3. Structure of Polyhedra

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Extreme Points, Vertices

Definition 3.1
For $S \subset \mathbb{R}^n$, $\bar{x} \in S$ is an *extreme point* of S if

$$\bar{x} = \lambda y + (1 - \lambda)z,\ y, z \in S,\ \lambda \in (0, 1) \implies y = z = \bar{x}.$$

Definition 3.2
For $S \subset \mathbb{R}^n$, $\bar{x} \in S$ is a *vertex* of S if there exists $c \in \mathbb{R}^n$ such that $\arg\max\{cx \mid x \in S\} = \{\bar{x}\}$.

Proposition 3.1

For any $S \subset \mathbb{R}^n$, if $\bar{x} \in S$ is a vertex of S, then it is an extreme point of S.

Proof

- Suppose that $\arg\max\{cx \mid x \in S\} = \{\bar{x}\}$, and suppose that $\bar{x} = \lambda y + (1-\lambda)z$, $y, z \in S$, $\lambda \in (0,1)$.
- Then $\lambda cy \leq \lambda c\bar{x}$ and $(1-\lambda)cz \leq (1-\lambda)c\bar{x}$, and therefore $c\bar{x} = \lambda cy + (1-\lambda)cz \leq c\bar{x}$.
- Hence, $cy = c\bar{x}$ and $cz = c\bar{x}$, and therefore $y = z = \bar{x}$.
- The converse does not hold *in general*.

Basic Feasible Solutions of Polyhedra

- For $A = \begin{bmatrix} a^1 & \cdots & a^m \end{bmatrix} \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^m$, consider the polyhedron

$$P = \{x \in \mathbb{R}^n \mid A^{\mathrm{T}} x \leq b\}.$$

Definition 3.3

For $P = \{x \in \mathbb{R}^n \mid A^{\mathrm{T}} x \leq b\}$, $\bar{x} \in P$ is a *basic feasible solution* of P if there exists a subset B of A with $\mathrm{rank}(B) = n$ such that $B^{\mathrm{T}} \bar{x} = b^B$ (where $b^B = (b_j)_{j \in B}$).

Proposition 3.2

For $P = \{x \in \mathbb{R}^n \mid A^{\mathrm{T}} x \leq b\}$*, the following statements are equivalent:*

1. $\bar{x} \in P$ *is a vertex of* P*.*
2. $\bar{x} \in P$ *is an extreme point of* P*.*
3. $\bar{x} \in P$ *is a basic feasible solution of* P*.*

Proof

- $1 \Rightarrow 2$: By Proposition 3.1.
- $2 \Rightarrow 3$ (not $3 \Rightarrow$ not 2):
  Suppose that $\bar{x} \in P$ is not a basic feasible solution of P.
- Let $B = \{a^j \in A \mid a^j \bar{x} = b_j\}$.

  By assumption, $\mathrm{rank}(B) < n$.
- Take any $w \in \ker(B)$, $w \neq 0$ (where $\ker(B) \neq \{0\}$ since $\mathrm{rank}(\ker(B)) = n - \mathrm{rank}(B) > 0$).
- Let $\varepsilon > 0$ be such that $a^j(\bar{x} \pm \varepsilon w) < b_j$ for all $j \in A \setminus B$, and let $y = \bar{x} + \varepsilon w$ and $z = \bar{x} - \varepsilon w$.
- Then, $y \neq \bar{x}$, $z \neq \bar{x}$, $y, z \in P$, and $\bar{x} = \frac{1}{2}y + \frac{1}{2}z$,

  which means that $\bar{x}$ is not an extreme point of P.

- $3 \Rightarrow 1$:
  Let $\bar{x} \in P$ be a basic feasible solution of P, and let $B \subset A$ be such that $B^{\mathrm{T}}\bar{x} = b^B$ and $\mathrm{rank}(B) = n$.
- Let $c = \sum_{j \in B} a^j$.
- Then we have $c\bar{x} = \sum_{j \in B} a^j \bar{x} = \sum_{j \in B} b_j$, and

  if $Ax \leq b$, then $cx = \sum_{j \in B} a^j x \leq \sum_{j \in B} b_j = c\bar{x}$.
- If $cx = c\bar{x}$, then $a^j x = b_j$ for all $j \in B$,

  but since $\mathrm{rank}(B) = n$, this implies that $x = \bar{x}$.
- Hence, $\bar{x}$ is a vertex of P.

Linear Programs

- Standard form:

$$(*)\quad \max_{x\in\mathbb{R}^n}\ cx \qquad \text{s.t.}\ \ Ax = b,\quad x \geq 0$$

  where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

- Any linear program can be converted into the standard form:
  - If x_j is unrestricted, then substitute $x_j = x_j^+ - x_j^-$ with $x_j^+, x_j^- \geq 0$.
  - If a constraint is $\sum_{j=1}^n a_{ij}x_j \leq b_i$, then add a slack variable $s_i \geq 0$ so that $\sum_{j=1}^n a_{ij}x_j + s_i = b_i$.
  - If the objective is $\min cx$, then replace it with $\max(-c)x$.

- x^* is a *feasible solution* of $(*)$ if $x^* \in \{x \mid Ax = b,\ x \geq 0\}$.
- x^* is an *optimal solution* of $(*)$ if it is a feasible solution of $(*)$ such that $cx^* = \max\{cx \mid Ax = b,\ x \geq 0\}$.
- $(*)$ is *feasible* (resp. *infeasible*) if $\{x \mid Ax = b,\ x \geq 0\} \neq \emptyset$ (resp. $= \emptyset$).
- $(*)$ is *unbounded* if $\{cx \mid Ax = b,\ x \geq 0\}$ is unbounded above.

Basic Solutions

- Consider the linear program $(*)$ ($A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$).
- We assume throughout that $\operatorname{rank}(A) = m$ (and thus $m \leq n$).

  (If $\operatorname{rank}(A) < m$, remove redundant rows.)

Definition 3.4

Any set B of m LI columns of A (also considered as an $m \times m$ matrix) is called a *basis* of A.

Definition 3.5

$x \in \mathbb{R}^n$ is called a *basic solution* of $(*)$ if it is a solution to $Ax = b$ such that there exists a basis B of A such that $x_j \neq 0$ only if $j \in B$.

Definition 3.6

$x \in \mathbb{R}^n$ is called a *basic feasible solution* of $(*)$ if it is a basic solution such that $x \geq 0$.

Proposition 3.3

$\bar{x}$ is a basic feasible solution of $(*)$ if and only if it is a basic feasible solution of $\{x \mid Ax \le b, -Ax \le -b, -Ix \le 0\}$.

- Therefore, by Proposition 3.2 we have the following.

Proposition 3.4

Denote $P = \{x \mid Ax = b,\ x \ge 0\}$. The following statements are equivalent:

1. *$\bar{x} \in P$ is a vertex of P.*
2. *$\bar{x} \in P$ is an extreme point of P.*
3. *$\bar{x} \in P$ is a basic feasible solution of $(*)$.*

Proposition 3.5

If $\{x \mid Ax = b,\ x \geq 0\} \neq \emptyset$*, then there exists a basic feasible solution, hence an extreme point.*

- By Proposition 2.6, part 1 (cone version of Carathéodory's Theorem), and Proposition 3.4.

Fundamental Theorem of Linear Programming

Proposition 3.6

If $\max\{cx \mid Ax = b, x \geq 0\}$ *has an optimal solution, then there exists an optimal solution that is an extreme point of* $\{x \mid Ax = b, x \geq 0\}$.

Proof

- Denote $P = \{x \mid Ax = b, x \geq 0\}$ ($\neq \emptyset$).
- Let $v^* = \max\{cx \mid x \in P\}$ and
  $P^* = \left\{x \;\middle|\; \begin{bmatrix} A \\ c^{\mathrm{T}} \end{bmatrix} x = \begin{bmatrix} b \\ v^* \end{bmatrix},\ x \geq 0 \right\}$ ($\neq \emptyset$),
  where $\operatorname{rank}\left(\begin{bmatrix} A \\ c^{\mathrm{T}} \end{bmatrix}\right) = m$ or $m + 1$.
- If $\operatorname{rank}\left(\begin{bmatrix} A \\ c^{\mathrm{T}} \end{bmatrix}\right) = m$, then $P^* = P$.
  Then by Proposition 3.5 applied to P,
  P has an extreme point, which is an optimal solution.

- If $\mathrm{rank}\left(\begin{bmatrix} A \\ c^{\mathrm{T}} \end{bmatrix}\right) = m + 1$,

  by Proposition 3.5 applied to P^*,
  P^* has an extreme point.

- Let $x^* \in P^*$ be an extreme point of P^*.

  We want to show that it is an extreme point of P.

- Let $x^* = \lambda y + (1 - \lambda)z$, $y, z \in P$, $\lambda \in (0, 1)$.

- Then $v^* = cx^* = \lambda cy + (1 - \lambda)cz$ while $cy \leq v^*$ and $cz \leq v^*$.

  Thus we must have $cy = cz = v^*$, i.e., $y, z \in P^*$.

- But since x^* is an extreme point of P^*, we must have
  $y = z = x^*$.

Duality

- Given the linear program

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

  multiply both sides of $Ax = b$ by y from the left:

$$yAx = yb.$$

- If y satisfies $c \leq yA$, then, since $x \geq 0$, we have

$$cx \leq yAx = yb.$$

- Thus, for any y such that $c \leq yA$, yb is an upper bound of $\{cx \mid Ax = b, x \geq 0\}$.

- Primal problem:

$$
\text{(P)} \quad \max_{x \in \mathbb{R}^n} \; cx \quad \text{s.t.} \quad Ax = b, \quad x \geq 0
$$

- Dual problem:

$$
\text{(D)} \quad \min_{y \in \mathbb{R}^m} \; yb \quad \text{s.t.} \quad yA \geq c, \quad (y : \text{unrestricted})
$$

Weak Duality

Proposition 3.7

If x and y are feasible solutions of (P) *and* (D), *respectively, then $cx \leq yb$.*

- Therefore, if feasible solutions x^* and y^* satisfy $cx^* = y^*b$, then they are optimal solutions of (P) and (D), respectively.

Strong Duality

Proposition 3.8

If (P) *and* (D) *are feasible,*
then both (P) *and* (D) *have optimal solutions, and*

$$\max\{cx \mid Ax = b,\ x \geq 0\} = \min\{yb \mid yA \geq c\}.$$

Proof

- (P) and (D) have optimal solutions if and only if there exist x and y such that $Ax = b$, $x \geq 0$, $yA \geq c$, $cx \geq yb$, i.e.,

$$\begin{bmatrix} A & 0 \\ 0 & -A^{\mathrm{T}} \\ -c^{\mathrm{T}} & b^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{matrix} = \\ \leq \\ \leq \end{matrix} \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix}, \; x \geq 0.$$

- The alternative is:

$$\begin{bmatrix} \lambda & \mu & \eta \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A^{\mathrm{T}} \\ -c^{\mathrm{T}} & b^{\mathrm{T}} \end{bmatrix} \begin{matrix} \geq & = \end{matrix} \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \lambda & \mu & \eta \end{bmatrix} \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix} < 0, \; \mu \geq 0, \eta \geq 0.$$

- We want to show that, whenever (P) and (D) are feasible,

$$\lambda A - \eta c \geq 0, \ -\mu A^{\mathrm{T}} + \eta b^{\mathrm{T}} = 0, \ \mu \geq 0, \ \eta \geq 0 \tag{1}$$

  implies

$$\lambda b - \mu c \geq 0. \tag{2}$$

- For $\eta > 0$, (1) implies that μ/η and λ/η are feasible solutions, and hence by weak duality, $(\mu/\eta)c \leq (\lambda/\eta)b$, or $\mu c \leq \lambda b$.

- For $\eta = 0$, let x and y be feasible solution.

  Then from (1) we have $\lambda b - \mu c \geq \lambda A x - \mu A^{\mathrm{T}} y \geq 0$.

Infeasibility and unboundedness

Lemma 3.9

1. *If* (P) *is infeasible, then* (D) *is either infeasible or unbounded.*
2. *If* (P) *is unbounded, then* (D) *is infeasible.*

- Thus, if (D) is feasible and bounded, then so is (P).
- The same results hold with (P) and (D) interchanged.

Proof

- Part 1:

  If (P) is infeasible, then by Farkas' Lemma, there exists $\hat{y}$ such that $\hat{y}A \geq 0$ and $\hat{y}b < 0$.

- If (D) is feasible, i.e., there exists y^0 such that $y^0 A \geq c$, then for $t > 0$, $y^0 + t\hat{y}$ is feasible (since $(y^0 + t\hat{y})A \geq c$), and $(y^0 + t\hat{y})b = y^0 b + t(\hat{y}b) \to -\infty$ as $t \to \infty$.

- Part 2: By weak duality.

Strong Duality

Proposition 3.10

If either (P) *or* (D) *has an optimal solution, then the other also has an optimal solution, and*

$$\max\{cx \mid Ax = b,\ x \geq 0\} = \min\{yb \mid yA \geq c\}.$$

Proof

- If either (P) or (D) has an optimal solution, then the other is feasible by Lemma 3.9(1).
- Then by Proposition 3.8, it also has an optimal solution.

Complementary Slackness

Proposition 3.11

If x^* and y^* are optimal solutions of (P) *or* (D), *respectively, then* $(y^*A - c)x^* = 0$.

Proof

- Since $cx^* = y^*b$, we have

$$\begin{aligned}(y^*A - c)x^* &= y^*Ax^* - cx^* \\ &= y^*b - cx^* = 0.\end{aligned}$$

Primal and Dual Problems in Various Forms

- $\max cx$ s.t. $Ax \leq b$, $x \geq 0$
  - $\rightarrow \max cx + 0s$ s.t. $Ax + s = b$, $x \geq 0$, $s \geq 0$
  - Dual:

    $\min yb$ s.t. $y \begin{bmatrix} A & I \end{bmatrix} \geq \begin{bmatrix} c & 0 \end{bmatrix}$

    $\iff yA \geq c$, $y \geq 0$
- $\max cx$ s.t. $Ax = b$
  - $\rightarrow \max c(x^+ - x^-)$ s.t. $A(x^+ - x^-) = b$, $x^+ \geq 0$, $x^- \geq 0$
  - Dual:

    $\min yb$ s.t. $y \begin{bmatrix} A & -A \end{bmatrix} \geq \begin{bmatrix} c & -c \end{bmatrix}$

    $\iff yA = c$

- $\max cx$ s.t. $Ax \leq b$
  - $\rightarrow \max c(x^+ - x^-) + 0s$ s.t. $A(x^+ - x^-) + s = b$, $x^+ \geq 0$, $x^- \geq 0$, $s \geq 0$
  - Dual:

    $\min yb$ s.t. $y \begin{bmatrix} A & -A & I \end{bmatrix} \geq \begin{bmatrix} c & -c & 0 \end{bmatrix}$

    $\iff yA = c$, $y \geq 0$

Primal	Dual
$\max cx$	$\min yb$
$Ax = b$	y: unrestricted
$Ax \leq b$	$y \geq 0$
$x \geq 0$	$yA \geq c$
x: unrestricted	$yA = c$

Farkas' Lemma from Duality Theorem

- Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
- Suppose that $yA \geq 0$, $yb < 0$ has no solution, i.e.,

$$yA \geq 0 \implies yb \geq 0. \qquad (*)$$

  We want to show that $Ax = b$, $x \geq 0$ has a solution.
- Consider the linear program: $\max 0x$ s.t. $Ax = b$, $x \geq 0$.
- Its dual problem is: $\min yb$ s.t. $yA \geq 0$.
- By $(*)$, $y = 0$ is an optimal solution of the dual problem.
- Therefore, by Proposition 3.10, the primal problem has a feasible solution.

Application: Zero-Sum Games

Definition 3.7

A *zero-sum game* is given by an $m \times n$ matrix $A = (a_{ij})$, where when Row player plays strategy $i \in \{1, \dots, m\}$ and Column player plays strategy $j \in \{1, \dots, n\}$, Row's payoff is a_{ij} and Column's payoff is $-a_{ij}$.

- The set of mixed strategies for Row:

$$\Delta^m = \{x \in \mathbb{R}^m \mid x \geq 0,\ \textstyle\sum_{i=1}^m x_i = 1\}.$$

- The set of mixed strategies for Column:

$$\Delta^n = \{y \in \mathbb{R}^n \mid y \geq 0,\ \textstyle\sum_{j=1}^n y_j = 1\}.$$

- Min-max value for Row:

$$\min_{y \in \Delta^n} \max_i (Ay)_i \ (= \min_{y \in \Delta^n} \max_{x \in \Delta^m} xAy)$$

- Max-min value for Column:

$$\max_{x \in \Delta^m} \min_j (xA)_j \ (= \max_{x \in \Delta^m} \min_{y \in \Delta^n} xAy)$$

- Consider the following linear programs:

$$\text{(LP-R)} \quad \min_{y,R} \; R \quad \text{s.t.} \quad Ay - \mathbf{1}R \le 0, \quad \mathbf{1}y = 1,\; y \ge 0$$

$$\text{(LP-C)} \quad \max_{x,C} \; C \quad \text{s.t.} \quad xA - C\mathbf{1} \ge 0, \quad x\mathbf{1} = 1,\; x \ge 0$$

- (LP-R) and (LP-C) are duals to each other.

  Both are feasible, and therefore by strong duality (Proposition 3.8), these have optimal solutions (x^*, R^*) and (y^*, C^*), and $R^* = C^*$.

Minimax Theorem

Proposition 3.12

$\min_{y \in \Delta^n} \max_{x \in \Delta^m} xAy = \max_{x \in \Delta^m} \min_{y \in \Delta^n} xAy.$

Proof

- Clearly, LHS $\geq$ RHS.

  We want to show LHS $\leq$ RHS.

- Let (y^*, R^*) and (x^*, C^*) be optimal solutions of (LP-R) and (LP-C), respectively, where $R^* = C^*$.

  Then we have $Ay^* \leq \mathbf{1}R^*$, and hence $xAy^* \leq R^*$ for all $x \in \Delta^m$, i.e., $\max_x xAy^* \leq R^*$.

  Hence $\min_y \max_x xAy \leq R^*$.

- Similarly, we have $\max_x \min_y xAy \geq C^*$.

- Since $R^* = C^*$, we have LHS $\leq$ RHS.

- (The argument above in fact shows that

$$x^*Ay^* \leq \max_x xAy^* \leq R^* = C^* \leq \min_y x^*Ay \leq x^*Ay^*,$$

  which holds as equality.)

Definition 3.8

A profile $(x^*, y^*) \in \Delta^m \times \Delta^n$ is a *Nash equilibrium* of the zero-sum game A if

$$x^* A y^* \geq x A y^* \quad \text{for all } x \in \Delta^m,$$
$$x^* A y^* \leq x^* A y \quad \text{for all } y \in \Delta^n.$$

Proposition 3.13

(x^*, y^*) is a Nash equilibrium of A and $x^* A y^* = R^* = C^*$ if and only if (x^*, R^*) and (y^*, C^*) are optimal solutions of (LP-R) *and* (LP-C)*, respectively.*

Proof

- "If": By the Minimax Theorem.
- "Only if": $(y, R) = (y^*, x^* A y^*)$ and $(x, C) = (x^*, x^* A y^*)$ are feasible solutions of (LP-R) and (LP-C) and give the same value ($x^* A y^*$), hence they are optimal solutions.

Integrality

- We discuss sufficient conditions under which
  - all extreme points of a polyhedron are integral (integer valued); and
  - a linear program has an integral optimal solution.

Unimodular Matrices

- A square integer matrix $A \in \mathbb{Z}^{m \times m}$ is called *unimodular* if $\det A = 1$ or -1.

Proposition 3.14

For $A \in \mathbb{Z}^{m \times m}$, A^{-1} exists and is an integer matrix if and only if it is unimodular.

- Example: $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is a unimodular matrix ($\det A = -1$).

  $\rightarrow A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

Proof

- If A^{-1} exists and is an integer matrix, then $(\det A) \times (\det A^{-1}) = \det I = 1$.

  Then by the integrality of A and A^{-1}, we must have $(\det A, \det A^{-1}) = (1, 1)$ or $(-1, -1)$.

- If A is unimodular, then $A^{-1} = A^* / (\det A) = A^* \times 1$ or (-1) for some A^* called the adjoint of A, which is constructed with $+$, $-$, and $\times$ of the entries of A, so is an integer matrix.

Totally Unimodular Matrices

Definition 3.9

$A \in \mathbb{Z}^{m \times n}$ is *totally unimodular* (*TUM*) if $\det B = 1, -1,$ or 0 for every square submatrix B of A.

- $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$: not TUM
- $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$: TUM

Integral Extreme Points

Proposition 3.15

Let $A \in \mathbb{Z}^{m \times n}$ *and* $b \in \mathbb{Z}^m$*, and assume that* $\operatorname{rank} A = m$*.*
If A *is TUM, then every extreme point of*
$\{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$ *is integral.*

Proof

- By Proposition 3.4, every extreme point w of $\{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$ is a basic feasible solution, i.e., there exists a basis B of A such that $w^B = B^{-1}b$ (where $w = [w^B|0]$).
- Since A is TUM, any such B is unimodular.
- Therefore, by Proposition 3.14, B^{-1} is integral, and so is w.

A Sufficient Condition for TUM

Proposition 3.16

Suppose that $A \in \mathbb{Z}^{m \times n}$ *satisfies the following property:*

1. *each entry is* 0, 1, *or* -1;
2. *each column contains at most two non-zero entries; and*
3. *if a column contains two non-zero entries, then they are of opposite sign (i.e.,* 1 *and* -1*).*

Then A *is TUM.*

Proof

- It suffices to show that for any $B \in \mathbb{Z}^{k \times k}$, if B satisfies the property in the proposition, then $\det B = 1, -1,$ or 0.
- Prove by induction.

  The claim obviously holds for $k = 1$.
- Suppose that the claim holds for $k - 1$.

  Let $B \in \mathbb{Z}^{k \times k}$ satisfy the property in the proposition.

- There are three cases:
  1. There is a column whose entries are all zero.

     In this case, $\det B = 0$.
  2. There is a column that has exactly one non-zero entry (which is 1 or -1).

     In this case, suppose that j is such a column and $b_{ij} = 1$ or -1.

     Let $C \in \mathbb{R}^{(k-1)\times(k-1)}$ be the submatrix of B obtained by removing row i and column j.

     Then $\det B = (\det C) \times b_{ij} = (\det C) \times 1$ or $(-1) = 1, -1,$ or 0 by the induction hypothesis.
  3. All columns have two non-zero entries (which are 1 and -1).

     In this case, the sum of all the row vectors is the zero vector, and hence $\det B = 0$.

Application: Doubly Stochastic Matrices

- $A = (x_{ij}) \in \mathbb{R}^{n \times n}$ is called a *doubly stochastic matrix* if

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= 1 \quad \text{for all } i = 1, \ldots, n, \\ \sum_{i=1}^{n} x_{ij} &= 1 \quad \text{for all } j = 1, \ldots, n, \\ x_{ij} &\geq 0 \quad \text{for all } i, j = 1, \ldots, n. \end{aligned}$$

- Example:

$$\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0 & 0.2 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}$$

- A doubly stochastic matrix that consists only of 0 and 1 is called a *permutation matrix*.
- A convex combination of doubly stochastic matrices is a doubly stochastic matrix.

Proposition 3.17 (Birkhoff-von Neumann Theorem)

Any doubly stochastic matrix is written as a convex combination of permutation matrices.

- Example:

$$\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0 & 0.2 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}$$

$$= 0.2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Proof

- The set D of doubly stochastic matrices is the polyhedron defined by

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= 1 && \text{for all } i = 1, \ldots, n, \\ \sum_{i=1}^{n} (-x_{ij}) &= -1 && \text{for all } j = 1, \ldots, n-1, \\ x_{ij} &\geq 0 && \text{for all } i, j = 1, \ldots, n. \end{aligned}$$

  (One equation is implied by the others.)

- $D \neq \emptyset$ has an extreme point (Proposition 3.5).

- Written in a matrix form $Ax = b$, $x \geq 0$, A is TUM:
  - The column for x_{ij}, $j \neq n$, has exactly one 1 and exactly one -1; and
  - The column for x_{in} has exactly one 1.
- Therefore, by Proposition 3.15, all the extreme points of D are integral, and hence are permutation matrices.
- Thus, by the Krein-Milman Theorem, every doubly stochastic matrix (i.e., element of D) is written as a convex combination of permutation matrices (i.e., extreme points of D).

Application: Efficient Assignment of Indivisible Goods

- Indivisible objects $i \in M$
- Agents $j \in N$
- $v_{ij} \geq 0$: monetary value of one unit of object i for agent j
- Each agent consumes at most one object.
- Assume $|M| \geq |N|$.
- Assignment: $(x_{ij})_{i \in M, j \in N}$ where $x_{ij} \in \{0, 1\}$

  $x_{ij} = 1 \iff i$ is consumed by j.

- Efficient assignment problem:

$$
\begin{aligned}
(\mathsf{P}^*) \quad \max \quad & \sum_{i\in M, j\in N} v_{ij} x_{ij} \\
\text{s. t.} \quad & \sum_{j\in N} x_{ij} \le 1 \quad \text{for all } i \in M \\
& \sum_{i\in M} x_{ij} \le 1 \quad \text{for all } j \in N \\
& x_{ij} \in \{0,1\} \quad \text{for all } i \in M, j \in N
\end{aligned}
$$

- Since there are finitely many feasible solutions, (P*) has an optimal solution (x^*_{ij}).
- Is there a price vector p^* that "supports" the assignment (x^*_{ij})

  (i.e., agents optimize against p^* and demand and supply balance)?

- Consider the relaxed problem where the 0-1 constraint is removed (converted into the standard form):

$$
\begin{aligned}
\text{(P)} \quad \max \quad & \sum_{i \in M, j \in N} v_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j \in N} x_{ij} + s_i = 1 && \text{for all } i \in M \\
& \sum_{i \in M} (-x_{ij}) - t_j = -1 && \text{for all } j \in N \\
& x_{ij} \geq 0, s_i \geq 0, t_j \geq 0 && \text{for all } i \in M, j \in N
\end{aligned}
$$

- Written in a matrix form, the constraint matrix is TUM:
  - The column for x_{ij} has exactly one 1 and exactly one -1;
  - The column for s_i has exactly one 1; and
  - The column for t_j has exactly one -1.

- Since the feasible region is nonempty, it has extreme points, which are all integral by Proposition 3.15.
- Since there is an optimal solution that is an extreme point by Proposition 3.6, (P) has an integral optimal solution (x_{ij}^*, s_i^*, t_j^*).
- Clearly, (x_{ij}^*) is an optimal solution of (P*).

- Now consider the dual problem of (P):

$$\text{(D)} \quad \min \quad \sum_{i \in M} p_i + \sum_{j \in N} \lambda_j$$
$$\text{s.t.} \quad p_i + \lambda_j \geq v_{ij} \qquad \text{for all } i \in M, j \in N$$
$$p_i \geq 0, \lambda_j \geq 0 \qquad \text{for all } i \in M, j \in N$$

- Let (p_i^*, λ_j^*) be an optimal solution of (D).
- Then (p_i^*) supports (x_{ij}^*):
  - By optimality, $\lambda_j^* = \max_{i \in M}(v_{ij} - p_i^*)$.
  - By complementary slackness, $(p_i^* + \lambda_j^* - v_{ij})x_{ij}^* = 0$.
  - Therefore, if $x_{ij}^* = 1$, then
    $v_{ij} - p_i^* = \lambda_j^* = \max_{h \in M}(v_{hj} - p_h^*)$,
    i.e., i maximizes $v_{hj} - p_h^*$, $h \in M$.