4. Lattices and Supermodularity

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Partially Ordered Sets

Definition 4.1

A binary relation $\precsim$ on a set X is a *partial order* if it satisfies the following:

- Transitivity:
  for all $x, y, z \in X$, if $x \precsim y$ and $y \precsim z$, then $x \precsim z$.
- Reflexivity:
  for all $x \in X$, $x \precsim x$.
- Antisymmetry:
  for all $x, y \in X$, if $x \precsim y$ and $y \precsim x$, then $x = y$.

- A *partially ordered set* (or *poset*) is a set X with a partial order $\precsim$ on X, denoted $(X, \precsim)$.

Examples

- $(\mathbb{R}, \leq)$, where $\leq$ is the usual order on $\mathbb{R}$.

  In fact, it is a totally ordered set:
  $\leq$ also satisfies *completeness*: for all $x, y \in \mathbb{R}$, $x \leq y$ or $y \leq x$.

- $(\mathbb{R}^n, \leq)$, where $\leq$ is the vector order on $\mathbb{R}^n$.

- $(2^X, \subset)$, where 2^X is the set of all subsets of a set X, and $\subset$ is set inclusion.

Upper/lower bounds, ...

- Let $(X, \precsim)$ be a partially ordered set, and let $S \subset X$.
- $x \in X$ is an *upper bound* of S if $y \precsim x$ for all $y \in S$.

  $x \in X$ is a *lower bound* of S if $x \precsim y$ for all $y \in S$.
- $x \in X$ is a *greatest* (or *largest*) *element* of S if $x \in S$, and x is an upper bound of S.

  $x \in X$ is a *least* (or *smallest*) *element* of S if $x \in S$, and x is a lower bound of S.
- $x \in X$ is a *maximal element* of S if $x \in S$, and $x \precsim y$ and $y \in S \implies y = x$.

  $x \in X$ is a *minimal element* of S if $x \in S$, and $y \precsim x$ and $y \in S \implies y = x$.

- If the set of upper bounds of S has a least element, then it is called the *least upper bound*, or *supremum*, of S, and denoted $\sup_X S$.

  That is, $x = \sup_X S$ if and only if

  1. $y \precsim x$ for all $y \in S$; and
  2. if $y \precsim z$ for all $y \in S$, then $x \precsim z$.

- If the set of lower bounds of S has a greatest element, then it is called the *greatest lower bound*, or *infimum*, of S, and denoted $\inf_X S$.

  That is, $x = \inf_X S$ if and only if

  1. $x \precsim y$ for all $y \in S$; and
  2. if $z \precsim y$ for all $y \in S$, then $z \precsim x$.

- $\sup_X S$ is a greatest (least) element of S if and only if $\sup_X S \in S$ ($\inf_X S \in S$).

(Abstract) Lattices

- For $x, y \in X$, write

$$x \vee_X y = \sup_X\{x, y\}, \quad x \wedge_X y = \inf_X\{x, y\}.$$

  (If there is no risk of confusion, we just write $x \vee y$ and $x \wedge y$.)

Definition 4.2

A partially ordered set $(X, \precsim)$ is a *lattice* if $x \vee_X y$ and $x \wedge_X y$ exist for all $x, y \in X$.

Example

- $(\mathbb{R}, \leq)$ is a lattice.
  - $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$
- $(\mathbb{R}^n, \leq)$ is a lattice.
  - $x \vee y \in \mathbb{R}^n$: the vector such that $(x \vee y)_i = \max\{x_i, y_i\}$
  - $x \wedge y \in \mathbb{R}^n$: the vector such that $(x \wedge y)_i = \min\{x_i, y_i\}$
- $(2^X, \subset)$ is a lattice.
  - $S \vee T = S \cup T$, $S \wedge T = S \cap T$
- $X = \{(0,0), (1,0), (0,1)\} \subset \mathbb{R}^2$

  $(X, \leq)$ is not a lattice.
  - The set of upper bounds of $\{(1,0), (0,1)\}$ is empty, so $(1,0) \vee_X (0,1)$ does not exist.

- $X = \{(0,0),(1,0),(0,1),(2,2)\} \subset \mathbb{R}^2$

  $(X, \leq)$ is a lattice.

  - $(1,0) \vee_X (0,1) = (2,2)$
  - Note that $(1,0) \vee_{\mathbb{R}^2} (0,1) = (1,1)$.

    $\implies X$ is not a *sublattice* of $(\mathbb{R}^2, \leq)$ (to be defined later).

- $X = \{(0,0),(1,0),(0,1)\} \cup \{(x_1,x_2) \mid x_1 = x_2,\ x_1 > 2\} \subset \mathbb{R}^2$

  $(X, \leq)$ is not a lattice.

  - The set of upper bounds of $\{(1,0),(0,1)\}$,
    $\{(x_1,x_2) \mid x_1 = x_2,\ x_1 > 2\}$, does not have a least element.

Sublattices

Definition 4.3

For a lattice $(X, \precsim)$, $K \subset X$ is a *sublattice* of $(X, \precsim)$ if $x \vee_X y \in K$ and $x \wedge_X y \in K$ for all $x, y \in K$.

- If $K \subset X$ is a sublattice of $(X, \precsim)$, then $(K, \precsim)$ is a lattice, but not vice versa.
- (According to this definition, Definition 7.1 in the textbook is defining a sublattice of $\mathbb{R}^n$.)
- $X = \{(0,0), (1,0), (0,1), (1,1)\}$ is a sublattice of $(\mathbb{R}^2, \leq)$.
- $X = \{(0,0), (1,0), (0,1), (2,2)\}$ is not a sublattice of $(\mathbb{R}^2, \leq)$.

Complete Lattices

Definition 4.4

A lattice $(X, \precsim)$ is *complete* if $\sup_X S$ and $\inf_X S$ exist for all $S \subset X$
(where $\sup_X \emptyset = \inf_X X$ and $\inf_X \emptyset = \sup_X X$ by convention).

- This property is called "compact" in the textbook.
  We follow the "standard" terminology here.
- Any lattice $(X, \precsim)$ with finite X is a complete lattice, but not always if X is infinite.
- $X = [0, 1] \subset \mathbb{R}$

  $(X, \leq)$ is a complete lattice.
- $X = [0, 1) \subset \mathbb{R}$

  $(X, \leq)$ is not a complete lattice.

- $X = [0,1) \cup \{2\} \subset \mathbb{R}$

  $(X, \leq)$ is a complete lattice.

  - $\sup_X [0,1) = 2$
  - Note that $\sup_{\mathbb{R}} [0,1) = 1$.

    $\implies$ X is not a complete sublattice of $(\mathbb{R}, \leq)$.

Complete Sublattices

Definition 4.5

For a lattice $(X, \precsim)$, $K \subset X$ is a complete sublattice of $(X, \precsim)$ if $\sup_X S$ and $\inf_X S$ exist in K for all $S \subset K$, $S \neq \emptyset$.

Complete (Sub-)Lattices in $\mathbb{R}^n$

Proposition 4.1

1. *If $K \subset \mathbb{R}^n$, $K \neq \emptyset$, is compact and $(K, \leq)$ is a lattice, then $(K, \leq)$ is a complete lattice.*
2. *For $X \subset \mathbb{R}^n$, suppose that $(X, \leq)$ is a lattice.*

   If $K \subset X$ is compact and a sublattice of $(X, \leq)$, then K is a complete sublattice of $(X, \leq)$.

- Part 1 is a special case of part 2. (Let $K = X$.)

Proof

2.

- Let $S \subset K$, $S \neq \emptyset$.

  We want to show that $\sup_X S$ exists in K.
  (The existence of $\inf_X S$ in K can be shown symmetrically.)

- Let $U \subset X$ be the set of upper bounds of S in X:
  $U = \{u \in X \mid s \leq u \text{ for all } s \in S\}$.

  For the moment, assume that $U \neq \emptyset$. We prove this later.

- For $(s, u) \in S \times U$, write $[s, u] = \{x \in \mathbb{R}^n \mid s \leq x \leq u\}$, which is a closed set.

- We want to show that $\bigcap_{(s,u) \in S \times U} [s, u] \cap K \neq \emptyset$.

  If $\underline{u} \in \bigcap_{(s,u) \in S \times U} [s, u] \cap K$, then

  - $\underline{u} \in K$ $(\subset X)$;
  - $s \leq \underline{u}$ for all $s \in S$: $\underline{u}$ is an upper bound of S;
  - $\underline{u} \leq u$ for all $u \in U$. $\therefore \sup_X S = \underline{u} \in K$.

- Take any $(s^1, u^1), \dots, (s^K, u^K) \in S \times U$.
- Since X is a lattice, $\overline{s} = \sup_X \{s^1, \dots, s^K\}$ exists in X.
- Since K is a sublattice of X, $\overline{s} \in K$.
- For each $k = 1, \dots, K$, since u^k is an upper bound of $\{s^1, \dots, s^K\}$, we have $\overline{s} \leq u^k$.
- Therefore $\bigcap_{k=1}^K [s^k, u^k] \cap K \neq \emptyset$.
- By the compactness of K, this implies that $\bigcap_{(s,u) \in S \times U} [s, u] \cap K \neq \emptyset$.
- Finally, we show that $U \neq \emptyset$.
  - Write $[s, \infty) = \{x \in \mathbb{R}^n \mid s \leq x\}$, which is a closed set.
  - By the compactness of K, a similar argument as above shows that $\bigcap_{s \in S} [s, \infty) \cap K \neq \emptyset$.
  - Thus $U = \bigcap_{s \in S} [s, \infty) \cap X \supset \bigcap_{s \in S} [s, \infty) \cap K \neq \emptyset$.

Complete (Sub-)Lattices in $\mathbb{R}^n$

Proposition 4.2

For a sublattice $K \subset \mathbb{R}^n$ of $(\mathbb{R}^n, \leq)$, K is a complete sublattice of $(\mathbb{R}^n, \leq)$ if and only if it is a compact set.

Proof

- "If" part:

  Follows from Proposition 4.1.

- "Only if" part:

  Boundedness: K is contained in a bounded set $\{x \in \mathbb{R}^n \mid \inf_{\mathbb{R}^n} K \leq x \leq \sup_{\mathbb{R}^n} K\}$.

  Closedness: If $\{x^k\} \subset K$ and $x^k \to x^*$, then let $y^k = \inf_{\mathbb{R}^n}\{x^m\}_{m \geq k} \in K$, and let $\bar{y} = \sup_{\mathbb{R}^n}\{y^k\} \in K$.

  Show that $x^* = \bar{y}$:

  - For any $\varepsilon > 0$, there exists k such that $\bar{y} - \varepsilon \mathbf{1} \leq y^k$, and hence $\bar{y} - \varepsilon \mathbf{1} \leq x^m$ for all $m \geq k$. Therefore, $\bar{y} - \varepsilon \mathbf{1} \leq x^*$. Since $\varepsilon > 0$ is arbitrary, this implies $\bar{y} \leq x^*$.
  - For any $\varepsilon > 0$, there exists k such that $x^* - \varepsilon \mathbf{1} \leq x^m$ for all $m \geq k$, and hence $x^* - \varepsilon \mathbf{1} \leq y^k$. Therefore, $x^* - \varepsilon \mathbf{1} \leq \bar{y}$. Since $\varepsilon > 0$ is arbitrary, this implies that $x^* \leq \bar{y}$.

Tarski's Fixed Point Theorem

- For partially ordered sets $(X, \precsim_X)$ and $(Y, \precsim_Y)$, a function $f\colon X \to Y$ is *non-decreasing* (or *isotone*, or *order-preserving*) if $f(x) \precsim_Y f(x')$ whenever $x \precsim_X x'$.

Proposition 4.3 (Tarski's Fixed Point Theorem)

Suppose that $(X, \precsim)$ is a complete lattice, and that $f\colon X \to X$ is a non-decreasing function. Let $X^* \subset X$ be the set of fixed points of f.

1. $\sup\{x \in X \mid x \precsim f(x)\}$ *and* $\inf\{x \in X \mid f(x) \precsim x\}$ *are the greatest and the least elements of X^*.*
   In particular, $X^* \neq \emptyset$.
2. $(X^*, \precsim)$ *is a complete lattice.*

- X^* is not a sublattice in X in general.

Proof

1.

- Let $X' = \{x \in X \mid x \precsim f(x)\}$.

  $X' \neq \emptyset$ since $\inf X \in X'$.

- Denote $x^* = \sup X' \in X$.

  We show that $x^* \in X^*$.

- Take any $x \in X'$, where $x \precsim f(x)$ and $x \precsim x^*$.

  By the monotonicity of f, we have $f(x) \precsim f(x^*)$, so that $x \precsim f(x^*)$.

  Since this holds for any $x \in X'$, we have

$$x^* \precsim f(x^*). \tag{1}$$

- By the monotonicity of f, (1) implies that $f(x^*) \precsim f(f(x^*))$.

  This means that $f(x^*) \in X'$.

  Therefore, we have

$$f(x^*) \precsim x^*. \tag{2}$$

- By (1) and (2), we have $x^* = f(x^*)$, i.e., $x^* \in X^*$.
- For any $x \in X^*$, we have $x \in X'$, and therefore $x \precsim x^*$.

  Thus, x^* is the greatest element of X^*.
- A symmetric argument shows that $\inf\{x \in X \mid f(x) \precsim x\}$ is the least element of X^*.

2.

- Take any $S \subset X^*$.
- Denote $\bar{s} = \sup_X S \in X$ and $Z = \{x \in X \mid \bar{s} \precsim x\}$.

  $(Z, \precsim)$ is a complete lattice.
- We have $f(Z) \subset Z$.
  - Take any $z \in Z$.
  - For any $x \in S$ ($\subset X^*$), we have $x = f(x) \precsim f(\bar{s}) \precsim f(z)$.
  - This shows that $\bar{s} = \sup_X S \precsim z$, i.e., $f(z) \in Z$.
- Thus, the restriction $f|_Z$ of f to Z is a non-decreasing function from the complete lattice Z to itself.
- Let Z^* ($\subset X^*$) denote the set of fixed points of $f|_Z$, which is the set of upper bounds of S in X^*.

  By part 1, Z^* has a least element, which is $\sup_{X^*} S$.

- A symmetric argument shows that $\inf_{X^*} S$ exists in X^*.
- Thus, $(X^*, \precsim)$ is a complete lattice.

Proposition 4.4

Suppose that $(X, \precsim_X)$ *is a complete lattice and* $(Y, \precsim_Y)$ *is a lattice, and that* $f\colon X \times Y \to X$ *is a non-decreasing function. Then the greatest and the least fixed points of* $f(\cdot, y)$ *are non-decreasing in* y.

Proof

- Let $\bar{x}^*(y) \in X$ denote the greatest fixed point of $f(\cdot, y)$.
- Let $y' \precsim y''$.

  Let $Z = \{x \in X \mid \bar{x}^*(y') \precsim x\}$.

  $(Z, \precsim)$ is a complete lattice.
- $f(\cdot, y'')$ maps Z into itself:

  for any $x \in Z$, we have

  $$\bar{x}^*(y') = f(\bar{x}^*(y'), y') \precsim f(x, y') \precsim f(x, y'').$$
- By Tarski's Fixed Point Theorem, $f(\cdot, y'')$ has a fixed point in Z, and its greatest fixed point in Z is the greatest fixed point in X,

  that is, $\bar{x}^*(y'') \in Z$, or $\bar{x}^*(y') \precsim \bar{x}^*(y'')$.

Proposition 4.5

Let $X \subset \mathbb{R}^n$ be a compact set, and suppose that $(X, \leq)$ has a least element $\underline{x}$. Suppose that $f\colon X \to X$ is non-decreasing and continuous.

Then the sequence $x^k = f(x^{k-1})$ with $x^0 = \underline{x}$ converges to the least fixed point.

Proof

- By the monotonicity of f, we have

$$\begin{aligned} x^0 &= \underline{x} \leq f(x^0) = x^1, \\ x^1 &= f(x^0) \leq f(x^1) = x^2, \\ x^2 &= f(x^1) \leq f(x^2) = x^3, \\ &\vdots \end{aligned}$$

- By the boundedness of X, x^k converges to some $\underline{x}^*$, and by the closedness of X, $\underline{x}^* \in X$.
- By the continuity of f, $\underline{x}^* = f(\underline{x}^*)$, i.e., $\underline{x}^*$ is a fixed point of f.

- Let $\hat{x} \in X$ be a fixed point of f.
- By the monotonicity of f, we have

$$
\begin{aligned}
x^0 &= \underline{x} \leq \hat{x}, \\
x^1 &= f(x^0) \leq f(\hat{x}) = \hat{x}, \\
x^2 &= f(x^1) \leq f(\hat{x}) = \hat{x}, \\
&\vdots
\end{aligned}
$$

- Therefore, $\underline{x}^* \leq \hat{x}$.
- This shows that $\underline{x}^*$ is the least fixed point.

Application: Games with Monotone Best Responses

- $\mathcal{I} = \{1, \ldots, I\}$: Set of players
- S_i: Set of strategies of player $i \in \mathcal{I}$

  Partially ordered by $\precsim_i$
- Assumption: For all $i \in \mathcal{I}$, $(S_i, \precsim_i)$ is a complete lattice.
- $\precsim$: Product partial order on $S = \prod_{i \in \mathcal{I}} S_i$

  $(s_i)_{i \in \mathcal{I}} \precsim (s'_i)_{i \in \mathcal{I}}$ if and only if $s_i \precsim_i s'_i$ for all $i \in \mathcal{I}$
- $\precsim_{-i}$: Product partial order on $S_{-i} = \prod_{j \neq i} S_j$

  $(s_j)_{j \neq i} \precsim_{-i} (s'_j)_{j \neq i}$ if and only if $s_j \precsim_j s'_j$ for all $j \neq i$
- $u_i \colon S \to \mathbb{R}$: Payoff function of player $i \in \mathcal{I}$
- Denote this game by G.

- Best response correspondence (in pure strategies) of player i:

$$b_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i \in S_i\}$$

- Assumption: G has monotone best responses,

  i.e., for all $i \in \mathcal{I}$,

  - for all $s_{-i} \in S_{-i}$, $b_i(s_{-i})$ has a greatest element $\bar{b}_i(s_{-i})$ and a least element $\underline{b}_i(s_{-i})$, and
  - $\bar{b}_i(s_{-i})$ and $\underline{b}_i(s_{-i})$ are non-decreasing in s_{-i}.

- (We will later discuss what conditions on the primitives of the game guarantee this assumption to hold.)

Examples

Coordination game:

	L_2	R_2
L_1	$4, 4$	$0, 2$
R_1	$2, 0$	$3, 3$

- $b_i(L_j) = L_i$, $b_i(R_j) = R_i$
- With orders $L_i \prec_i R_i$, the best responses are non-decreasing.

Battle of the sexes:

	L_2	R_2
L_1	$0, 0$	$2, 1$
R_1	$1, 2$	$0, 0$

- $b_i(L_j) = R_i$, $b_i(R_j) = L_i$
- With orders $L_1 \prec_1 R_1$ and $R_2 \prec_2 L_2$, the best responses are non-decreasing.

Matching pennies:

	L_2	R_2
L_1	$1, -1$	$-1, 1$
R_1	$-1, 1$	$1, -1$

- $b_1(L_2) = L_1$, $b_1(R_2) = R_1$

  $b_2(L_1) = R_2$, $b_2(R_1) = L_2$
- With any orders, the best responses cannot be non-decreasing simultaneously for both players.

Existence of Pure-Strategy Nash Equilibria

Proposition 4.6

Suppose that the game G has monotone best responses.
Then G has a pure-strategy Nash equilibrium.
In particular, there are a greatest and a least pure-strategy Nash equilibria.

Proof

- The function $\bar{b}\colon S \to S$ defined by $\bar{b}(s) = (\bar{b}_1(s_{-1}), \ldots, \bar{b}_n(s_{-i}))$ is a non-decreasing function from the complete lattice S to itself.
- By Tarski's Fixed Point Theorem, a greatest fixed point of $\bar{b}$ exists, which is the greatest pure-strategy Nash equilibrium.

Supermodular Functions

Definition 4.6

For a lattice $(X, \precsim)$, a function $f\colon X \to \mathbb{R}$ is said to be *supermodular* if

$$f(x) + f(x') \leq f(x \vee x') + f(x \wedge x')$$

for all $x, x' \in X$.

- f is said to be *strictly supermodular* if $f(x) + f(x') < f(x \vee x') + f(x \wedge x')$ whenever neither $x \precsim x'$ nor $x' \precsim x$.
- f is said to be (strictly) *submodular* if $-f$ is (strictly) supermodular.

Example

- Let $X = \{x \in \mathbb{R}^2 \mid \underline{x} \leq x \leq \overline{x}\}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^2$.

  Suppose that $f\colon X \to \mathbb{R}$ is supermodular.

- Consider (x_1', x_2'') and (x_1'', x_2') with $x_1' \leq x_1''$ and $x_2' \leq x_2''$.

  By the supermodularity of f, we have

$$
\begin{aligned}
& f(x_1', x_2'') + f(x_1'', x_2') \\
& \leq f(x_1' \vee x_1'', x_2'' \vee x_2') + f(x_1' \wedge x_1'', x_2'' \wedge x_2') \\
& = f(x_1'', x_2'') + f(x_1', x_2'),
\end{aligned}
$$

  or $f(x_1'', x_2') - f(x_1', x_2') \leq f(x_1'', x_2'') - f(x_1', x_2'')$,

  that is, f satisfies *increasing differences* in (x_1, x_2).

- Conversely, if f satisfies increasing differences in (x_1, x_2), then it is supermodular.

Example: Submodular Functions on $\mathbb{R}^2$

- Let $X = \{x \in \mathbb{R}^2 \mid \underline{x} \leq x \leq \overline{x}\}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^2$.

  Suppose that $f\colon X \to \mathbb{R}$ is submodular (i.e., $-f$ is supermodular).

- Define the partial order $\leq^*$ on $\mathbb{R}^2$ by $(x_1', x_2') \leq^* (x_1'', x_2'') \iff x_1' \leq x_1'',\ x_2' \geq x_2''$.

- Then f is supermodular with respect to $\leq^*$:

  If $(x_1', x_2') \leq^* (x_1'', x_2'')$, then

$$
\begin{aligned}
f(x_1'', x_2') - f(x_1', x_2') &= -[(-f(x_1'', x_2')) - (-f(x_1', x_2'))] \\
&\leq -[(-f(x_1'', x_2'')) - (-f(x_1', x_2''))] \\
&= f(x_1'', x_2'') - f(x_1', x_2'').
\end{aligned}
$$

- This "trick" does not work with more than two variables.

Proposition 4.7

Let $X = \{x \in \mathbb{R}^n \mid \underline{x} \leq x \leq \overline{x}\}$ *for some* $\underline{x}, \overline{x} \in \mathbb{R}^n$, $\underline{x} \ll \overline{x}$, *and suppose that* $f\colon X \to \mathbb{R}$ *is twice continuously differentiable on* $\operatorname{int} X$ *and continuous on* X.
Then f *is supermodular if and only if for all* $i, j = 1, \ldots, n$, $i \neq j$ $\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0$ *for all* $x \in \operatorname{int} X$.

- Example:
  $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$, $\alpha_1, \alpha_2 \geq 0$, is supermodular on $\mathbb{R}^2_+$.

Optimization

Proposition 4.8

Let $(X, \precsim)$ *be a lattice.*

- *If* $f: X \to \mathbb{R}$ *is supermodular, then* $\arg\max_{x \in X} f(x)$ *is a sublattice of* X*.*
- *If* f *is strictly supermodular, then* $\arg\max_{x \in X} f(x)$ *is a chain, i.e., for any* $x, x' \in \arg\max_{x \in X} f(x)$*,* $x \precsim x'$ *or* $x' \precsim x$*.*

Proof

1.

- Suppose that $x, x' \in \arg\max_{x \in X} f(x)$.
- By supermodularity, we have

$$0 \leq f(x) - f(x \wedge x') \leq f(x \vee x') - f(x') \leq 0,$$

  which must hold with equality.

- Thus, $x \vee x', x \wedge x' \in \arg\max_{x \in X} f(x)$.

2.

- If $x, x' \in \arg\max_{x \in X} f(x)$, then we have $f(x \vee x') + f(x \wedge x') \leq f(x) + f(x')$.
- If neither $x \precsim x'$ nor $x' \precsim x$, then this contradicts the strict supermodularity.

Proposition 4.9

Let X and Y be lattices, and suppose that $f\colon X \times Y \to \mathbb{R}$ is supermodular.
Assume that $v(y) = \sup_{x \in X} f(x, y)$ is finite for all $y \in Y$.
Then v is supermodular.

Proof

- Let $y, y' \in Y$.
- For any $x, x' \in X$, we have

$$\begin{aligned} & v(y \vee_Y y') + v(y \wedge_Y y') \\ & \geq f(x \vee_X x', y \vee_Y y') + f(x \wedge_X x', y \wedge_Y y') \\ & \geq f(x, y) + f(x', y'). \end{aligned}$$

- Since this holds for all $x, x' \in X$, it follows that $v(y \vee_Y y') + v(y \wedge_Y y') \geq v(y) + v(y')$.

Monotone Comparative Statics

Definition 4.7

For partially ordered sets $(X, \precsim_X)$ and $(Y, \precsim_Y)$, a function $f\colon X \times Y \to \mathbb{R}$ satisfies *increasing differences* in (x, y) if

$$f(x'', y') - f(x', y') \leq f(x'', y'') - f(x', y'')$$

whenever $x' \precsim_X x''$ and $y' \precsim_Y y''$.

Proposition 4.10

Suppose that

- $X \subset \mathbb{R}^n$, $X \neq \emptyset$, *is compact and a lattice (with respect to* $\leq$*);*
- $(Y \precsim_Y)$ *is a partially ordered set; and*
- $f\colon X \times Y \to \mathbb{R}$ *satisfies the following:*
  - $f(\cdot, y)$ *is continuous for each* $y \in Y$*;*
  - $f(\cdot, y)$ *is supermodular for each* $y \in Y$*; and*
  - $f(x, y)$ *satisfies increasing differences in* (x, y)*.*

Then

1. *for each* $y \in Y$, $\arg\max_{x \in X} f(x, y)$ *is a nonempty, complete sublattice of* X*; and*
2. *its greatest and least elements are non-decreasing in* y*.*

Proof

- Write $X^*(y) = \arg\max_{x \in X} f(x, y)$.
- By the compactness of X and the continuity of $f(\cdot, y)$, $X^*(y)$ is nonempty and compact.
- By the compactness, the lattice X is a complete lattice by Proposition 4.1.
- Thus, together with the supermodularity of $f(\cdot, y)$, $X^*(y)$ is a sublattice of X by Proposition 4.8.
- Again by Proposition 4.1, $X^*(y)$ is a complete sublattice.
- Denote the greatest and the least elements of $X^*(y)$ by $\overline{x}^*(y)$ and $\underline{x}^*(y)$, respectively.

- Suppose that $y' \precsim_Y y''$.

  If $x' \in X^*(y')$ and $x'' \in X^*(y'')$, then

$$
\begin{aligned}
0 &\leq f(x', y') - f(x' \wedge_X x'', y') && \text{(by } x' \in X^*(y')\text{)} \\
&\leq f(x' \vee_X x'', y') - f(x'', y') && \text{(by supermodularity)} \\
&\leq f(x' \vee_X x'', y'') - f(x'', y'') && \text{(by increasing differences)} \\
&\leq 0 && \text{(by } x'' \in X^*(y'')\text{)}
\end{aligned}
$$

  which must hold with equality.

- Thus, $x' \wedge_X x'' \in X^*(y')$ and $x' \vee_X x'' \in X^*(y'')$.
- In particular, we must have
  - $\underline{x}^*(y') \precsim_X \underline{x}^*(y') \wedge_X \underline{x}^*(y'')$, so that $\underline{x}^*(y') \precsim_X \underline{x}^*(y'')$; and
  - $\overline{x}^*(y') \vee_X \overline{x}^*(y'') \precsim_X \overline{x}^*(y'')$, so that $\overline{x}^*(y') \precsim_X \overline{x}^*(y'')$.

Supermodular Games

- $\mathcal{I} = \{1, \ldots, I\}$: Set of players
- $S_i \subset \mathbb{R}^{n_i}$: Set of strategies of player $i \in \mathcal{I}$

  Partially ordered by $\leq$ on $\mathbb{R}^{n_i}$
  - $S_i \subset \mathbb{R}^{n_i}$: compact
- $u_i \colon S \to \mathbb{R}$: Payoff function of player $i \in \mathcal{I}$ $\quad (S = \prod_{j \in \mathcal{I}} S_j)$
  - $u_i(s_i, s_{-i})$: continuous in s_i for each s_{-i} and continuous in s_{-i} for each s_i
- Denote this game by G.
- G is called a *supermodular game* if for each $i \in \mathcal{I}$,
  - S_i is a complete lattice;
  - $u_i(s_i, s_{-i})$ is supermodular in s_i for each s_{-i}; and
  - $u_i(s_i, s_{-i})$ satisfies increasing differences in (s_i, s_{-i}).

Smooth Supermodular Games

- The game is supermodular if the following are satisfied:

  For each $i \in \mathcal{I}$:

  - $S_i = \{s_i \in \mathbb{R}^{n_i} \mid \underline{s}_i \leq s_i \leq \overline{s}_i\}$ for some $\underline{s}_i \ll \overline{s}_i$;
  - u_i is twice continuously differentiable on $\operatorname{int} S$, and continuous on S;
  - for all $s \in \operatorname{int} S$, $\frac{\partial^2 u_i}{\partial s_{ik} \partial s_{i\ell}}(s) \geq 0$ for all $k, \ell = 1, \ldots, n_i$, $k \neq \ell$.
  - for all $s \in \operatorname{int} S$, $\frac{\partial^2 u_i}{\partial s_{ik} \partial s_{jm}}(s) \geq 0$ for all $j \neq i$, and for all $k = 1, \ldots, n_i$ and all $m = 1, \ldots, n_j$.

Proposition 4.11

Suppose that the game G is a supermodular game.

1. G *has monotone best responses,*

   i.e., the greatest and the least best responses $\overline{b}_i(s_{-i})$ *and* $\underline{b}_i(s_{-i})$ *are well defined and non-decreasing in* s_{-i}*.*

2. G *has a greatest and a least pure-strategy Nash equilibria* $\overline{s}^*$ *and* $\underline{s}^*$*.*

3. *Let* $\overline{s} = (\overline{s}_i)_{i \in \mathcal{I}}$ *and* $\underline{s} = (\underline{s}_i)_{i \in \mathcal{I}}$ *be the greatest and the least strategy profiles. Then the sequences* $\overline{s}^k = \overline{b}(\overline{s}^{k-1})$, $\overline{s}^0 = \overline{s}$ *and* $\underline{s}^k = \underline{b}(\underline{s}^{k-1})$, $\underline{s}^0 = \underline{s}$ *converge to* $\overline{s}^*$ *and* $\underline{s}^*$*, respectively (where* $\overline{b}(s) = (\overline{b}_i(s_{-i}))_{i \in \mathcal{I}}$ *and* $\underline{b}(s) = (\underline{b}_i(s_{-i}))_{i \in \mathcal{I}}$*).*

Proof

- Part 1: By Proposition 4.10.
- Part 2: By Proposition 4.6.
- Part 3:

  $\{\underline{s}^k\}$ is increasing and bounded above, and thus converges to some $s^* \in S$.
- For any $s_i \in S_i$, $u_i(\underline{s}_i^k, \underline{s}_{-i}^{k-1}) \geq u_i(s_i, \underline{s}_{-i}^{k-1})$ for all k.

  By continuity, letting $k \to \infty$ we have

  $$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*). \qquad (*)$$

  Thus, s^* is a Nash equilibrium.
- For any Nash equilibrium $\hat{s}$, $\underline{s}^0 \leq \hat{s}$, $\underline{s}^1 = \underline{b}(\underline{s}^0) \leq \underline{b}(\hat{s}) \leq \hat{s}$, ..., and hence $s^* \leq \hat{s}$, i.e., s^* is the least Nash equilibrium.

- Proof of $(*)$:

  Suppose that $f(x, y)$ is continuous in x for each y and in y for each y and satisfies increasing differences in (x, y).

  If $\{(x^k, y^k)\}$ is non-decreasing and converges to (x^*, y^*), then $\lim_{k\to\infty} f(x^k, y^k) = f(x^*, y^*)$.

  - $f(x^*, y^*) - f(x^k, y^k) =$
    $f(x^*, y^*) - f(x^*, y^k) + f(x^*, y^k) - f(x^k, y^k)$,
    where by increasing differences,

$$f(x^*, y^0) - f(x^k, y^0) \le f(x^*, y^k) - f(x^k, y^k) \le f(x^*, y^*) - f(x^k, y^*).$$

  - Therefore,

$$\begin{aligned} &[f(x^*, y^*) - f(x^*, y^k)] + [f(x^*, y^0) - f(x^k, y^0)] \\ &\le f(x^*, y^*) - f(x^k, y^k) \\ &\le [f(x^*, y^*) - f(x^*, y^k)] + [f(x^*, y^*) - f(x^k, y^*)], \end{aligned}$$

    where the left and the right hand sides go to 0 as $k \to \infty$ by continuity in x (for $y = y^0, y^*$) and in y (for $x = x^*$).

Example: Bertrand Game with Differentiated Products

- Firms: $\mathcal{I} = \{1, \ldots, I\}$
- Strategy space of i: $S_i = [0, \bar{p}_i]$ (prices)
- $d_i(p_i, p_{-i})$: Demand for i's product
  - $\frac{\partial d_i}{\partial p_i} < 0$
  - $\frac{\partial d_i}{\partial p_j} > 0$, $j \neq i$ (substitutability)
- C_i: Total cost
  - $C_i'' \geq 0$
- Payoff function of i:

$$u_i(p_i, p_{-i}) = p_i d_i(p_i, p_{-i}) - C_i(d_i(p_i, p_{-i}))$$

- Cross derivatives:

$$\frac{\partial^2 u_i}{\partial p_i \partial p_j} = (p_i - C_i') \frac{\partial^2 d_i}{\partial p_i \partial p_j} + \left(1 - C_i'' \frac{\partial d_i}{\partial p_i}\right) \frac{\partial u_i}{\partial p_j}$$

  Second term > 0

- With linear demand $d_i(p_i, p_{-i}) = a_i - b_i p_i + g_{ij} \sum_{j \neq i} p_j$, $b_i, g_{ij} > 0$ (Problem 7.7), we have $\frac{\partial^2 d_i}{\partial p_i \partial p_j} = 0$ and therefore $\frac{\partial^2 u_i}{\partial p_i \partial p_j} > 0$, so that the game is supermodular.

Example: Cournot Game with Two Firms

- Firms: $\mathcal{I} = \{1, 2\}$
- Strategy space of 1: $S_1 = [0, \bar{x}_1]$ (quantities)

  Strategy space of 2: $S_2 = [-\bar{x}_2, 0]$ (negative of quantities)
- $P(Q)$: Inverse demand
  - $Q = x_1 + (-x_2)$: total supply
  - $P' < 0$
- C_i: Total cost
- Payoff functions:

$$u_1(x_1, x_2) = P(x_1 - x_2)x_1 - C_1(x_1)$$
$$u_2(x_1, x_2) = P(x_1 - x_2)(-x_2) - C_2(-x_2)$$

- Cross derivatives:

$$\frac{\partial^2 u_1}{\partial x_1 \partial x_2} = -P''(x_1 - x_2)x_1 - P'(x_1 - x_2)$$
$$\frac{\partial^2 u_2}{\partial x_2 \partial x_1} = P''(x_1 - x_2)x_2 - P'(x_1 - x_2)$$

$-P' > 0$

- With linear inverse demand $P(Q) = 1 - Q$ (Problem 7.8), we have $P'' = 0$ and therefore $\frac{\partial^2 u_i}{\partial x_i \partial x_{-i}} > 0$, so that the game is supermodular.

Application: Stable Matchings

- M: Set of men
- W: Set of women
- Assume $|M| = |W|$.
- Each $m \in M$ has a strict preference ordering $>^m$ over $W \cup \{m\}$;
  each $w \in W$ has a strict preference ordering $>^w$ over $M \cup \{w\}$.

  $x >^i y \cdots i$ ranks x above y.
- Assume
  - for all $m \in M$, $w >^m m$ for all $w \in W$; and
  - for all $w \in W$, $m >^w w$ for all $m \in M$.
- Write $x \geq^i y$ for "not $y >^i x$" ($\iff$ "$x >^i y$ or $x = y$").

- A *matching* is a function $\mu\colon M \cup W \to M \cup W$ such that
  - $\mu(m) \in W \cup \{m\}$ for all $m \in M$;
  - $\mu(w) \in M \cup \{w\}$ for all $w \in W$; and
  - $\mu(m) = w$ if and only if $\mu(w) = m$.
- A pair $(m, w) \in M \times W$ is a *blocking pair* for matching μ if $w >^m \mu(m)$ and $m >^w \mu(w)$.
- Matching μ is *stable* if there is no blocking pair for μ.

  (By assumption, individual rationality is satisfied.)

- Example:

M				W			
m_1:	w_2	w_1	w_3	w_1:	m_1	m_3	m_2
m_2:	w_1	w_3	w_2	w_2:	m_3	m_1	m_2
m_3:	w_1	w_2	w_3	w_3:	m_1	m_3	m_2

  - $\{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$ is not stable.
    $\because (m_1, w_2)$ is a blocking pair.
  - $\{(m_1, w_1), (m_2, w_3), (m_2, w_3)\}$ is stable.

Proposition 4.12

There exists a stable matching.
Moreover, there exist

- *a stable matching that is most preferred by all* $m \in M$ *and least preferred by all* $w \in W$ *among all stable matchings; and*
- *a stable matching that is most preferred by all* $w \in W$ *and least preferred by all* $m \in M$ *among all stable matchings.*

- First (formulated and) proved by Gale and Shapley (1962) via the "deferred acceptance algorithm".
- We prove by Tarski's Fixed Point Theorem.

- A *semi-matching* (or *pre-matching*) is a function $\mu\colon M \cup W \to M \cup W$ such that
  - $\mu(m) \in W \cup \{m\}$ for all $m \in M$; and
  - $\mu(w) \in M \cup \{w\}$ for all $w \in W$.
- X_0: Set of all semi-matchings
  $X \subset X_0$: Set of all matchings
- Define the function $f\colon X_0 \to X_0$ by the following: for $\mu \in X_0$,

$$f(\mu)(m) = \underset{>^m}{\arg\max}\{w \in W \mid m \geq^w \mu(w)\} \cup \{m\},$$
$$f(\mu)(w) = \underset{>^w}{\arg\max}\{m \in M \mid w \geq^m \mu(m)\} \cup \{w\}.$$

- X^*: Set of fixed points of f

Proposition 4.13

Any fixed point of f is a matching, i.e., $X^* \subset X$.

Proof

- Suppose that $f(\mu) = \mu$.
- Suppose that $\mu(m) = w$.
- Then $f(\mu)(m) = w$, which implies $m \geq^w \mu(w)$.
- Also $f(\mu)(w) \geq^w m$, and therefore $\mu(w) \geq^w m$.
- Hence $\mu(w) = m$.
- A symmetric argument shows that $\mu(w) = m \implies \mu(m) = w$.

Proposition 4.14

μ is a stable matching if and only if it is a fixed point of f, i.e., $\mu \in X^*$.

Proof

- Suppose that $\mu \in X^*$ ($\subset X$).

  If $w >^m \mu(m)$, then $f(\mu)(w) \geq^w m$, hence $\mu(w) \geq^w m$.

  Hence there is no blocking pair.

- Suppose that $\mu \in X \setminus X^*$.

  Suppose that there exists $m \in M$ such that $w = \mu(m) \neq w' = f(\mu)(m)$.

  Then $m >^{w'} \mu(w')$ and $w' >^m \mu(m)$.

  Hence (m, w') is a blocking pair.

Proposition 4.15

f has a fixed point, i.e., $X^* \neq \emptyset$.

Proof

- Define the partial order $\succsim$ on X_0 as follows: $\mu \succsim \nu$ if and only if
  - $\mu(m) \geq^m \nu(m)$ for all $m \in M$, and
  - $\nu(w) \geq^w \mu(w)$ for all $w \in W$.
- Then $(X_0, \succsim)$ is a complete lattice.

- $f\colon X_0 \to X_0$ is non-decreasing:

  Suppose that $\mu \succsim \nu$.

  - By $\nu(w) \geq^w \mu(w)$, we have $m \geq^w \nu(w) \implies m \geq^w \mu(w)$.

    Therefore, $f(\mu)(m) \geq^m f(\nu)(m)$.
  - By $\mu(m) \geq^m \nu(m)$, we have $w \geq^m \mu(m) \implies w \geq^m \nu(m)$.

    Therefore, $f(\nu)(w) \geq^w f(\mu)(w)$.
- Thus, by Tarski's Fixed Point Theorem, $X^* \neq \emptyset$.

  In particular, X^* has a greatest element (best for M and worst for W) and a least element (worst for M and best for W).

Problem 3, Homework 4

1. By Tarski's Fixed Point Theorem, f has a greatest fixed point $\bar{x}^*$.
2. - Let $X' = \{x \in X \mid \bar{x}^* \leq x\}$.
   - For $x \in X'$, we have $g(x) \geq f(x) \geq f(\bar{x}^*) = \bar{x}^*$, so that $g(x) \in X'$.
   - Thus, g maps the compact convex set X' to X'.
   - By Brouwer's Fixed Point Theorem, g has a fixed point x^{**} in X'.
   - For any fixed point x^* of f, we have $x^* \leq \bar{x}^* \leq x^{**}$.