \dots, ya^n) \in \mathbb{R}^n$ (instead of $y^{\mathrm{T}} A$)

- The *span* of A: $\mathrm{span}(A) = \{Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n\}$.
- The *kernel* (or *null space*) of A: $\ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$.
  $(\ker(A^{\mathrm{T}}) = \{y \in \mathbb{R}^m \mid yA = 0\})$

Proposition 1.3

For $A \in \mathbb{R}^{m \times n}$,

$$\mathrm{rank}(A) = \mathrm{rank}(A^{\mathrm{T}}).$$

Proposition 1.4

For $A \in \mathbb{R}^{m \times n}$,

$$\mathrm{rank}[\mathrm{span}(A)] + \mathrm{rank}[\ker(A^{\mathrm{T}})] = m.$$

- A square matrix $A \in \mathbb{R}^{n \times n}$ is *non-singular* if $\operatorname{rank}(A) = n$, i.e., the columns of A are LI.
- For $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$ such that $AB = BA = I$ is called the *inverse* of A and denoted A^{-1}.
- $A \in \mathbb{R}^{n \times n}$ is *invertible* if A^{-1} exists.
- $A \in \mathbb{R}^{n \times n}$ is invertible if and only if it is non-singular.

Fundamental Theorem of Linear Algebra

Proposition 1.5

Let $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$. *Either*

1. $Ax = b$ *has a solution, or*
2. $yA = 0$, $yb \neq 0$ *has a solution,*

but not both.

- A theorem of the form "Either . . . , or . . . , but not both" is called a "theorem of the alternative".
- Condition 1 is equivalent to "$b \in \text{span}(A)$".

Proof

- Write $[A|b] = (a^1, \ldots, a^n, b) \in \mathbb{R}^{m\times(n+1)}$.
- Recall

$$
\begin{aligned}
\operatorname{span}(A) &= \{Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n\},\\
\operatorname{span}([A|b]) &= \{Ax + \lambda b \in \mathbb{R}^m \mid x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}\},\\
\ker(A^{\mathrm{T}}) &= \{y \in \mathbb{R}^m \mid yA = 0\},\\
\ker([A|b]^{\mathrm{T}}) &= \{y \in \mathbb{R}^m \mid yA = 0,\ yb = 0\}.
\end{aligned}
$$

Clearly, $\operatorname{span}(A) \subseteq \operatorname{span}([A|b])$, $b \in \operatorname{span}([A|b])$, and $\ker(A^{\mathrm{T}}) \supseteq \ker([A|b]^{\mathrm{T}})$.

- By Proposition 1.4, we have

$$
\begin{aligned}
&\text{rank}[\text{span}(A)] + \text{rank}[\ker(A^{\mathrm{T}})] \\
&= m = \text{rank}[\text{span}([A|b])] + \text{rank}[\ker([A|b]^{\mathrm{T}})].
\end{aligned}
$$

- Therefore, we have

$$
\begin{aligned}
& b \in \text{span}(A) \\
&\iff \text{span}(A) = \text{span}([A|b]) \\
&\iff \text{rank}[\text{span}(A)] = \text{rank}[\text{span}([A|b])] \\
&\iff \text{rank}[\ker(A^{\mathrm{T}})] = \text{rank}[\ker([A|b]^{\mathrm{T}})] \\
&\iff \ker(A^{\mathrm{T}}) = \ker([A|b]^{\mathrm{T}}) \\
&\iff yb = 0 \text{ whenever } yA = 0.
\end{aligned}
$$

Cones

Definition 1.1

$C \subset \mathbb{R}^n$ is a *cone* if

$$x \in C, \; \lambda > 0 \Rightarrow \lambda x \in C.$$

- Some textbooks define cones with "$\lambda \geq 0$".

- For a finite set $S = \{x^1, x^2, \dots, x^J\} \subset \mathbb{R}^n$, a non-negative linear combination

$$\sum_{j \in S} \lambda_j x^j$$

with $\lambda_1, \dots, \lambda_J \geq 0$ is called a *conic combination* of S.

- For $S \subset \mathbb{R}^n$, the set of all conic combinations of finite subsets of S is called the *conic hull* of S and denoted by $\mathrm{cone}(S)$.
- $\mathrm{cone}(S)$ is a cone.

Definition 1.2

For $A \in \mathbb{R}^{m \times n}$, the conic hull of the column vectors of A is called the *finite cone* generated by the columns of A and denoted by $\mathrm{cone}(A)$, i.e.,

$$\mathrm{cone}(A) = \{Ax \in \mathbb{R}^m \mid x \geq 0\}.$$

Farkas' Lemma

- In Mathematics II, we proved Farkas' Lemma by the separating hyperplane theorem + the closedness of a finite cone.
- Here we prove Farkas' Lemma by an algebraic argument.
- We will later prove separating hyperplane theorems from Farkas' Lemma.

Fundamental Theorem of Linear Inequalities

Proposition 1.6

Let $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$*. Let* $r = \operatorname{rank}(A)$*. Either*

1. b *is written as a conic combination of LI columns of* A*, or*
2. *there exist an LI set of* $r-1$ *columns of* A*,* $\{a^{j_1}, \ldots, a^{j_{r-1}}\}$*, and* $y \in \mathbb{R}^m$ *such that* $ya^{j_t} = 0$ *for all* $t = 1, \ldots, r-1$*,* $yA \geq 0$*, and* $yb < 0$*.*

but not both.

- From Schrijver, *Theory of Linear and Integer Programming*, Theorem 7.1.

Proof

- Both conditions cannot hold simultaneously:

  If $b = Ax$ for some $x \geq 0$ and there is y as in Condition 2, then we would have

  $$0 > yb = y(Ax) = (yA)x \geq 0.$$

- If $b \notin \operatorname{span}(A)$, then by the Fundamental Theorem of Linear Algebra (Proposition 1.5), there is y such that $yA = 0$ and $yb < 0$. $\quad \rightarrow$ Condition 2

- In the following, we assume that $b \in \operatorname{span}(A)$.

- Consider the following procedure (recall $r = \text{rank}(A)$):

0. Let $D = \{a^{j_1}, \dots, a^{j_r}\}$ be any linearly independent set of r column vectors of A.
1. Write $b = \sum_{j \in D} \lambda_j a^j$.

   If $\lambda_j \geq 0$ for all $j \in D$, then stop. $\rightarrow$ Condition 1
2. Otherwise, choose the smallest index h among $\{j_1, \dots, j_r\}$ with $\lambda_h < 0$.

   Let y be such that $ya = 0$ for all $a \in D \setminus \{a^h\}$ and $ya^h \neq 0$. (Such y exists by Proposition 1.5 since $a^h \notin \text{span}(D \setminus \{a^h\})$.)

   Normalize y so that $ya^h = 1$.

   Observe that $yb = y\left(\sum_{j \in D} \lambda_j a^j\right) = \lambda_h < 0$.
3. If $ya \geq 0$ for all $a \in A$, then stop. $\rightarrow$ Condition 2
4. Otherwise, choose the smallest index w such that $ya^w < 0$.

   Let $D \leftarrow (D \setminus \{a^h\}) \cup \{a^w\}$, which is linearly independent (since $a^w \notin \text{span}(D \setminus \{a^h\})$), and go to Step 1.

- Let D^k denote the set D at the start of the kth iteration of this procedure.
- We want to show that this procedure stops with finitely many iterations.
- Assume the contrary.

  Then, since there are finitely many linearly independent sets of columns of A, we have $D^k = D^\ell$ for some k and ℓ, $k < \ell$.
- Let s be the largest index for which a^s is removed at one of the iterations $k, k+1, \ldots, \ell - 1$, say iteration p.
- Since $D^\ell = D^k$, a^s is inserted in D at some iteration q, $k \leq q \leq \ell - 1$.

- Write $D^p = \{a^{j_1}, \ldots, a^{j_r}\}$.
- At iteration p (where a^s is removed): $b = \sum_{j \in D^p} \lambda_j^p a^j$, where
  - $\lambda_s^p < 0$; and
  - $\lambda_{j_t}^p \geq 0$ for all $j_t < s$.
- At iteration q (where a^s is inserted): Let y^q be as in Step 2. Then
  - $y^q a^s < 0$; and
  - $y^q a^j \geq 0$ for all $j < s$.
- For all $j_t > s$, we have $a^{j_t} \in D^\ell$, hence $a^{j_t} \in D^k$, and hence $a^{j_t} \in D^q$. Therefore,
  - $y^q a^{j_t} = 0$ for all $j_t > s$.
- Therefore, we have $y^q b = y^q \left(\sum_{j \in D^p} \lambda_j^p a^j \right) = \sum_{j \in D^p} \lambda_j^p (y^q a^j) > 0$, which contradicts $y^q b < 0$.

Farkas' Lemma

Proposition 1.7 (Farkas' Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Either

1. $Ax = b$, $x \geq 0$ *has a solution, or*
2. $yA \geq 0$, $yb < 0$ *has a solution,*

but not both.

- Condition 1 is equivalent to "$b \in \mathrm{cone}(A)$".

Proof

- Both conditions cannot hold simultaneously.
- If Condition 1 does not hold, then by Proposition 1.6, Condition 2 holds.

Example 6

- $A = \begin{bmatrix} 4 & 1 & -5 \\ 1 & 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example 7

- $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

Variant of Farkas' Lemma

Proposition 1.8 (Gale's Theorem)

Let $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$. *Either*

1. $Ax \leq b$ *has a solution, or*
2. $yA = 0$, $y \geq 0$, $yb < 0$ *has a solution,*

but not both.

Proof

- $\exists\, x : Ax \leq b$

  $$\iff \exists\, x, s : Ax + s = b,\ \ s \geq 0$$

  $$\iff \exists\, z, z', s : A(z - z') + s = b,\ \ z \geq 0, z' \geq 0, s \geq 0$$

  $$\iff \exists\, z, z', s : \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} z \\ z' \\ s \end{bmatrix} = b,\ \ \begin{bmatrix} z \\ z' \\ s \end{bmatrix} \geq 0$$

- The Farkas alternative is

  $y \begin{bmatrix} A & -A & I \end{bmatrix} \geq 0$, $yb < 0$, or

  $yA \geq 0$, $-yA \geq 0$, $yI \geq 0$, $yb < 0$, i.e.,

  $yA = 0$, $y \geq 0$, $yb < 0$.

Variant of Farkas' Lemma

Proposition 1.9 (Farkas' Lemma: Inequality version)

Let $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$. *Either*

1. $Ax \leq b$, $x \geq 0$ *has a solution, or*
2. $yA \geq 0$, $y \geq 0$, $yb < 0$ *has a solution,*

but not both.

Proof

- Condition 1 is written as $\exists\, x : \begin{bmatrix} A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ 0 \end{bmatrix}$.
- By Proposition 1.8, its alternative is:

  $\exists\, y, z : \begin{bmatrix} y & z \end{bmatrix} \begin{bmatrix} A \\ -I \end{bmatrix} = 0,\ \begin{bmatrix} y & z \end{bmatrix} \geq 0,\ \begin{bmatrix} y & z \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} < 0$, or

  $\exists\, y, z : yA = z,\ y \geq 0, z \geq 0,\ yb < 0$

  $\iff \exists\, y : yA \geq 0,\ y \geq 0,\ yb < 0.$

Application: No Arbitrage

- Assets: $1, \dots, m$
- States: $\Omega = \{1, \dots, n\}$
- a_{ij}: payoff to asset i in state j

  $A = (a_{ij}) \in \mathbb{R}^{m \times n}$
- $y \in \mathbb{R}^m$: portfolio
  - $y_i > 0$: long position in asset i
  - $y_i < 0$: short position in asset i
- Ex post wealth vector of y:

$$yA = \left(\sum_{i=1}^{m} y_i a_{ij} \right)_{j \in \Omega}$$

- $p \in \mathbb{R}^m$, $p \geq 0$, $p \neq 0$: vector of asset prices
- p satisfies the *no arbitrage condition* if there exists no $y \in \mathbb{R}^m$ such that

$$yA \geq 0, \;\; yp < 0.$$

- By Farkas' Lemma, the no arbitrage condition is equivalent to the condition that there exists $\hat{\pi} \in \mathbb{R}^n$ such that

$$p = A\hat{\pi}, \;\; \hat{\pi} \geq 0.$$

- By $p \neq 0$, we have $\hat{\pi} \neq 0$. So by normalization we have

$$p^* = A\pi,$$

  where $p^* = \frac{p}{\sum_j \hat{\pi}_j}$ and $\pi = \frac{\hat{\pi}}{\sum_j \hat{\pi}_j}$.
- $\pi \in \mathbb{R}^n$: *risk-neutral probability distribution*

Option Pricing

- States: $\Omega = \{g, b\}$
- Assets
  - Stock (risky asset): (uS^0, dS^0), $\quad u > 1 > d$
  - Bond (safe asset): (rB, rB), $\quad r > 1$

    Assume $u > r$ $(> 1 > d)$
  - Call option (derivative asset): $(\max\{0, uS^0 - K\}, \max\{0, dS^0 - K\})$

    K: strike price

    option to buy the stock at price K
- Asset price vector: $p = (S^0, B, p_3)$

  Determine p_3 so that p satisfies the no arbitrage condition.

- Payoff matrix:

$$A = \begin{bmatrix} uS^0 & dS^0 \\ rB & rB \\ \max\{0, uS^0 - K\} & \max\{0, dS^0 - K\} \end{bmatrix}$$

- (Un-normalized) risk-neutral probability distribution $\hat{\pi}$:

$$A\hat{\pi} = p$$

- From the first two equations:

$$uS^0\hat{\pi}_1 + dS^0\hat{\pi}_2 = S^0,$$
$$rB\hat{\pi}_1 + rB\hat{\pi}_2 = B.$$

This system has a unique solution

$$\hat{\pi}_1 = \frac{r-d}{r(u-d)},\ \hat{\pi}_2 = \frac{u-r}{r(u-d)} \qquad (> 0 \text{ by } u > r > d).$$

- For no arbitrage,

$$p_3 = \hat{\pi}_1 \max\{0, uS^0 - K\} + \hat{\pi}_2 \max\{0, dS^0 - K\}.$$

Application: Cooperative Games

- A cooperative game (with transferable utility) is a pair (v, N) where
  - $N = \{1, \ldots, n\}$: finite set of players; and
  - $v \colon 2^N \to \mathbb{R}$, where

    for $S \subset N$, $v(S)$ represents the monetary value of coalition S.

Definition 1.3

The *core* of game (v, N) is the set

$$C(v,N) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{j \in N} x_j = v(N),\ \sum_{j \in S} x_j \geq v(S) \text{ for all } S \subset N \right\}.$$

- That is, the core is the set of allocations $x \in \mathbb{R}^n$ such that
  - the value $v(N)$ of the grand coalition N is allocated without waste; and
  - there is no coalition that "blocks" x,

    i.e., there is no coalition S such that $\sum_{j \in S} x_j < v(S)$.
- The core may be empty.

  In what games is the core non-empty?

- Let $B(N)$ be the set of vectors $(y_S)_{S\subset N} \in \mathbb{R}^{2^n}$ such that

$$\sum_{S:i\in S} y_S = 1 \text{ for all } i \in N,$$
$$y_S \geq 0 \text{ for all } S \subset N.$$

- For example, the vector $(y_S)_{S\subset N} \in \mathbb{R}^{2^n}$ such that

$$y_N = 1$$
$$y_S = 0 \text{ for all } S \neq N$$

is in $B(N)$.

Thus, $B(N) \neq \emptyset$.

Proposition 1.10 (Bondareva-Shapley)

$C(v, N) \neq \emptyset$ *if and only if*

$$v(N) \geq \sum_{S \subset N} v(S) y_S \text{ for all } (y_S)_{S \subset N} \in B(N). \qquad (*)$$

- Game (v, N) is said to be *balanced* if $(*)$ holds. (Or equivalently, (v, N) is balanced if $C(v, N) \neq \emptyset$.)

Proof

- The condition $C(v, N) \neq \emptyset$ is written as

  $\exists\, x \in \mathbb{R}^n : \sum_{j \in N} x_j \leq v(N),\ \sum_{j \in S}(-x_j) \leq -v(S)\ \forall\, S \subset N$

  $\iff\ \exists\, x \in \mathbb{R}^n : Ax \leq b$, where

$$A = \begin{bmatrix} 1 & & \cdots & & 1 \\ & & \vdots & & \\ 0 & \cdots & \underbrace{-1 \cdots -1}_{S} & \cdots & 0 \\ & & \vdots & & \end{bmatrix} \in \mathbb{R}^{(1+2^n)\times n},$$

$$b = \begin{bmatrix} v(N) \\ \vdots \\ -v(S) \\ \vdots \end{bmatrix}$$

- By Proposition 1.8 (Gale's Theorem), its alternative is:

$$\exists\, y \in \mathbb{R}^{1+2^n} : yA = 0,\ y \geq 0,\ yb < 0$$

$$\iff \exists\, y = (y_0, \ldots, y_S \ldots) \in \mathbb{R}^{1+2^n} :$$

$$y_0 - \sum_{S:i\in S} y_S = 0 \quad \forall\, i \in N$$

$$y_0 \geq 0,\ y_S \geq 0 \quad \forall\, S \subset N$$

$$y_0 v(N) - \sum_{S\subset N} y_S v(S) < 0$$

(For such y, $y_0 > 0$, so let $y_0 = 1$.)

$$\iff \exists\, (y_S)_{S\subset N} \in \mathbb{R}^{2^n} :$$

$$\sum_{S:i\in S} y_S = 1 \quad \forall\, i \in N$$

$$y_S \geq 0 \quad \forall\, S \subset N$$

$$v(N) - \sum_{S\subset N} y_S v(S) < 0$$

- $\iff \exists\,(y_S)_{S\subset N} \in B(N) : v(N) < \sum_{S\subset N} y_S v(S)$
- The negation of this condition is:

  $\forall\,(y_S)_{S\subset N} \in B(N) : v(N) \geq \sum_{S\subset N} y_S v(S)$

Example

- $N = \{1, 2, 3\}$
- $v(S) = \begin{cases} 1 & \text{if } |S| = 2 \\ 0 & \text{if } |S| \leq 1 \end{cases}$
- Determine the condition for $v(N)$ under which $C(v, N) \neq \emptyset$.

Necessity

- $C(v, N) \neq \emptyset$ means that there exists $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_1 + x_2 + x_3 = v(N), \quad (1)$$
$$x_1 + x_2 \geq v(\{1,2\}) = 1, \quad (2)$$
$$x_2 + x_3 \geq v(\{2,3\}) = 1, \quad (3)$$
$$x_3 + x_1 \geq v(\{3,1\}) = 1, \quad (4)$$
$$x_1 \geq v(\{1\}) = 0,$$
$$x_2 \geq v(\{2\}) = 0,$$
$$x_3 \geq v(\{3\}) = 0.$$

- Adding (2)–(4) and applying (1), we have $2v(N) = 2(x_1 + x_2 + x_3) = 3$, or $v(N) \geq \frac{3}{2}$.

Sufficiency

- Conversely, suppose that $v(N) \geq \frac{3}{2}$.

  We want to show that the balancedness condition is satisfied.

- Let $(y_S) \in B(N)$, i.e., $y_S \geq 0$ for all $S \subset N$ and $\sum_{S:i\in S} y_S = 1$ for all $i \in N$.

- Then we have

$$
\begin{aligned}
&y_{\{1,2\}}v(\{1,2\}) + y_{\{2,3\}}v(\{2,3\}) + y_{\{3,1\}}v(\{3,1\}) + y_N v(N) \\
&= \frac{1}{2}[(y_{\{1,2\}} + y_{\{2,3\}}) + (y_{\{2,3\}} + y_{\{3,1\}}) + (y_{\{3,1\}} + y_{\{1,2\}})] + y_N v(N) \\
&\leq \frac{1}{2}[(1 - y_N) + (1 - y_N) + (1 - y_N)] + y_N v(N) \\
&= \frac{3}{2} + \underbrace{y_N}_{\leq 1} \underbrace{\left(v(N) - \frac{3}{2}\right)}_{\geq 0} \leq v(N).
\end{aligned}
$$

- Thus, $C(v, N) \neq \emptyset$ if and only if $v(N) \geq \frac{3}{2}$.

Application: Afriat's Theorem

- We are given a dataset of price vector-consumption vector pairs: $\mathcal{D} = \{(p^1, x^1), \ldots, (p^n, x^n)\}$, where $p^i \in \mathbb{R}^m_{++}$ and $x^i \in \mathbb{R}^m_+$.
- Is there a utility function that rationalizes $\mathcal{D}$?

  A function $u \colon \mathbb{R}^m_+ \to \mathbb{R}$ is a utility function that rationalizes $\mathcal{D}$ if for all $i = 1, \ldots, n$,

  $$p^i x \leq p^i x^i \implies u(x) \leq u(x^i)$$

  (i.e., $x^i \in \arg\max\{u(x) \mid p^i x \leq p^i x^i\}$).
- The answer is trivially yes: let u be a constant function.
- Is there a locally insatiable utility function that rationalizes $\mathcal{D}$?

- u is *locally insatiable* if for any $x \in \mathbb{R}^m_+$ and $\varepsilon > 0$, there exists $x' \in \mathbb{R}^m_+$ such that $d(x', x) < \varepsilon$ and $u(x') > u(x)$.
- For a locally insatiable utility function u,

  if $x^* \in \arg\max\{u(x) \mid px \leq I\}$, then $px^* = I$.

Necessary Condition

- Suppose that a locally insatiable u rationalizes $\mathcal{D}$.
- Then
  - $p^i x^j \leq p^i x^i \implies u(x^j) \leq u(x^i)$;
  - $p^i x^j < p^i x^i \implies u(x^j) < u(x^i)$ (by local insatiability).
- Denote $a_{ij} = p^i(x^j - x^i)$. $\qquad (a_{ij} \leq 0 \iff p^i x^j \leq p^i x^i)$
- If we have a cycle

$$a_{i_1 i_2} \leq 0, a_{i_2 i_3} \leq 0, \ldots, a_{i_{k-1} i_k} \leq 0, a_{i_k i_1} \leq 0,$$

  then $u(x^{i_1}) \leq u(x^{i_2}) \leq \cdots \leq u(x^{i_{k-1}}) \leq u(x^{i_k}) \leq u(x^{i_1})$,
  and therefore, $a_{i_1 i_2} = 0, a_{i_2 i_3} = 0, \ldots, a_{i_{k-1} i_k} = 0, a_{i_k i_1} = 0$.
  - Call this necessary condition "Afriat Condition" (AC).

Afriat Condition

- Dataset $\mathcal{D} = \{(p^1, x^1), \ldots, (p^n, x^n)\}$ is said to satisfy AC if there exists no cycle $(i_1, i_2, \ldots, i_k, i_1)$ such that

$$a_{i_1 i_2} \leq 0, a_{i_2 i_3} \leq 0, \ldots, a_{i_{k-1} i_k} \leq 0, a_{i_k i_1} \leq 0$$

  with at least one "$\leq$" holding with "$<$".

- (This is equivalent to the following:

  for any cycle $(i_1, i_2, \ldots, i_k, i_1)$ such that $\sum_{\ell=1}^{k} a_{i_\ell i_{\ell+1}} < 0$, there exists some ℓ such that $a_{i_\ell i_{\ell+1}} > 0$.)

- AC is a necessary condition for the rationalizability of $\mathcal{D}$.

  Afriat's theorem shows that it is also sufficient.

Afriat's Theorem

Proposition 1.11

$\mathcal{D}$ is rationalized by a locally insatiable utility function if and only if it satisfies AC.

- Consider the following system of inequalities:

$$\text{(P)}\quad \begin{aligned} & s_i \geq 1 && \text{for all } i, \\ & y_i + a_{ij} s_i - y_j \geq 0 && \text{for all } (i,j),\ i \neq j. \end{aligned}$$

- The alternative:

$$\begin{aligned} & z_i + \sum_j a_{ij} w_{ij} = 0 && \text{for all } i, \\ & \sum_j w_{ij} - \sum_k w_{ki} = 0 && \text{for all } i, \\ & \sum_i z_i > 0, && \\ & z_i \geq 0 && \text{for all } i, \\ & w_{ij} \geq 0 && \text{for all } (i,j),\ i \neq j. \end{aligned}$$

- This is equivalent to:

$$
\text{(D)} \quad
\begin{aligned}
&\sum_j a_{ij} w_{ij} \leq 0 && \text{for all } i, \\
&\sum_j a_{ij} w_{ij} < 0 && \text{for some } i, \\
&\sum_j w_{ij} = \sum_k w_{kj} && \text{for all } i, \\
&w_{ij} \geq 0 && \text{for all } (i,j),\ i \neq j.
\end{aligned}
$$

Proposition 1.12

The following conditions are equivalent:

1. $\mathcal{D}$ *is rationalized by a locally insatiable utility function.*
2. (P) *is feasible.*
3. (D) *is infeasible.*
4. $\mathcal{D}$ *satisfies AC.*

Proof

- $1 \Rightarrow 4$: Already verified.
- $2 \Leftrightarrow 3$: By Proposition 1.8 (Gale's Theorem).

Proof of "2 ⇒ 1"

- Suppose that (P) is feasible, i.e., there exist y_i and s_i such that

  $$y_j \leq y_i + p^i(x^j - x^i)s_i, \quad s_i \geq 1$$

  for all i and j.

- Define $u\colon \mathbb{R}^m_+ \to \mathbb{R}$ by

  $$u(x) = \min\{y_1 + p^1(x - x^1)s_1, \ldots, y_n + p^n(x - x^n)s_n\},$$

  which is strictly increasing (and hence locally insatiable).

  (In fact, it is also continuous and concave).

- Then,

  $$\begin{aligned} u(x^j) &= \min\{y_j, y_i + p^i(x^j - x^i)s_i, i \neq j\} \\ &= y_j \quad \text{(by (P))}. \end{aligned}$$

- Therefore, if $p^j x \leq p^j x^j$ (or $p^j(x - x^j) \leq 0$), then

$$\begin{aligned} u(x) &\leq y^j + p^j(x - x^j)s^j \\ &\leq y^j = u(x^j). \end{aligned}$$

- This shows that the function u rationalizes $\mathcal{D}$.

Proof of "4 ⇒ 3" ("not 3 ⇒ not 4")

Lemma 1.13

If (D) *is feasible, then there exists a feasible solution* (w_{ij}) *such that there exists no cycle* $(i_1, \ldots, i_k, i_1)$ *such that* $a_{i_\ell i_{\ell+1}} = 0$ *and* $w_{i_\ell i_{\ell+1}} > 0$ *for all* $\ell = 1, \ldots, k$.

- Let (w_{ij}) be a feasible solution of (D): $\sum_j a_{ij} w_{ij} \leq 0$ for all i with "$<$" for some i, $\sum_j w_{ij} = \sum_k w_{ki}$ for all i, and $w_{ij} \geq 0$ for all i, j.
- Suppose that there exists a cycle $(i_1, \ldots, i_k, i_1)$ such that $a_{i_\ell i_{\ell+1}} = 0$ and $w_{i_\ell i_{\ell+1}} > 0$ for all $\ell = 1, \ldots, k$.

  Write $C = \{(i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, i_1)\}$.
- Let $\varepsilon = \min_\ell w_{i_\ell i_{\ell+1}}$, and define (w'_{ij}) by

$$w'_{ij} = \begin{cases} w_{ij} - \varepsilon & \text{if } (i,j) \in C, \\ w_{ij} & \text{if } (i,j) \notin C. \end{cases}$$

- Then, (w'_{ij}) also is a feasible solution of (D):
  - $w'_{ij} \geq 0$ for all i, j;
  - $a_{ij}w'_{ij} = a_{ij}w_{ij}$, so that $\sum_j a_{ij}w'_{ij} \leq 0$ for all i, with "$<$" for some i;
  - for each $i \in \{i_1, \ldots, i_k\}$, there are exactly one j such that $(i, j) \in C$ and exactly one k such that $(k, i) \in C$, so that $\sum_j w'_{ij} - \sum_k w'_{ki} = \sum_j w_{ij} - \sum_k w_{ki} = 0$.
- By construction, $w'_{i_\ell i_{\ell+1}} = 0$ for some ℓ.
- Since there are finitely many possible cycles, by repeating this procedure we obtain a desired feasible solution.

  [End of Lemma 1.13]

- Now suppose that (D) is feasible.

  Let (w_{ij}) be a feasible solution of (D) as in Lemma 1.13:

  - $\sum_j a_{ij} w_{ij} \leq 0$ for all i with "$<$" for some i,
    $\sum_j w_{ij} = \sum_k w_{ki}$ for all i, and $w_{ij} \geq 0$ for all i, j, and
  - if $a_{i_\ell i_{\ell+1}} \leq 0$ and $w_{i_\ell i_{\ell+1}} > 0$ for all $\ell = 1, \ldots, k$,
    then $a_{i_\ell i_{\ell+1}} < 0$ for some ℓ.

- Let i_1 be such that $\sum_j a_{i_1 j} w_{i_1 j} < 0$.

- Then there is some i_2 such that $a_{i_1 i_2} < 0$ and $w_{i_1 i_2} > 0$.

  Then $\sum_j w_{i_2 j} = \sum_k w_{k i_2} > 0$, so that $w_{i_2 j} > 0$ for some j.

- Then there is some i_3 such that $a_{i_2 i_3} \leq 0$ and $w_{i_2 i_3} > 0$.

  (Otherwise, we would have $\sum_j a_{i_2 j} w_{i_2 j} > 0$.)

  $\ldots$

- Proceeding this way, we have a sequence:

$$\begin{aligned} a_{i_1 i_2} < 0, \;\; w_{i_1 i_2} > 0 \\ a_{i_2 i_3} \leq 0, \;\; w_{i_2 i_3} > 0 \\ a_{i_3 i_4} \leq 0, \;\; w_{i_3 i_4} > 0 \\ \vdots \end{aligned}$$

- Since there are finitely many indices, we eventually repeat an index.
- If the repeated index is i_1, then we have a violation of AC.
- Otherwise, we have a cycle of nonpositive a_{ij}'s,
but by the choice of (w_{ij}), there must be a negative a_{ij},
which is again a violation of AC.

Problem 2.9

Proposition 1.14

Let $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$. *Either*

1. $Ax \ll b$ *has a solution, or*
2. $yA = 0$, $yb \leq 0$, $y \geq 0$, $y \neq 0$ *has a solution,*

but not both.

Proof

- Condition 2 is rewritten as:

$$\exists\, y \in \mathbb{R}^m : yA = 0,\ yb \leq 0,\ y \geq 0,\ y \neq 0$$

$$\iff \exists\, y \in \mathbb{R}^m, s \in \mathbb{R} :$$
$$yA = 0,\ yb + s = 0,\ s \geq 0,\ y \geq 0,\ y\mathbf{1} = 1$$

($\mathbf{1}$ is the vector of ones)

$$\iff \exists\, y \in \mathbb{R}^m, s \in \mathbb{R} :$$
$$\begin{bmatrix} y & s \end{bmatrix} \begin{bmatrix} A & b & \mathbf{1} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} y & s \end{bmatrix} \geq 0$$

- By Farkas' Lemma, its alternative is:

$$\exists\, x \in \mathbb{R}^n, z \in \mathbb{R}, w \in \mathbb{R} :$$
$$\begin{bmatrix} A & b & \mathbf{1} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ w \end{bmatrix} \geq 0,\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \\ w \end{bmatrix} < 0$$

- $$\iff \exists\, x \in \mathbb{R}^n, z \in \mathbb{R}, w \in \mathbb{R} : \\ Ax + zb + w\mathbf{1} \geq 0,\ \ z \geq 0,\ \ w < 0$$
  $$\iff \exists\, x \in \mathbb{R}^n, z \in \mathbb{R}, w \in \mathbb{R} : \\ Ax + zb \geq (-w)\mathbf{1},\ \ z \geq 0,\ \ -w > 0$$
  $$\iff \exists\, x \in \mathbb{R}^n, z \in \mathbb{R} : Ax + zb \gg 0,\ \ z \geq 0$$
  $$\iff \exists\, x \in \mathbb{R}^n : Ax \ll b$$
- The last equivalence holds because:
  - If $z = 0$, then $\exists\, x : Ax \gg 0$.

    $A(-tx) \ll 0$ can be arbitrarily small as $t \to \infty$.

    Hence, $\exists\, x' : Ax' \ll b$.
  - If $z > 0$, then $\exists\, x, z : A\left(\frac{x}{-z}\right) \ll b$.

    Hence, $\exists\, x' : Ax' \ll b$.
  - For the converse, let $x = -x'$ and $z = 1$.

Problem 2.7

Proposition 1.15 (Gordan's Theorem)

Let $A \in \mathbb{R}^{m \times n}$. Either

1. $Ax = 0$, $x \geq 0$, $x \neq 0$ *has a solution, or*
2. $yA \gg 0$ *has a solution,*

but not both.

- Special case of Problem 2.9:

  Let $b = 0$ (and replace A with $-A^{\mathrm{T}}$).

Problem 2.6

Proposition 1.16

Let $A \in \mathbb{R}^{m \times n}$*. Either*

1. $Ax = 0$, $x \geq 0$, $\mathbf{1}x = 1$ *has a solution, or*
2. $yA \gg 0$ *has a solution,*

but not both.

- Condition 1 is equivalent to Condition 1 in Problem 2.7.

Problem 2.8

- $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a *column Markov matrix* (or *column stochastic matrix*) if $a_{ij} \geq 0$ for all i, j, and $\sum_{i=1}^{n} a_{ij} = 1$ for all j.
- It represents the transition probabilities of a Markov chain:

  a_{ij}: probability that the random variable changes from j to i.
- $x \in \mathbb{R}^n$ is a *probability vector* if $x_j \geq 0$ for all j, and $\sum_{j=1}^{n} x_j = 1$.
- A probability vector x is a *steady state vector* (or *stationary distribution*) of a column Markov matrix A if $Ax = x$.

Proposition 1.17

Every Markov matrix has a steady state vector.

Proof

- Let $A \in \mathbb{R}^{n \times n}$ be a column Markov matrix.
- We want to show that A has a steady state vector x.

  This is rewritten as:

  $\exists\, x : Ax = x,\ x \geq 0,\ \mathbf{1}x = 1$

  $\iff \exists\, x : (A - I)x = 0,\ x \geq 0,\ \mathbf{1}x = 1$
- By Problem 2.6, its alternative is:

  $\exists\, y : y(A - I) \gg 0$

  $\iff \exists\, y : yA \gg y$

  It suffices to show that this condition does not hold.

- Take any $y \in \mathbb{R}^n$.

  Let j^* be such that $y_{j^*} \geq y_j$ for all j.

- Then we have

$$
\begin{aligned}
(yA)_{j^*} &= \sum_{i=1}^{n} y_i a_{ij^*} \\
&\leq \sum_{i=1}^{n} y_{j^*} a_{ij^*} \quad (\because a_{ij^*} \geq 0 \text{ for all } i) \\
&= y_{j^*} \sum_{i=1}^{n} a_{ij^*} \\
&= y_{j^*}. \quad (\because \sum_{i=1}^{n} a_{ij^*} = 1)
\end{aligned}
$$

- Therefore, $yA \gg y$ cannot hold.

Variant of Farkas' Lemma

Proposition 1.18 (Ville's Theorem)

Let $A \in \mathbb{R}^{m \times n}$*. Either*

1. $Ax \gg 0$, $x \geq 0$ *has a solution, or*

2. $yA \leq 0$, $y \geq 0$, $y \neq 0$ *has a solution,*

but not both.

Proof

- First,

  $$\exists\, x : Ax \gg 0,\ \ x \geq 0$$

  $$\iff \exists\, x' : Ax' \gg 0,\ \ x' \gg 0$$

  $\because$ Given x such that $Ax \gg 0$, $x \geq 0$, let $x' = x + \varepsilon\mathbf{1}$ for sufficiently small $\varepsilon > 0$.

- Therefore, by Gordan's Theorem (Proposition 1.15), its alternative is

  $$\exists\, y, z : \begin{bmatrix} y & z \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} = 0,\ \ \begin{bmatrix} y & z \end{bmatrix} \geq 0,\ \ \begin{bmatrix} y & z \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix}$$

  $$\iff \exists\, y, z : yA = -z,\ \ y \geq 0,\ \ z \geq 0,\ \ y \neq 0$$

  $$\iff \exists\, y : yA \leq 0,\ \ y \geq 0,\ \ y \neq 0$$

Hyperplanes and Half-Spaces

Definition 1.4

- A *hyperplane* is a set of the form $\{x \in \mathbb{R}^n \mid hx = \beta\}$ with $h \in \mathbb{R}^n$, $h \neq 0$, and $\beta \in \mathbb{R}$.
- A *half-space* is a set of the form $\{x \in \mathbb{R}^n \mid hx \leq \beta\}$ with $h \in \mathbb{R}^n$, $h \neq 0$, and $\beta \in \mathbb{R}$.

Polyhedral Cones

Definition 1.5

A cone $C \subset \mathbb{R}^m$ is *polyhedral* if there exists $A \in \mathbb{R}^{m \times n}$ such that $C = \{x \in \mathbb{R}^m \mid A^{\mathrm{T}} x \leq 0\}$.

- That is, cone C is polyhedral if it is the intersection of finitely many half spaces.
- $\mathbb{R}^m$ is polyhedral by letting A be an "$m \times 0$ matrix".
- A cone $C \subset \mathbb{R}^m$ is *finitely generated* if there exists $A \in \mathbb{R}^{m \times n}$ such that $C = \mathrm{cone}(A)$.

Farkas-Minkowski-Weyl Theorem

Proposition 1.19 (Farkas-Minkowski-Weyl Theorem)

A cone is polyhedral if and only if it is finitely generated.

Proof

- Suppose that a cone $C \subset \mathbb{R}^m$ is finitely generated, i.e., $C = \text{cone}(A)$ for some $A = [a^1, \ldots, a^n] \in \mathbb{R}^{m \times n}$.
- We first consider the case where $\text{rank}(A) = m$ (so $m \leq n$).
- For each LI subset S of $m-1$ columns of A, define

$$
\begin{aligned}
F^S = \{y \in \mathbb{R}^m \mid \|y\| = 1,&\\
ya^j = 0 \text{ for all } j \in S,&\\
ya^j \leq 0 \text{ for all } j \notin S\}.&
\end{aligned}
$$

- For each such S, since $\text{rank}[\ker(S^{\mathrm{T}})] = 1$, there are only two y's such that $\|y\| = 1$ and $yS = 0$, and hence $|F^S| \leq 2$.
- Let $F = \bigcup F^S$ where the union is taken over all LI subsets S of $m-1$ columns of A.

  Then F is a finite set, since there are finitely many such S's.

- Consider F as the $m \times |F|$ matrix that consists of the vectors in F as columns.

  Write $D = \{x \in \mathbb{R}^m \mid F^{\mathrm{T}} x \leq 0\}$.

- By construction, $ya^j \leq 0$ for all $a^j \in A$ and all $y \in F$.

  Hence, $\mathrm{cone}(A) \subset D$.

- If $b \notin \mathrm{cone}(A)$, then by Proposition 1.6 (Fundamental Theorem of Linear Inequalities), there exists $y \in \mathbb{R}^m$ such that $yA \leq 0$, $yS = 0$ for some LI subset S of $m-1$ columns of A, and $yb > 0$, where by normalization we can assume $\|y\| = 1$.

  I.e., there is some $y \in F$ such that $yb > 0$, which implies that $b \notin D$.

- This shows that $\mathrm{cone}(A) = D$, i.e., $\mathrm{cone}(A)$ is polyhedral.

- Then consider the case where $r = \mathrm{rank}(A) < m$.

  Let $A' \in \mathbb{R}^{r\times n}$ be the matrix that consists the first r rows of A, and assume without loss of generality that $\mathrm{rank}(A') = r$.

- By the previous case, there exists a finite set $F' \subset \mathbb{R}^r$ (considered as an $r \times |F'|$ matrix) such that $\mathrm{cone}(A') = \{x' \in \mathbb{R}^r \mid (F')^{\mathrm{T}} x \leq 0\}$.

- Extend the r-dimensional vectors in F' to m-dimensional vectors by adding $m - r$ zeros (so $\begin{bmatrix} F' \\ O_{(m-r)\times|F'|} \end{bmatrix}$ is referred to again as F').

- Fix any basis $F'' = \{z^1, \ldots, z^{m-r}\}$ of $\ker(A^{\mathrm{T}})$ (note that $\mathrm{rank}[\ker(A^{\mathrm{T}})] = m - r$).

  Let $D = \{x \in \mathbb{R}^m \mid (F')^{\mathrm{T}} x \leq 0,\ (F'')^{\mathrm{T}} x = 0\}$.

- By construction, $ya^j \leq 0$ for all $a^j \in A$ and all $y \in F$.

  Hence, $\text{cone}(A) \subset D$.

- Suppose that $b \in \text{span}(A) \setminus \text{cone}(A)$.

  Then $b' \notin \text{cone}(A')$, where $b' \in \mathbb{R}^r$ is the vector that consists of the first r components of b.

  Then there is some $y \in F'$ such that $yb > 0$, and hence, $b \notin D$.

- Suppose that $b \notin \text{span}(A)$.

  Then there exists $z \in \mathbb{R}^m$ such that $zA = 0$ and $zb \neq 0$.

  Then there must be some $z^j \in F''$ such that $z^j b \neq 0$, and hence, $b \notin D$.

- This shows that $\text{cone}(A) = D$.

- Finally, let $F = F' \cup F'' \cup (-F'')$. Then we have $D = \{x \in \mathbb{R}^m \mid F^{\mathrm{T}} x \leq 0\}$, and thus $\text{cone}(A)$ is polyhedral.

- For the converse, suppose that C is polyhedral, i.e., $C = \{x \in \mathbb{R}^m \mid B^{\mathrm{T}} x \leq 0\}$ for some $B = [b^1, \ldots, b^n] \in \mathbb{R}^{m \times n}$.
- By the "if" part, there exists $G = [z^1, \ldots, z^\ell] \in \mathbb{R}^{m \times \ell}$ such that $\mathrm{cone}(B) = \{x \in \mathbb{R}^m \mid G^{\mathrm{T}} x \leq 0\}$.

  We want to show that $C = \mathrm{cone}(G)$.
- For all $z \in G$ and all $b \in B$ ($\subset \mathrm{cone}(B)$), $zb \leq 0$.

  Hence, for all $z \in G$, $z \in C$, and therefore, $\mathrm{cone}(G) \subset C$.
- Suppose that $x \notin \mathrm{cone}(G)$.

  Then by Farkas' Lemma, there exists $y \in \mathbb{R}^m$ such that $yG \leq 0$ and $yx > 0$,

  i.e., there exists $y \in \mathrm{cone}(B)$ such that $yx > 0$.
- Then there must be $b^j \in B$ such that $b^j x > 0$, i.e., $x \notin C$.
- This shows that $C \subset \mathrm{cone}(G)$.

Polyhedra and Polytopes

Definition 1.6
$P \subset \mathbb{R}^m$ is called a *polyhedron* if there exist $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$ such that $P = \{x \in \mathbb{R}^m \mid A^{\mathrm{T}} x \leq b\}$.

- I.e., it is the intersection of finitely many half-spaces.

Definition 1.7

For a finite set $S = \{x^1, x^2, \ldots, x^J\} \subset \mathbb{R}^n$, the vector

$$\sum_{j \in S} \lambda_j x^j$$

with $\lambda_1, \ldots, \lambda_J \geq 0$, $\sum_{j=1}^{J} \lambda_j = 1$ is called a *convex combination* of S.

Definition 1.8

For $S \subset \mathbb{R}^n$, the set of all convex combinations of finite subsets of S is called the *convex hull* of S and denoted by $\mathrm{conv}(S)$.

Definition 1.9

$P \subset \mathbb{R}^n$ is called a *polytope* if there exists a finite $S \subset \mathbb{R}^n$ such that $P = \mathrm{conv}(S)$.

Resolution Theorem

Proposition 1.20 (Resolution Theorem)
$P \subset \mathbb{R}^m$, $P \neq \emptyset$, is a polyhedron if and only if $P = Q + C$ for some polytope Q and some finitely generated cone C.

- $(Q + C = \{q + c \mid q \in Q,\ c \in C\})$

- As a corollary, we have the following (let $C = \{0\}$):

Proposition 1.21
$P \subset \mathbb{R}^m$, $P \neq \emptyset$, is a bounded polyhedron if and only if it is a polytope.

Proof of Proposition 1.20

- The "only if" part:

  Suppose that $P = \{x \in \mathbb{R}^m \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$.

- Consider the polyhedral cone $\hat{P} = \{(x, u) \in \mathbb{R}^m \times \mathbb{R} \mid u \geq 0,\ Ax - ub \leq 0\}$.

- By Proposition 1.19, it is finitely generated, i.e., $\hat{P} = \mathrm{cone}(\{(x^1, u_1), \ldots, (x^J, u_J)\})$ for some $(x^1, u_1), \ldots, (x^J, u_J) \in \mathbb{R}^m \times \mathbb{R}$, where $u^j \geq 0$.

- Let $J^+ = \{j \mid u_j > 0\}$ and $J^0 = \{j \mid u_j = 0\}$.

  By normalization, we let $u_j = 1$ for $j \in J^+$.

- Let $Q = \mathrm{conv}(\{x^j \mid j \in J^+\})$ and $C = \mathrm{cone}(\{x^j \mid j \in J^0\})$.

  We want to show that $P = Q + C$.

- Indeed,

$$
\begin{aligned}
x \in P \iff & (x,1) \in \hat{P} \\
\iff & (x,1) = \sum_{j \in J^+} \lambda_j (x^j, 1) + \sum_{j \in J^0} \lambda_j (x^j, 0) \\
& \quad \text{for some } \lambda_1, \ldots, \lambda_J \geq 0 \\
\iff & x = \sum_{j \in J^+} \lambda_j x^j + \sum_{j \in J^0} \lambda_j x^j \\
& \quad \text{for some } \lambda_1, \ldots, \lambda_J \geq 0 \text{ with } \sum_{j \in J^+} \lambda_j = 1 \\
\iff & x \in \operatorname{conv}(\{x^j \mid j \in J^+\}) + \operatorname{cone}(\{x^j \mid j \in J^0\}).
\end{aligned}
$$

- The "if" part:

  Suppose that $P = \text{conv}(\{x^1, \ldots, x^J\}) + \text{cone}(\{y^1, \ldots, y^K\})$ for some $x^1, \ldots, x^J, y^1, \ldots, y^K \in \mathbb{R}^m$.

- Define $\bar{P} = \text{cone}(\{(x^1, 1), \ldots, (x^J, 1), (y^1, 0), \ldots, (y^K, 0)\})$.

- By Proposition 1.19, the cone $\bar{P}$ is polyhedral, i.e., $\bar{P} = \{(x, u) \in \mathbb{R}^m \times \mathbb{R} \mid Ax - ub \leq 0\}$ for some $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$.

- Let $P' = \{x \in \mathbb{R}^m \mid Ax \leq b\}$.

  We want to show that $P = P'$.

- Indeed, $x \in P \iff (x, 1) \in \bar{P} \iff Ax - b \leq 0$.

Application: Linear Production Model

- Inputs: $\{1, \ldots, m\}$

  Input vector: $x \in \mathbb{R}_+^m$

- Outputs: $\{1, \ldots, n\}$

  Output vector: $y \in \mathbb{R}^n$

- $P \in \mathbb{R}^{m \times n}$: Production matrix

  $y = xP$

- $b \in \mathbb{R}_+^k$: Resource/capacity vector

- $C \in \mathbb{R}_+^{m \times k}$: Consumption matrix ($C \neq O$)

  Resource constraint: $xC \leq b$

- Input space: $X = \{x \in \mathbb{R}^m \mid xC \leq b,\ x \geq 0\}$

- Output space: $Y = \{y \in \mathbb{R}^n \mid y = xP,\ x \in X\}$

Lemma 1.22

Y is a polyhedron,
i.e., $Y = \{y \in \mathbb{R}^n \mid yD \leq r\}$ for some $D \in \mathbb{R}^{n \times \ell}$ and $r \in \mathbb{R}^\ell$.

Proof

- X is a bounded polyhedron (since $C \geq 0$ and $X \subset \mathbb{R}^m_+$), and hence is a polytope by the Resolution Theorem,

  i.e., $X = \mathrm{conv}(E)$ for some finite set $E \subset \mathbb{R}^m$.
- Then $Y = \mathrm{conv}\{xP \mid x \in E\}$; thus Y is a polytope.
- By the Resolution Theorem, Y is a (bounded) polyhedron.

- $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$, $y' \neq y$.

Proposition 1.23

$y^* \in Y$ is efficient if and only if there exists $p \gg 0$ such that $y^* p \geq yp$ for all $y \in Y$.

- With "$p \geq 0$, $p \neq 0$" in place of "$p \gg 0$":
  - the "if" part is false;
  - the "only if" part holds whenever Y is a convex set (not only for polyhedron Y).

Proof

- The "if" part:

  If y^* is not efficient, i.e., $y' - y^* \geq 0, \neq 0$ for some $y' \in Y$, then for any $p \gg 0$, we have $(y' - y^*)p > 0$ or $y'p > y^*p$.

- The "only if" part:

  Suppose that $y^* \in Y$ is efficient.

- By Lemma 1.22, Y is written as $Y = \{y \in \mathbb{R}^n \mid yD \leq r\}$ for some $D \in \mathbb{R}^{n\times\ell}$ and $r \in \mathbb{R}^\ell$.

- Write $D = [S|T]$ and $r = [r^S|r^T]$ so that $y^*S = r^S$ and $y^*T \ll r^T$.

- $S \neq \emptyset$ by the efficiency of y^*:

  If $S = \emptyset$, i.e., $y^*D \ll r$, then $(y^* + \varepsilon\mathbf{1})D \leq r$ for sufficiently small $\varepsilon > 0$, where $y^* + \varepsilon\mathbf{1} \gneq y^*$.

- $zS \leq 0$, $z \geq 0$, $z \neq 0$ has no solution by the efficiency of y^*:

  If there exists such z, then $(y^* + \varepsilon z)D \leq r$ for sufficiently small $\varepsilon > 0$, where $y^* + \varepsilon z \gneq y^*$.
- By Ville's Theorem (Problem 5 in Homework 1), $S\lambda \gg 0$, $\lambda \geq 0$ has a solution.
- For a solution λ, let $p = S\lambda$ ($\gg 0$).
- Then for any $y \in Y$ (where $yS \leq r$), we have

$$y^* p = y^* S\lambda = r^S \lambda,$$
$$yp = yS\lambda \leq r^S \lambda,$$

  as desired.