Mathematics II
Daisuke Oyama
April 17, 2024

Homework 2

Due on April 24

1. Prove Proposition 2.12.

2. Prove Proposition 2.13.

3. Suppose that a sequence $\{x^m\}$ in $\mathbb{R}^N$ converges to $\bar{x} \in \mathbb{R}^N$. Show that the set $\{x^m \mid m \in \mathbb{N}\} \cup \{\bar{x}\}$ is compact.

4. Suppose that $A \subset \mathbb{R}^N$ is closed and $B \subset \mathbb{R}^N$ is compact. Show that the set $A + B = \{x \in \mathbb{R}^N \mid z = a + b \text{ for some } a \in A \text{ and } b \in B\}$ is closed. Find a counter-example when B is only assumed to be closed.

5. We want to prove Proposition 2.14: A function $f\colon X \to \mathbb{R}^K$ is continuous at $\bar{x} \in X$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(*) \quad \|x - \bar{x}\| < \delta,\ x \in X \Longrightarrow \|f(x) - f(\bar{x})\| < \varepsilon.$$

Complete the proof by continuing the following:

Proof of the "if" part: Suppose that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $(*)$ holds. Take any sequence $\{x^m\}_{m=1}^{\infty}$ with $x^m \in X$ for all $m \in \mathbb{N}$ such that $x^m \to \bar{x}$ as $m \to \infty$. We want to show that $f(x^m) \to f(\bar{x})$ as $m \to \infty$. Fix any $\varepsilon > 0$. ...

Proof of the "only if" part: Suppose that there exists some $\varepsilon > 0$ such that for any $\delta > 0$, there exists some $x \in X$ such that $\|x - \bar{x}\| < \delta$ and $\|f(x) - f(\bar{x})\| \geq \varepsilon$. Then for each $m \in \mathbb{N}$, let $x^m \in X$ be such that ...

6.

(1) Prove Proposition 2.16.
(2) Prove Proposition 2.17.

7. For a nonempty subset A of $\mathbb{R}^N$ and for $x \in \mathbb{R}^N$, denote

$$d(x, A) = \inf\{\|y - x\| \mid y \in A\}.$$

Prove the following:

(1) For any $x \in \mathbb{R}^N$, there exists $\bar{y} \in \operatorname{Cl} A$ such that $d(x, A) = \|\bar{y} - x\|$.
(2) Show that the function $f\colon \mathbb{R}^N \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.
(3) $d(x, A) = 0$ if and only if $x \in \operatorname{Cl} A$.

8. Let $X \subset \mathbb{R}^N$ be a nonempty set, and $f\colon X \to \mathbb{R}$ a continuous function.

Prove the following:

(1) If X is closed, then the set

$$\operatorname*{arg\,max}_{x \in X} f(x) = \{x \in X \mid f(x) \geq f(y) \text{ for all } y \in X\}$$

is closed.

(2) If X is compact, then $\operatorname{arg\,max}_{x \in X} f(x)$ is compact.

9.

(1) Prove Proposition 2.24.

(2) Prove Proposition 2.26.

10. Let $X \subset \mathbb{R}^N$ be a nonempty set. For a function $f\colon X \to \mathbb{R}$, the *hypograph* and the *epigraph* of f are the sets

$$\begin{aligned}
\operatorname{hyp} f &= \{(x, y) \in X \times \mathbb{R} \mid y \leq f(x)\}, \\
\operatorname{epi} f &= \{(x, y) \in X \times \mathbb{R} \mid y \geq f(x)\},
\end{aligned}$$

respectively. Prove the following:

(1) f is upper semi-continuous if and only if $\operatorname{hyp} f$ is closed relative to $X \times \mathbb{R}$.

(2) f is lower semi-continuous if and only if $\operatorname{epi} f$ is closed relative to $X \times \mathbb{R}$.

11. Prove Proposition 3.2 by using Proposition 3.1.

12.

(1) Give an example of a correspondence that is upper semi-continuous, has a closed graph, but is not compact-valued.

(2) Give an example of a correspondence that is upper semi-continuous, but whose graph is not closed.

(Specify the domain and the codomain when you define a function/correspondence.)

13. Let X and Y be nonempty subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively. For a correspondence $F\colon X \to Y$ and $B \subset \mathbb{R}^K$, write

$$\begin{aligned}
F^{-1}(B) &= \{x \in X \mid F(x) \subset B\}, \\
F_{-1}(B) &= \{x \in X \mid F(x) \cap B \neq \emptyset\}.
\end{aligned}$$

$F^{-1}(B)$ is called the *upper inverse image* (or *strong inverse image*) of B under F, while $F_{-1}(B)$ is called the *lower inverse image* (or *weak inverse image*) of B under F.

Prove the following:

(1) F is upper semi-continuous if and only if $F^{-1}(B)$ is open for any open set $B \subset Y$.

(2) F is lower semi-continuous if and only if $F_{-1}(B)$ is open for any open set $B \subset Y$.

14.

(1) Prove Proposition 3.9.

(2) Prove Proposition 3.10.

15. Define the correspondences $B\colon \mathbb{R}^N_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N_+$ by

$$B(p, w) = \{x \in \mathbb{R}^N_+ \mid p \cdot x \leq w\}.$$

(1) Show that B is upper semi-continuous.

(2) Show that B is lower semi-continuous.