6. Matroids and Polymatroids

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- We first study polymatroids,
- and then study matroids as a special case of polymatroids.

Submodular Functions

- $E = \{1, \ldots, n\}$: Finite set

Definition 6.1

Let $f\colon 2^E \to \mathbb{R}$.

- f is *non-decreasing* if $S \subset T \implies f(S) \leq f(T)$.
- f is *submodular* if $-f$ is supermodular (with respect to $\subset$), i.e.,

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$$

for all $S, T \subset E$.

Proposition 6.1

For $f: 2^E \to \mathbb{R}$, the following statements are equivalent:

1. *f is submodular.*
2. $f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T)$
   for all $i \in E$ and all $S \subset T \subset E \setminus \{i\}$.
3. $f(S) + \sum_{i \in T \setminus S}(f(S \cup \{i\}) - f(S)) \geq f(T)$
   for all $S \subset T \subset E$.
4. $f(S \cup \{i\}) - f(S) \geq f(S \cup \{i, j\}) - f(S \cup \{j\})$
   for all $i, j \in E$, $i \neq j$, and all $S \subset E \setminus \{i, j\}$.

- If f and g are submodular, then for $\alpha, \beta \geq 0$, $\alpha f + \beta g$ is submodular.
- If f is submodular and non-decreasing, then for any $k \in \mathbb{R}$, $g(S) = \min\{f(S), k\}$ is submodular.
- If f is submodular, then $g(S) = f(E \setminus S)$ is submodular.
- If f is submodular, then $g(S) = \min_{T \supset S} f(T)$ is submodular and non-decreasing.

Polymatroids

- For $x = (x_e)_{e \in E} \in \mathbb{R}^E$, we write $x(S) = \sum_{e \in S} x_e$ for $S \subset E$.

Definition 6.2

Let $f\colon 2^E \to \mathbb{R}$ be a submodular function.
The set

$$P(f) = \{x \in \mathbb{R}^E \mid x \geq 0,\ x(S) \leq f(S) \text{ for all } S \subset E\}$$

is called the *polymatroid* associated with (E, f).

- $P(f) \neq \emptyset$ if and only if $f(S) \geq 0$ for all $S \subset E$,

  in particular, if $f(\emptyset) = 0$ and f is non-decreasing.

- The set

$$P^*(f) = \{x \in \mathbb{R}^E \mid x(S) \leq f(S) \text{ for all } S \subset E\}$$

  is called the *extended polymatroid* associated with (E, f).
  $(P(f) = P^*(f) \cap \mathbb{R}^E_+)$
- The set

$$B(f) = \{x \in \mathbb{R}^E \mid x(S) \leq f(S) \text{ for all } S \subset E,\ x(E) = f(E)\}$$

  is called the *base polytope* of $P^*(f)$.
  $(B(f) = P^*(f) \cap \{x \in \mathbb{R}^E \mid x(E) = f(E)\})$
- In the following, we assume that $f(\emptyset) = 0$, and that f is non-decreasing when we talk about $P(f)$.

Cores of Convex Games

- The *core* of a cooperative game $v\colon 2^E \to \mathbb{R}$ is the set

  $$C(v) = \{x \in \mathbb{R}^E \mid x(S) \geq v(S) \text{ for all } S \subset E,\ x(E) = v(E)\}.$$

- $v\colon 2^E \to \mathbb{R}$ is a *convex game* if v is supermodular and $v(\emptyset) = 0$.
- For v, define $v^{\#}\colon 2^E \to \mathbb{R}$ by

  $$v^{\#}(S) = v(E) - v(E \setminus S).$$

  - $v^{\#}$ is submodular if and only if v is supermodular.
  - $x(E) = v(E) \iff x(E) = v^{\#}(E)$.
  - Since $v(S) = v^{\#}(E) - v^{\#}(E \setminus S) = x(E) - v^{\#}(E \setminus S)$, $x(S) \geq v(S)$ for all $S \subset E \iff x(S) \leq v^{\#}(S)$ for all $S \subset E$.
- Therefore, $C(v) = B(v^{\#})$.

Pareto Frontier of a Polymatroid

- Let $f\colon 2^E \to \mathbb{R}$ be a submodular function.

Proposition 6.2

$P^*(f) = \{x \in \mathbb{R}^E \mid x \leq y$ *for some* $y \in B(f)\}$.

Corollary 6.3

$B(f) \neq \emptyset$.

(Alternative proof of the nonemptiness of the core)

Corollary 6.4

$\max\{x(E) \mid x \in P^*(f)\} = \max\{x(E) \mid x \in B(f)\} = f(E)$.

Proof of Proposition 6.2

- Take any $x \in P^*(f)$.
- Then $\arg\max\{y(E) \mid y \in P^*(f), x \leq y\} \neq \emptyset$, since $\{y \in \mathbb{R}^E \mid y \in P^*(f), x \leq y\}$ is a nonempty compact set.
- Take any y in this argmax set.
- Denote $\mathcal{S}_y = \{S \subset E \mid y(S) = f(S)\}$,

  which is closed under $\cup$ and $\cap$.
- By the optimality of y, for each $e \in E$, there exists $S^e \in \mathcal{S}_y$ such that $e \in S^e$.
- Then $E = \bigcup_{e \in E} S^e \in \mathcal{S}_y$, i.e., $y(E) = f(E)$.

  Hence $y \in B(f)$.

Extreme Points of a Base Polytope

- Suppose that $f\colon 2^E \to \mathbb{R}$ is a submodular function with $f(\emptyset) = 0$, where $E = \{1, \ldots, n\}$.
- Consider the base polytope:

$$B(f) = \{x \in \mathbb{R}^n \mid x(S) \leq f(S) \text{ for all } S \subset E,\ x(E) = f(E)\}.$$

- For a permutation $\sigma = (i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$, define $x^\sigma \in \mathbb{R}^n$ by

$$x^\sigma_{i_k} = f(\{1, \ldots, i_k\}) - f(\{1, \ldots, i_{k-1}\}) \quad (k = 1, \ldots, n).$$

Proposition 6.5

The points written as x^σ are precisely the extreme points of $B(f)$.

- Proposition 5.3

Extreme Points of an Extended Polymatroid

Proposition 6.6

The points written as x^σ are precisely the extreme points of $P^*(f)$.

Proof

- First, the basic feasible points x^σ of $B(f)$ are basic feasible points, hence extreme points, of $P^*(f)$.
- Second, for any $x \in P^*(f) \setminus B(f)$, there exists $y \in B(f)$ such that $x \leq y$.

  Therefore, any extreme point of $P^*(f)$ is contained in $B(f)$.
- Thus, x^σ are precisely the extreme points of $P^*(f)$.

Extreme Points of a Polymatroid

- Suppose in addition that f is non-decreasing.
- For any sequence $\gamma = (i_1, \ldots, i_m)$ of distinct elements of E, define $x^\gamma \in \mathbb{R}^n_+$ by

$$x^\gamma_{i_k} = \begin{cases} f(\{1, \ldots, i_k\}) - f(\{1, \ldots, i_{k-1}\}) & \text{if } k = 1, \ldots, m, \\ 0 & \text{if } k = m+1, \ldots, n. \end{cases}$$

Proposition 6.7

The points written as x^γ are precisely the extreme points of $P(f)$.

- By a similar argument as in the proof of Proposition 5.3.

Integrality of Extreme Points

Proposition 6.8

If f is integer-valued, then all the extreme points of $B(f)$, $P^*(f)$, and $P(f)$ are integral.

Linear Programming on Polymatroids

- Given $f\colon 2^E \to \mathbb{R}$, let $P(f)$ be the polymatroid associated with (E, f).

  A vector $w = (w_e)_{e \in E} \in \mathbb{R}^n$ is given.

- Consider the linear program:

$$\max_{x \in P(f)} wx,$$

or explicitly,

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & x(S) \le f(S) && \text{for all } S \subset E \\ & x_e \ge 0 && \text{for all } e \in E. \end{aligned}$$

- Label the elements of E as $\{1,\ldots,n\}$ so that $w_1 \geq \cdots \geq w_k > 0 \geq w_{k+1} \geq \cdots \geq w_n$.
- Define $S^0 = \emptyset$, and $S^\ell = \{1,\ldots,\ell\}$, $\ell = 1,\ldots,n$.
- Define $x^* \in \mathbb{R}^n$ by

$$\begin{aligned} x_i^* &= f(S^i) - f(S^{i-1}) && (1 \leq i \leq k), \\ x_i^* &= 0 && (k+1 \leq i \leq n). \end{aligned}$$

Proposition 6.9

Suppose that f is a non-decreasing and submodular function with $f(\emptyset) = 0$.
Then x^* is an optimal solution to $\max_{x\in P(f)} wx$.

Proof

- Feasibility: By Proposition 6.7.
- To prove the optimality, consider the dual problem:

$$\begin{aligned} \min \quad & \sum_{S \subset E} y_S f(S) \\ \text{s. t.} \quad & \sum_{S \ni e} y_S \geq w_e \quad && \text{for all } e \in E \\ & y_S \geq 0 && \text{for all } S \subset E. \end{aligned}$$

- Define $y^* = (y^*_S)_{S \subset E}$ by
  - $y^*_{S^\ell} = w_\ell - w_{\ell+1}$ for $\ell = 1, \dots, k-1$;
  - $y^*_{S^k} = w_k$; and
  - $y^*_S = 0$ for $S \neq S^1, \dots, S^k$.
- Then y^* is feasible and satisfies $\sum_{S \subset E} y^*_S f(S) = \sum_{e \in E} w_e x^*_e$.

Proposition 6.10

Suppose that f is a submodular function with $f(\emptyset) = 0$ and that $w_e > 0$ for all $e \in E$ (so that $k = n$).
Then x^* is an optimal solution to $\max_{x \in B(f)} wx$ and $\max_{x \in P^*(f)} wx$.

Extreme Points of a Polymatroid (Alternative Proof)

- Suppose that $\bar{x}$ is an extreme point of $P(f)$.
- Since it is a vertex of $P(f)$, there exists $w \in \mathbb{R}^n$ such that $\arg\max_{x \in P(f)} wx = \{\bar{x}\}$.
- Without loss, assume $w_1 \geq \cdots \geq w_k > 0 \geq w_{k+1} \geq \cdots \geq w_n$.
- By Proposition 6.9, x^* is an optimal solution to $\max_{x \in P(f)} wx$.
- Thus, $\bar{x} = x^*$.

Polymatroid Intersection Theorem

- Given $f\colon 2^E \to \mathbb{R}$ and $g\colon 2^E \to \mathbb{R}$, define the function $f|g\colon 2^E \to \mathbb{R}$ by

$$(f|g)(S) = \min_{T \subset S} f(T) + g(S \setminus T).$$

  - If f and g are submodular, then so is $f|g$.
  - If $f(\emptyset) = g(\emptyset) = 0$, then $(f|g)(\emptyset) = 0$.
  - If f and g are non-decreasing, then so is $f|g$.
  - If f and g are integer-valued, then so is $f|g$.

Proposition 6.11

$P(f) \cap P(g) = P(f|g)$ *and* $P^*(f) \cap P^*(g) = P^*(f|g)$.

Proof

- Suppose that $x(S) \leq f(S)$ and $x(S) \leq g(S)$ for all $S \subset E$.

  Then for any $S \subset E$ and $T \subset S$, from $x(T) \leq f(T)$ and $x(S \setminus T) \leq g(S \setminus T)$, it follows that $x(S) \leq f(T) + g(S \setminus T)$.
- If $x(S) \leq f(T) + g(S \setminus T)$ for all $S \subset E$ and $T \subset S$, then for any $S \subset E$, $x(S) \leq f(S)$ and $x(S) \leq g(S)$.

Proposition 6.12

If f and g are integer-valued, then all the extreme points of $P(f) \cap P(g)$ and $P^*(f) \cap P^*(g)$ are integral.

Discrete Separation Theorem

Proposition 6.13

Suppose that $f\colon 2^E \to \mathbb{R}$ *and* $g\colon 2^E \to \mathbb{R}$ *are submodular and supermodular functions, respectively.*

- *If* $g(S) \le f(S)$ *for all* $S \subset E$*, then there exists* $x \in \mathbb{R}^E$ *such that* $g(S) \le x(S) \le f(S)$ *for all* $S \subset E$*.*
- *If in addition* f *and* g *are integer-valued, then there exists* $x \in \mathbb{Z}^E$ *such that* $g(S) \le x(S) \le f(S)$ *for all* $S \subset E$*.*

Proof

- Define the function $g^{\#}\colon 2^E \to \mathbb{R}$ by $g^{\#}(S) = g(E) - g(E \setminus S)$, which is submodular.
- Take any $z \in B(f|g^{\#})$ ($\neq \emptyset$), where $(f|g^{\#})(S) = \min_{T \subset S} f(T) + g^{\#}(S \setminus T)$.
- Then
  - $z(S) \leq f(S)$ for all $S \subset E$;
  - $z(S) \leq g^{\#}(S) = g(E) - g(E \setminus S)$ for all $S \subset E$; and
  - $z(E) = \min_{T \subset E} f(T) + g(E) - g(T) = g(E)$, since $f(T) - g(T) \geq 0$ for all T and $f(\emptyset) - g(\emptyset) = 0$;

  and therefore,
  - $z(S) \leq z(E) - g(E \setminus S)$ for all $S \subset E$, and hence $z(S) \geq g(S)$ for all $S \subset E$.
- If f and g are integer-valued, then we can take an integer vector as $z \in B(f|g^{\#})$.

Sum of Base Polytopes/Polymatroids

Proposition 6.14

1. $B(f) + B(g) = B(f + g)$.
2. $P(f) + P(g) = P(f + g)$.

Proof

- We only prove part 1; the proof of part 2 is similar.
- "$\subset$": Immediate.
- "$\supset$": Any extreme point of $B(f+g)$ is written as z^σ for some permutation $\sigma = (i_1, \ldots, i_n)$ where $z^\sigma_{i_k} = (f+g)(\{i_1, \ldots, i_k\}) - (f+g)(\{i_1, \ldots, i_{k-1}\})$.
- z^σ is written as $z^\sigma = x^\sigma + y^\sigma$, where $x^\sigma_{i_k} = f(\{i_1, \ldots, i_k\}) - f(\{i_1, \ldots, i_{k-1}\}) \in B(f)$, and $y^\sigma_{i_k} = g(\{i_1, \ldots, i_k\}) - g(\{i_1, \ldots, i_{k-1}\}) \in B(g)$, so that $z^\sigma \in B(f) + B(g)$.
- Then $B(f+g) = \mathrm{conv}\{z^\sigma \mid \sigma \in \Pi\} \subset B(f) + B(g)$ since $B(f) + B(g)$ is a convex set.

Sum of Extended Polymatroids

Proposition 6.15

$P^*(f) + P^*(g) = P^*(f + g)$.

Proof

- "$\subset$": Immediate.
- Suppose that $z \in P^*(f + g)$.

  Then there exists $z' \in B(f + g) = B(f) + B(g)$ such that $z \leq z'$, where $z' = x' + y'$ for some $x' \in B(f)$ and $y' \in B(g)$.
- Then $z = x' + (z - x')$, where $x' \in P^*(f)$ and $z - x' \in P^*(g)$ since $z - x' \leq y'$.
- Thus $z \in P^*(f) + P^*(g)$.

Matroids

Definition 6.3

For a finite set E and a family $\mathcal{I} \subset 2^E$ of subsets of E, $(E, \mathcal{I})$ is called an *independence system* if

1. $\emptyset \in \mathcal{I}$; and
2. $A \subset B \in \mathcal{I} \implies A \in \mathcal{I}$.

- Elements in $\mathcal{I}$ are called *independent sets*.
- Elements in $2^E \setminus \mathcal{I}$ are called *dependent sets*.

Definition 6.4

For an independence system $(E, \mathcal{I})$ and for $T \subset E$,
$B \subset T$ is called a *basis* of T or *maximal* in T if $B \in \mathcal{I}$ and $B \cup \{j\} \notin \mathcal{I}$ for any $j \in T \setminus B$.

Definition 6.5

An independence system $(E, \mathcal{I})$ is called a *matroid* if for all $T \subset E$, all bases of T have the same size, i.e., $|B| = |B'|$ whenever B and B' are bases of T.

Examples

- $E = \{1, 2, 3\}$

  $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}\}$

  $(E, \mathcal{I})$ is an independence system that is not a matroid.

  $\because$ $\{1\}$ and $\{2, 3\}$ are bases of E with different cardinality.

- $E = \{1, 2, 3\}$

  $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}\}$

  $(E, \mathcal{I})$ is a matroid.

- E: Finite set of n-dimensional vectors

  $\mathcal{I}$: All linearly independent subsets of E

  $(E, \mathcal{I})$ is a matroid.

Lemma 6.16

Let $(E,\mathcal{I})$ *be an independence system.*
For any $T \subset E$ *and an independent set* $S \subset T$*, there exists a basis* B *of* T *such that* $S \subset B$*.*

Proposition 6.17

Let $(E,\mathcal{I})$ be an independence system.
$(E,\mathcal{I})$ is a matroid if and only if for any $A, B \in \mathcal{I}$ with $|A| < |B|$, there exists $j \in B \setminus A$ such that $A \cup \{j\} \in \mathcal{I}$.

- "Only if":

  If there exist $A, B \in \mathcal{I}$ such that $|A| < |B|$ and $A \cup \{j\} \notin \mathcal{I}$ for any $j \in B \setminus A$, then A is a basis of $A \cup B$, while there is a basis of $A \cup B$ that contains B, that is, $(E,\mathcal{I})$ is not a matroid.

- "If":

  If $(E,\mathcal{I})$ is not a matroid, then there exist $S \subset E$ and bases A, B of S such that $|A| < |B|$, where $A \cup \{j\} \notin \mathcal{I}$ for any $j \in S \setminus A$, in particular, for any $j \in B \setminus A$.

Rank Functions

Definition 6.6

For an independence system $(E, \mathcal{I})$, the function $r\colon 2^E \to \mathbb{R}$ defined by

$$r(S) = \max\{|T| \mid T \in \mathcal{I},\ T \subset S\}$$

is called the *rank function* of $(E, \mathcal{I})$.

- If $(E, \mathcal{I})$ is a matroid, $r(S)$ equals the cardinality of any basis of S.
- If r is a rank function of $(E, \mathcal{I})$, then $\mathcal{I} = \{T \mid r(T) = |T|\}$.

  Therefore, we can write (E, r) for $(E, \mathcal{I})$.

Proposition 6.18

The rank function r of an independence system satisfies the following properties:

- r *is integer valued, and* $r(\emptyset) = 0$.
- $0 \leq r(S \cup \{j\}) - r(S) \leq 1$ *for any* $S \subset E$ *and* $j \in E \setminus S$.

Proposition 6.19

An independence system $(E, \mathcal{I})$ *is a matroid if and only if its rank function* r *is submodular.*

Proof

- Suppose that $(E, \mathcal{I})$ is a matroid.
- We want to show that $r(S \cup \{j\}) - r(S) \geq r(S \cup \{k, j\}) - r(S \cup \{k\})$ for any $j \neq k$ and $S \subset E \setminus \{k, j\}$.

  It suffices to consider the case where $r(S \cup \{j\}) - r(S) = 0$.
- Let $B \subset S$ be a basis of S, so that $r(S) = |B|$, and either $r(S \cup \{k\}) = |B|$ or $r(S \cup \{k\}) = |B| + 1$.

  Note that $B \cup \{j\} \notin \mathcal{I}$, since $r(S \cup \{j\}) = r(S) = |B|$. Therefore $B \cup \{k, j\} \notin \mathcal{I}$ as well.
- If $r(S \cup \{k\}) = |B|$, then $B \cup \{k\} \notin \mathcal{I}$, and hence $r(S \cup \{k, j\}) = |B|$.
- If $r(S \cup \{k\}) = |B| + 1$, then $B \cup \{k\} \in \mathcal{I}$, and hence $r(S \cup \{k, j\}) = |B| + 1$.

- Suppose that r is submodular.
- Let $S \subset E$.

  We want to show that $|B| = r(S)$ for any basis B of S.
- For $A \subset S$, $A \in \mathcal{I}$, suppose that $|A| < r(S)$.

  Then by submodularity of r,
  $r(A) = |A| < r(S) \leq r(A) + \sum_{j \in S \setminus A}(r(A \cup \{j\}) - r(A))$,
  which implies that $r(A \cup \{j\}) > r(A)$ for some $j \in S \setminus A$.
- This implies that $A \cup \{j\} \in \mathcal{I}$ for some $j \in S \setminus A$, so that A is not a basis of S.
- Thus, if B is a basis of S, then $|B| = r(S)$;
  in particular, any basis of S has the same cardinality.

  This shows that $(E, \mathcal{I})$ is a matroid.

Matroid Rank Functions

Definition 6.7

A function $r\colon 2^E \to \mathbb{R}$ is called a *matroid rank function* if it satisfies the following properties:

- r is integer valued, and $r(\emptyset) = 0$.
- $0 \leq r(S \cup \{j\}) - r(S) \leq 1$ for any $S \subset E$ and $j \in E \setminus S$.
- r is submodular.

- Proposition 6.19 implies that if r is a matroid rank function, then the independence system (E, r) is a matroid.

Matroids and Polymatroids

- Let $r\colon 2^E \to \mathbb{R}$ be a matroid rank function:
  i.e., an integer-valued submodular function with $r(\emptyset) = 0$ such that $0 \le r(S \cup \{i\}) - r(S) \le 1$ for all $S \subset E$.
- Let $\mathcal{I} = \{T \subset E \mid |T| = r(T)\}$.

  Then $(E, \mathcal{I})$ is a matroid with r as its rank function (Proposition 6.19).
- Identify 2^E and $\{0,1\}^E$:

  Identify $S \subset E$ with $x = (x_e)_{e \in E} \in \{0,1\}^E$ such that $x_e = 1$ if and only if $e \in S$, and vice versa.
- Then $\mathcal{I}$ is identified with the set of integer points in $P(r)$:
  $\{x \in \{0,1\}^E \mid x(S) \le r(S) \text{ for all } S \subset E\}$.

  (Note $r(\{e\}) \le 1$ for any $e \in E$.)

- For $T \in \mathcal{I}$, let $x^T \in \{0,1\}^E$ be such that $x_e = 1$ if and only if $e \in T$.

  Then for any $S \subset E$, $x^T(S) = |S \cap T| = r(S \cap T) \leq r(S)$ since $S \cap T \in \mathcal{I}$, hence $x^T \in P(r)$.

- For $x \in P(r) \cap \{0,1\}^E$, let $T^x \subset E$ be such that $e \in T$ if and only if $x_e = 1$.

  Then $|T^x| \leq r(T^x)$, hence $T^x \in \mathcal{I}$ ($|S| \geq r(S)$ by definition).

Matroid Intersection Theorem

Proposition 6.20

Let $M_1 = (E, \mathcal{I}_1)$ *and* $M_2 = (E, \mathcal{I}_2)$ *be matroids with rank functions* r_1 *and* r_2*, respectively. Then*

$$\max\{|J| \mid J \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{S \subset E} r_1(S) + r_2(E \setminus S).$$

Proof

- Write $P^{\mathbb{Z}}(f)$ for the set of integer points in $P(f)$.
- $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_1 \cap \mathcal{I}_2$ can be identified with $P^{\mathbb{Z}}(r_1)$, $P^{\mathbb{Z}}(r_2)$, and $P^{\mathbb{Z}}(r_1) \cap P^{\mathbb{Z}}(r_2)$, respectively.
- By the Polymatroid Intersection Theorem, $P^{\mathbb{Z}}(r_1) \cap P^{\mathbb{Z}}(r_2) = P^{\mathbb{Z}}(r_1|r_2)$, where $(r_1|r_2)(S) = \min_{T \subset S} r_1(T) + r_2(S \setminus T)$, which is a matroid rank function.
- Hence,

$$
\begin{aligned}
&\max\{|J| \mid J \in \mathcal{I}_1 \cap \mathcal{I}_2\} \\
&= \max\{z(E) \mid z \in P^{\mathbb{Z}}(r_1) \cap P^{\mathbb{Z}}(r_2)\} \\
&= \max\{z(E) \mid z \in P^{\mathbb{Z}}(r_1|r_2)\} = (r_1|r_2)(E).
\end{aligned}
$$

Application: Hall's Marriage Theorem

- A: Set of agents
- G: Set of goods
- $|A| = |G| = n$
- $D_i \subset G$: Set of acceptable goods for agent $i \in A$
- An assignment is a set $T \subset A \times G$ such that $|T| = n$, and if $(i, j), (i', j') \in T$, then $i \neq i'$ and $j \neq j'$.
- A *feasible assignment* is an assignment T such that $j \in D_i$ for all $(i, j) \in T$.
- If a feasible assignment exists, then clearly it is necessary that $|B| \leq \left|\bigcup_{i \in B} D_i\right|$ for all $B \subset A$.
- In fact, this condition is also sufficient.

Proposition 6.21

A feasible assignment exists if and only if

$$|B| \le \left| \bigcup_{i \in B} D_i \right| \text{ for all } B \subset A. \tag{$*$}$$

Proof

- We show the sufficiency of condition $(*)$.
- Let $E = \{(i,j) \in A \times G \mid j \in D_i\}$.
- Define $\mathcal{I}_A = \{T \subset E \mid (i,j),(i',j') \in T \implies i \neq i'\}$.

  Then $M_A = (E, \mathcal{I}_A)$ is a matroid.

  Let r_A be its rank function.
- Define $\mathcal{I}_G = \{T \subset E \mid (i,j),(i',j') \in T \implies j \neq j'\}$.

  Then $M_G = (E, \mathcal{I}_G)$ is a matroid.

  Let r_G be its rank function.
- $T \subset A \times G$ is a feasible assignment if and only if $T \in \mathcal{I}_A \cap \mathcal{I}_G$ and $|T| = n$.
- In light of the Matroid Intersection Theorem, it suffices to show that $\min_{S \subset E} r_A(S) + r_G(E \setminus S) \geq n$.

- Take any $S \subset E$.
- We have $r_A(S) = |S \cap A|$
  $(S \cap A = \{i \in A \mid (i,j) \in S \text{ for some } j \in G\})$,
- and

$$
\begin{aligned}
r_G(E \setminus S) &= |\{j \in G \mid (i,j) \in E \setminus S \text{ for some } i \in A\}| \\
&= |\{j \in G \mid j \in D_i \text{ and } (i,j) \notin S \text{ for some } i \in A\}| \\
&\geq |\{j \in G \mid j \in D_i \text{ for some } i \notin S \cap A\}| \\
&= \left|\bigcup_{i \notin S \cap A} D_i\right| \\
&\geq |A \setminus (S \cap A)| \qquad \text{(by } (*)) \\
&= |A| - |S \cap A|.
\end{aligned}
$$

- Therefore, we have $r_A(S) + r_G(E \setminus S) \geq |A| = n$ as desired.

Matroid Partition Theorem

Definition 6.8

Let $M_i = (E, \mathcal{I}_i)$, $i = 1, \ldots, k$, be a collection of matroids.
$J \subset E$ is *partitionable* with respect to $\{M_i\}_{i=1}^k$ if there exists a partition $\{J^1, \ldots, J^k\}$ of J such that $J^i \in \mathcal{I}_i$ for all $i \in 1, \ldots, k$.

Proposition 6.22

Let $M_i = (E, \mathcal{I}_i)$, $i = 1, \ldots, k$, be a collection of matroids, with corresponding rank functions r_i.
Then

$$\max\{|J| \mid J\text{: partitionable}\} = \min_{S \subset E} |E \setminus S| + \sum_{i=1}^{k} r_i(S).$$

Proof

- $J^1, \ldots, J^k$ are disjoint if and only if $(\mathbf{1}_{J^1} + \cdots + \mathbf{1}_{J^k})_e \leq 1$ for all $e \in E$, where $\mathbf{1}_S \in \mathbb{R}^E$ is such that $(\mathbf{1}_S)_e = 1$ if $e \in S$ and $(\mathbf{1}_S)_e = 0$ if $e \notin S$.
- This condition is equivalent to $\mathbf{1}_{J^1} + \cdots + \mathbf{1}_{J^k} \in P(r_0)$, where $r_0(S) = |S|$.
- Therefore, J is partitionable if and only if $\mathbf{1}_J \in P(r_0) \cap \sum_{i=1}^k P(r_i)$.
- But by the Polymatroid Intersection Theorem, $P(r_0) \cap \sum_{i=1}^k P(r_i) = P(r_0 | \sum_{i=1}^k r_i)$, where $(r_0 | \sum_{i=1}^k r_i)(S) = \min_{T \subset S} |T| + \sum_{i=1}^k r_i(S \setminus T)$, which is a matroid rank function.

- Therefore,

$$
\begin{aligned}
&\max\{|J| \mid J\text{: partitionable}\} \\
&= \max\{z(E) \mid z \in P^{\mathbb{Z}}(r_0) \cap \textstyle\sum_{i=1}^{k} P^{\mathbb{Z}}(r_i)\} \\
&= \max\{z(E) \mid z \in P(r_0 | \textstyle\sum_{i=1}^{k} r_i)\} \\
&= (r_0 | \sum_{i=1}^{k} r_i)(E).
\end{aligned}
$$

Matroid Packing Theorem

Definition 6.9

A collection of matroids $M_i = (E, \mathcal{I}_i)$, $i = 1, \ldots, k$, can be *packed* into E if there exist disjoint sets $B_1, \ldots, B_k$ such that B_i is basis in M_i for each i.

Proposition 6.23

A collection of matroids $M_i = (E, \mathcal{I}_i)$, $i = 1, \ldots, k$, with the corresponding rank functions r_i can be packed into E if and only if

$$\min_{S \subset E} |E \setminus S| + \sum_{i=1}^{k} r_i(S) = \sum_{i=1}^{k} r_i(E) \text{ for all } S \subset E. \qquad (**)$$

- Recall $r_i(E) = |B_i|$ for any basis in M_i.

Proof

- By the Matroid Partition Theorem, $(**)$ holds if and only if there exists a partitionable J with respect to $\{M_i\}_i$ such that $|J| = \sum_i |B_i|$ for any basis B_i in M_i for all i,

  or equivalently, there exist disjoint sets $F_i \in \mathcal{I}_i$ such that $\sum_i |F_i| = \sum_i |B_i|$ for any basis B_i in M_i, in particular for a basis $B_i \supset F_i$, for all i.

- The above equality holds if and only if $F_i = B_i$,

  thus this condition holds if and only if there exist disjoint bases B_i in M_i.

Application: Efficient Assignment of Indivisible Goods

- M: Set of indivisible objects
- N: Set of agents
- $v_j(S)$: monetary value of $S \subset M$ for $j \in N$

  Assume:
  - $v_j(\emptyset) = 0$ (normalization)
  - v_j is non-decreasing.
- Demand correspondence of $j \in N$:

  $D_j(p) = \{S \subset M \mid v_j(S) - p(S) \geq v_j(T) - p(T) \text{ for all } T \subset M\}$

  $(p(S) = \sum_{i \in S} p_i)$
- Assignment: $(y_j(S))_{S \subset M, j \in N}$ where $y_j(S) \in \{0, 1\}$

  $y_j(S) = 1 \iff S \subset M$ is consumed by $j \in N$.

Substitutes Condition (Condition S)

Definition 6.10
v satisfies condition (S) if
for any p, p' with $p \leq p'$ and any $S \in D(p)$,
there exists $B \in D(p')$ such that $\{i \in S \mid p_i = p'_i\} \subset B$.

- Unit demand case:

  For each $j \in N$, fix $a^j \in \mathbb{R}^M_+$ and let $v_j(S) = \max_{i \in S} a^j_i$.

  Then v_j satisfies condition (S).

Proposition 6.24

Suppose that v is non-decreasing.
If v satisfies condition (S), then it is submodular.

Single Improvement Property (Condition SI)

Definition 6.11

v satisfies condition (SI) if
for any p and any $S \notin D(p)$, there exists $B \subset M$ such that

$$v(B) - p(B) > v(S) - p(S)$$

and $|S \setminus B|, |B \setminus S| \leq 1$.

Proposition 6.25

Suppose that v is non-decreasing.
v satisfies condition (S) if and only if it satisfies condition (SI).

- In the following, we assume that each v_j satisfies condition (SI).

Efficient Assignment Problem

- Integer program:

$$
\begin{aligned}
(\mathsf{P}^*) \quad \max \quad & \sum_{S\subset M, j\in N} v_j(S) y_j(S) \\
\text{s. t.} \quad & \sum_{S\ni i, j\in N} y_j(S) \le 1 && \text{for all } i \in M \\
& \sum_{S\subset M} y_j(S) \le 1 && \text{for all } j \in N \\
& y_j(S) \in \{0,1\} && \text{for all } S \subset M, j \in N
\end{aligned}
$$

- Since there are finitely many feasible solutions, (P*) has an optimal solution $(y_j^*(S))$.

- Is there a price vector p^* that "supports" the assignment $(y_j^*(S))$?

- Relaxed problem:

$$
\begin{aligned}
\text{(P)} \quad \max \quad & \sum_{S \subset M, j \in N} v_j(S) y_j(S) \\
\text{s. t.} \quad & \sum_{S \ni i, j \in N} y_j(S) \leq 1 \quad \text{for all } i \in M \\
& \sum_{S \subset M} y_j(S) \leq 1 \qquad \text{for all } j \in N \\
& y_j(S) \geq 0 \qquad \qquad \text{for all } S \subset M, j \in N
\end{aligned}
$$

- If (P) has an integral optimal solution, then it is an optimal solution of (P^*).
- Let V_{LP} denote the optimal value of (P).

- Dual problem:

$$
\begin{aligned}
\text{(D)} \quad \min \quad & \sum_{i \in M} p_i + \sum_{j \in N} \lambda_j \\
\text{s. t.} \quad & \sum_{i \in S} p_i + \lambda_j \geq v_j(S) && \text{for all } S \subset M, j \in N \\
& p_i \geq 0, \lambda_j \geq 0 && \text{for all } i \in M, j \in N
\end{aligned}
$$

- Given (p_i), it is optimal to set $\lambda_j = \max_{S \subset M}(v_j(S) - p(S))$.
- Let

$$
V_p = p(M) + \sum_{j \in N} \max_{S \subset M}(v_j(S) - p(S)).
$$

- By the Duality Theorem, $V_{\text{LP}} = \min_{p \geq 0} V_p$.

Matroids

- We will show that for an optimal solution p^* to (D), there exist disjoint sets B_j, $j \in N$, such that $B_j \in D_j(p^*)$.
- Define

$$K_j(p) = \min\{|B| \mid B \in D_j(p)\},$$
$$D_j^*(p) = \arg\min\{|B| \mid B \in D_j(p)\}.$$

- For each $j \in N$ and p, define the independence system $(M, \mathcal{I}_j(p))$ by $T \in \mathcal{I}_j \iff T \subset B$ for some $B \in D_j^*(p)$.
- The rank function $r_j(\cdot|p)$:

$$\begin{aligned} r_j(S|p) &= \max\{|T| \mid T \in \mathcal{I},\ T \subset S\} \\ &= \max\{|T| \mid T \subset B \cap S \text{ for some } B \in D_j^*(p)\} \\ &= \max\{|B \cap S| \mid B \in D_j^*(p)\} \end{aligned}$$

Proposition 6.26

For each $j \in N$ *and* $p \in \mathbb{R}_+^M$, $(M, \mathcal{I}_j(p))$ *is a matroid.*

Matroid Packing

Proposition 6.27

Let p^* *be an optimal solution to* $\min_{p \geq 0} V_p$.
Then there exist disjoint sets B_j, $j \in N$, *such that* $B_j \in D_j(p^*)$.

- I.e., matroids $(M, \mathcal{I}_j(p^*))$, $j \in N$, can be packed into M.

Proof

- In light of the Matroid Packing Theorem, it suffices to show that for all $S \subset M$,

$$|M \setminus S| + \sum_{j \in N} r_j(S|p^*) \geq \sum_{j \in N} r_j(M|p^*).$$

- Since $r_j(S|p^*) = \max_{B \in D_j^*(p^*)} |B \cap S|$ and $r_j(M|p^*) = |B|$ for any $B \in D_j^*(p^*)$, this is equivalent to the condition: for all $S \subset M$,

$$|M \setminus S| + \sum_{j \in N} |B_j \cap S| \geq \sum_{j \in N} |B_j|.$$

for some $B_j \in D_j^*(p^*)$, $j \in N$.

- This is equivalent to the condition: for all $T \subset M$,

$$|T| \geq \sum_{j \in N} |B_j \cap T| \qquad (***)$$

for some $B_j \in D_j^*(p^*)$, $j \in N$.

- Fix any $T \subset M$, and for $\varepsilon > 0$, defined p' by $p'_i = p^*_i + \varepsilon$ for $i \in T$ and $p'_i = p^*_i$ for $i \notin T$.
- For each $j \in N$, pick any $B'_j \in D_j(p')$.
- Then we have

$$
\begin{aligned}
V_{p^*} &\leq V_{p'} = p'(M) + \sum_{j \in N} [v_j(B'_j) - p'(B'_j)] \\
&= p^*(M) + \varepsilon|T| + \sum_{j \in N} [v_j(B'_j) - (p^*(B'_j) + \varepsilon|B'_j \cap T|)] \\
&= p^*(M) + \sum_{j \in N} [v_j(B'_j) - p^*(B'_j)] + \varepsilon \left(|T| - \textstyle\sum_{j \in N} |B'_j \cap T| \right) \\
&\leq p^*(M) + \sum_{j \in N} \max_B [v_j(B) - p^*(B)] + \varepsilon \left(|T| - \textstyle\sum_{j \in N} |B'_j \cap T| \right) \\
&= V_{p^*} + \varepsilon \left(|T| - \textstyle\sum_{j \in N} |B'_j \cap T| \right).
\end{aligned}
$$

- Therefore, we have $|T| \geq \sum_{j \in N} |B'_j \cap T|$.
- Now let $\varepsilon \to 0$.

  Then by continuity, we have $|T| \geq \sum_{j \in N} |B_j \cap T|$ for some $B_j \in D_j(p^*)$, $j \in N$.
- Then by Lemma 8.35 in the textbook, we have $|T| \geq \sum_{j \in N} |B_j \cap T|$ for some $B_j \in D^*_j(p^*)$, $j \in N$, as desired.

Claim 1

There exists an optimal solution p^* to $\min_{p \geq 0} V_p$ such that

- there exist disjoint sets B_j, $j \in N$, such that $B_j \in D_j(p^*)$, and
- $p_i^* = 0$ for all $i \notin \bigcup_{j \in N} B_j$.

Proposition 6.28

Let p^* be as in the Claim.
Then the assignment $(y_j^*(S))$ defined by $y_j^*(B_j) = 1$ (and $y_j^*(S) = 0$ otherwise) is an optimal solution to (P*) *and is supported by p^*.*

- By weak duality