Mathematics II
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Homework 4

Due on May 8

1. For $a > 0$, $a \neq 1$, let $f(x) = a^x$.

(1) Compute $f'(0)$.

(2) Compute $\displaystyle\lim_{c \to 1} \frac{a^{1-c} - 1}{1 - c}$.

2. Compute $\displaystyle\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x}$.

3. Consider the function $\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(1) Show that f is continuous at 0.

(2) Show that f is differentiable at 0.

4. The following statement is *false*:

If a differentiable function $f\colon I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is a nonempty open interval, is strictly increasing, then $f'(x) > 0$ for all $x \in I$.

Find a counter-example.

5. Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), where $a < b$. Show that if $f'(x) \neq 1$ for all $x \in (a, b)$, then f has at most one fixed point on $[a, b]$.

6. Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), where $a < b$, and that $f(a) > 0 > f(b)$. Assume that $f'(x) < 0$ whenever $f(x) = 0$. Show that there exists a unique $x \in [a, b]$ such that $f(x) = 0$.

7. Let $I \subset \mathbb{R}$ be a nonempty open interval, and for a function $f\colon I \to \mathbb{R}$ and $\bar{x} \in I$, suppose that f is differentiable on I and f' is differentiable at $\bar{x}$. Show directly using the Mean Value Theorem (and without using Taylor's Theorem) that if $\bar{x}$ is a local maximizer of f, then $f''(\bar{x}) \leq 0$.

(Hint: construct a sequence $\{x^m\}$ with $x^m \searrow \bar{x}$ such that $f'(x^m) \leq 0$.)

8. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} \dfrac{2x_1x_2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0,0), \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

(1) Compute $\dfrac{\partial f}{\partial x_1}$ and $\dfrac{\partial f}{\partial x_2}$.

(2) Show that f is not continuous at $(0,0)$.

9. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} x_1x_2\dfrac{x_1^2 - x_2^2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0,0), \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

(1) Compute $\dfrac{\partial^2 f}{\partial x_2 \partial x_1}$ and $\dfrac{\partial^2 f}{\partial x_1 \partial x_2}$.

(2) Show that $\dfrac{\partial^2 f}{\partial x_2 \partial x_1}$ is not continuous at $(0,0)$.

(3) Show that $\dfrac{\partial f}{\partial x_1}$ is not differentiable at $(0,0)$.

10. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} -\dfrac{1}{x_2^3}x_1^2(x_1 - 2x_2)^2 & \text{if } 2x_2 < x_1 < 0, \\ \dfrac{1}{x_2^3}x_1^2(x_1 - 2x_2)^2 & \text{if } 0 < x_1 < 2x_2, \\ -x_1^2(x_1 - 2x_2)^2 & \text{otherwise.} \end{cases}$$

(1) Compute $\dfrac{\partial f}{\partial x_1}$ and $\dfrac{\partial f}{\partial x_2}$.

(2) Show that f is continuous at $(0,0)$.

(3) Show that f is not differentiable at $(0,0)$.

11. For a function $f\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^N$ is a nonempty open set, the *directional derivative* of f at $x \in U$ with respect to $d \in \mathbb{R}^N$ is defined by

$$f'(x; d) = \lim_{\lambda \searrow 0} \frac{f(x + \lambda d) - f(x)}{\lambda}$$

if the limit exists. Show that if f is differentiable, then $f'(x; d) = \nabla f(x) \cdot d$ for all $x \in U$ and $d \in \mathbb{R}^N$.

12. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} \sqrt{x_1^2 + x_2^2}\sin\left(\dfrac{x_2^2}{x_1}\right) & \text{if } x_1 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Show that f is continuous at $(0,0)$.

(2) Compute the directional derivative $f'((0,0);d)$.

(3) Show that f is not differentiable at $(0,0)$.

13. Let

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(1) Determine the condition under which M is negative definite.

(2) Determine the condition under which M is negative semi-definite.

14. Let $f \colon \mathbb{R}^2_{++} \to \mathbb{R}$ be defined by

$$f(x_1, x_2) = (x_1)^{\alpha_1}(x_2)^{\alpha_2},$$

where $\alpha_1, \alpha_2 \geq 0$.

(1) Determine the condition on α_1 and α_2 under which f is concave.

(2) Determine the condition on α_1 and α_2 under which f is quasi-concave.