5. Cores of Convex Games

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Reference

- L. S. Shapley, "Cores of Convex Games," *International Journal of Game Theory* 1, 11-26, 1971.

Cooperative Games

- $N = \{1, \dots, n\}$: Set of players
- A (*cooperative*) *game* is a function $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ (or pair (v, N)).
- A game v is *additive* if $v(S) + v(T) = v(S \cup T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$.
- A game v is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$.

  (A game v is *subadditive* if $-v$ is superadditive.)
- For a vector $x \in \mathbb{R}^n$, we write $x(S) = \sum_{i \in S} x_i$ for $S \subset N$.
- With a game v fixed, we write $H_S = \{x \in \mathbb{R}^n \mid x(S) = v(S)\}$ for $S \subset N$.

Convex Games

- A game v is *convex* if it is supermodular with respect to $\subset$:

  $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subset N$.

- A game v is *strictly convex* if it is strictly supermodular with respect to $\subset$:

  $v(S) + v(T) < v(S \cup T) + v(S \cap T)$ whenever neither $S \subset T$ nor $T \subset S$.

Proposition 5.1

For $v\colon 2^N \to \mathbb{R}$, the following statements are equivalent:

1. *v is convex.*
2. $v(S \cup \{i\}) - v(S) \le v(T \cup \{i\}) - v(T)$
   for all $i \in N$ and all $S \subset T \subset N \setminus \{i\}$.
3. $v(S) + \sum_{i \in T \setminus S} (v(S \cup \{i\}) - v(S)) \le v(T)$
   for all $S \subset T \subset N$.
4. $v(S \cup \{i\}) - v(S) \le v(S \cup \{i, j\}) - v(S \cup \{j\})$
   for all $i, j \in N$, $i \neq j$, and all $S \subset N \setminus \{i, j\}$.

Proof

- $1 \Rightarrow 2$:

$$v(S \cup \{i\}) + v(T) \leq v((S \cup \{i\}) \cup T) + v((S \cup \{i\}) \cap T) = v(T \cup \{i\}) + v(S).$$

- $2 \Rightarrow 1$:

Write $S \setminus T = \{i_1, \ldots, i_K\}$. Then we have

$$\begin{aligned} & v(S) - v(S \cap T) \\ & = \sum_{k=1}^{K} [v((S \cap T) \cup \{i_1, \ldots, i_k\}) - v((S \cap T) \cup \{i_1, \ldots, i_{k-1}\})] \\ & \leq \sum_{k=1}^{K} [v(T \cup \{i_1, \ldots, i_k\}) - v(T \cup \{i_1, \ldots, i_{k-1}\})] \\ & = v(S \cup T) - v(T). \end{aligned}$$

- Other equivalences: Homework exercise

Core

Definition 5.1

The *core* of a game $v\colon 2^N \to \mathbb{R}$ is the set

$$C(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \subset N\}.$$

- For $S \subset N$, write $C_S = C(v) \cap H_S$ $(= \{x \in C(v) \mid x(S) = v(S)\})$.

  $(C_N = C(v))$

Example: $n = 2$

- $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = 1$, $v(\{1,2\}) = 3$

Example: $n = 3$

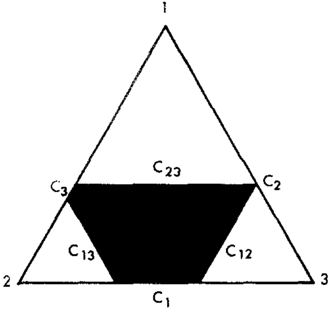


Fig. 1: Core configuration of a 3-person convex game

(Shapley 1971, page 17)

Example: $n = 4$

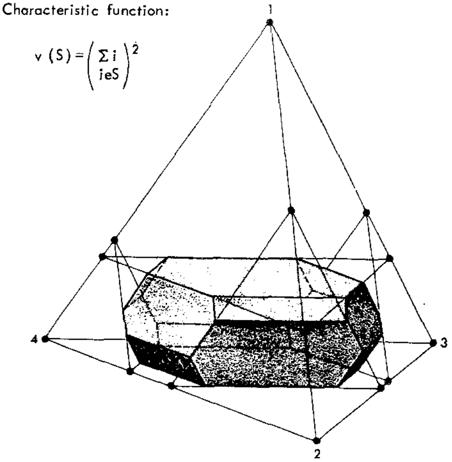
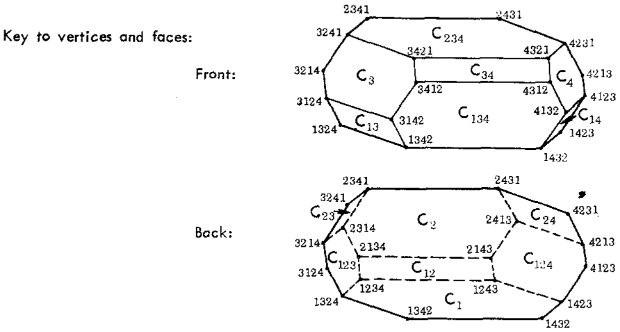


Fig. 2: Core of a four-person convex game

(Shapley 1971, page 17)



(Shapley 1971, page 17)

Nonemptiness of the Core

- Write Π for the set of all permutations of $N = \{1, \ldots, n\}$.
- For $\sigma = (i_1, \ldots, i_n) \in \Pi$, define $\alpha^\sigma \in \mathbb{R}^n$ by

$$\alpha^\sigma_{i_k} = v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, v_{k-1}\}).$$

  - α^σ is called the *marginal contribution vector* associated with σ.

Proposition 5.2

Let v be a convex game. Then $\alpha^\sigma \in C(v)$ for any $\sigma \in \Pi$.
In particular, $C(v) \neq \emptyset$.

Proof

- Without loss, consider $\sigma = (1, \ldots, n) \in \Pi$.
- Take any $S \subset N$, and denote $S = \{i_1, \ldots, i_m\}$, where $i_1 < \cdots < i_m$.

  Note that $\{i_1, \ldots, i_{k-1}\} \subset \{1, \ldots, i_{k-1}\}$.
- Then by convexity, we have

$$\begin{aligned}\alpha^\sigma(S) &= \sum_{k=1}^{m} v(\{1, \ldots, i_k\}) - v(\{1, \ldots, i_{k-1}\}) \\ &\geq \sum_{k=1}^{m} v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\}) \\ &= v(\{i_1, \ldots, i_m\}) - v(\emptyset) = v(S).\end{aligned}$$

Binding Inequalities

- Let v be a convex game.
- For $x \in C(v)$, define $\mathcal{S}_x = \{S \subset N \mid x(S) = v(S)\}$.
- $\mathcal{S}_x$ is closed under $\cup$ and $\cap$.
  - $\because \mathcal{S}_x = \arg\max\{v(S) - x(S) \mid S \subset N\}$, where $v - x$ is supermodular, which is a sublattice of 2^N by Proposition 4.8.
- If v is strictly convex, then $\mathcal{S}_x$ is a chain.

Extreme Points of the Core

Proposition 5.3

Let v be a convex game. Then the points written as α^σ are precisely the extreme points of $C(v)$.

Corollary 5.4

Let v be a convex game. Then $C(v) = \mathrm{conv}\{\alpha^\sigma \mid \sigma \in \Pi\}$.

Proof of Proposition 5.3

- Let $\sigma = (i_1, \ldots, i_n) \in \Pi$.
- $\alpha^\sigma \in C(v)$ is the solution to the LI system of n equations:

$$\begin{aligned} x(\{i_1\}) &= v(\{i_1\}) \\ x(\{i_1, i_2\}) &= v(\{i_1, i_2\}) \\ &\vdots \\ x(\{i_1, \ldots, i_n\}) &= v(\{i_1, \ldots, i_n\}). \end{aligned}$$

  Thus, α^σ is a basic feasible solution of $C(v)$.
- Therefore, by Proposition 3.2, α^σ is an extreme point of $C(v)$.

- Suppose that $x \in C(v)$ is an extreme point of $C(v)$.
- Let $\emptyset = S^0 \subsetneq S^1 \cdots \subsetneq S^m = N$ be a maximal chain in $\mathcal{S}_x$.

  $(\mathcal{S}_x = \{S \subset N \mid x(S) = v(S)\})$.
- If $m = n$, then $x = \alpha^\sigma$ for some $\sigma \in \Pi$.

  Assume that $m < n$.
- Then for some $k = 1, \ldots, m$, $S^k \setminus S^{k-1}$ contains at least two elements.

  Take any $i, j \in S^k \setminus S^{k-1}$, $i \neq j$.

- If $i, j \in S$ for all $S \in \mathcal{S}_x$, then $x + \varepsilon(e^i - e^j), x - \varepsilon(e^i - e^j) \in C(v)$ for some sufficiently small $\varepsilon > 0$, and hence x would not be an extreme point.
- Thus assume that there exists $S \in \mathcal{S}_x$ such that $i \in S$ and $j \notin S$.
- Then let $T = (S^{k-1} \cup S) \cap S^k$.

  Then $T \in \mathcal{S}_x$ and $S^{k-1} \subsetneqq T \subsetneqq S^k$ (since $i \in T$ and $j \notin T$).

  This contradicts the maximality of m.
- Hence, $m = n$, and therefore $x = \alpha^\sigma$ for some $\sigma \in \Pi$.

Shapley Value

- The *Shapley value* of a game v is the vector $\varphi \in \mathbb{R}^n$ defined by

$$\varphi_i = \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n - |S| + 1)!}{n!} (v(S \cup \{i\}) - v(S)).$$

- For $i \in N$ and $\sigma \in \Pi$, write $S(i, \sigma) \subset S \setminus \{i\}$ for the set of players that appear before i in σ.

  Then φ_i is written as

$$\begin{aligned}\varphi_i &= \frac{1}{n!} \sum_{S \subset N \setminus \{i\}} \sum_{\sigma \in \Pi : S(i,\sigma) = S} (v(S \cup \{i\}) - v(S)) \\ &= \frac{1}{n!} \sum_{\sigma \in \Pi} (v(S(i, \sigma) \cup \{i\}) - v(S(i, \sigma))),\end{aligned}$$

  and hence $\varphi = \frac{1}{n!} \sum_{\sigma \in \Pi} \alpha^\sigma$.

- Therefore, if v is convex, then $\varphi \in C(v)$.