2. Continuous Functions and Compact Sets

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Euclidean Norm in $\mathbb{R}^N$

- For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, the Euclidean norm of x is denoted by $|x|$ or $\|x\|$, i.e.,

$$|x| = \sqrt{(x_1)^2 + \cdots + (x_N)^2},$$

or

$$\|x\| = \sqrt{(x_1)^2 + \cdots + (x_N)^2}.$$

- We follow MWG to use $\|\cdot\|$.
- For all $x, y \in \mathbb{R}^N$:
  - $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$;
  - $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{R}$;
  - $\|x + y\| \leq \|x\| + \|y\|$ (triangular inequality).

Convergence in $\mathbb{R}^N$

- A *sequence* in $\mathbb{R}^N$ is a function from $\mathbb{N}$ to $\mathbb{R}^N$.

  A sequence is denoted by $\{x^m\}_{m=1}^{\infty}$, or simply $\{x^m\}$, or x^m.

- Notation (in this course):
  For $A \subset \mathbb{R}^N$, if $x^m \in A$ for all $m \in \mathbb{N}$, then we write $\{x^m\}_{m=1}^{\infty} \subset A$.

Definition 2.1

A sequence $\{x^m\}_{m=1}^{\infty}$ *converges* to $\bar{x} \in \mathbb{R}^N$ if
for any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$\|x^m - \bar{x}\| < \varepsilon \text{ for all } m \geq M.$$

In this case, we write $\lim_{m\to\infty} x^m = \bar{x}$ or $x^m \to \bar{x}$ (as $m \to \infty$).

- $\bar{x}$ is called the *limit* of $\{x^m\}_{m=1}^{\infty}$.
- A sequence that converges to some $\bar{x} \in \mathbb{R}^N$ is said to be *convergent*.
- $\lim_{m\to\infty} x^m = \bar{x}$ if and only if $\lim_{m\to\infty} \|x^m - \bar{x}\| = 0$.

Convergence in $\mathbb{R}^N$

Proposition 2.1

For a sequence $\{x^m\}$ *in* $\mathbb{R}^N$*, where* $x^m = (x_1^m, \ldots, x_N^m)$*,*
$x^m \to \bar{x} = (\bar{x}_1, \ldots, \bar{x}_N) \in \mathbb{R}^N$
if and only if $x_i^m \to \bar{x}_i \in \mathbb{R}$ *for all* $i = 1, \ldots, N$*.*

- Thus, the definition in MWG (M.F.1) and that in Debreu (1.6.e) are equivalent.

Completeness of $\mathbb{R}^N$

- A sequence $\{x^m\}$ in $\mathbb{R}^N$ is a *Cauchy sequence* if for any $\varepsilon > 0$, there exists a natural number M such that

  $$\|x^m - x^n\| < \varepsilon \text{ for all } m, n \geq M.$$

- A convergent sequence is a Cauchy sequence.

Proposition 2.2 (Completeness of $\mathbb{R}^N$)
Every Cauchy sequence in $\mathbb{R}^N$ is convergent.

Proof

- Let $\{x^m\}$ be a Cauchy sequence in $\mathbb{R}^N$.
- Then, for each $i = 1, \ldots, N$, $\{x_i^m\}$ is a Cauchy sequence in $\mathbb{R}$, and hence is convergent (by the completeness of $\mathbb{R}$);

  denote its limit by $\bar{x}_i$.
- Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N)$.

  Then $x^m \to \bar{x}$ by Proposition 2.1.

Open Sets and Closed Sets in $\mathbb{R}^N$

- For $x \in \mathbb{R}^N$, the ε-open ball around x:

$$B_\varepsilon(x) = \{y \in \mathbb{R}^N \mid \|y - x\| < \varepsilon\}.$$

Definition 2.2

- $A \subset \mathbb{R}^N$ is an open set if for any $x \in A$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$.
- $A \subset \mathbb{R}^N$ is a closed set if $\mathbb{R}^N \setminus A$ is an open set.

Examples:

- $\{x \in \mathbb{R}^2 \mid x_1 + x_2 < 1\}$ is an open set.
  $\{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 1\}$ is a closed set.
- $B_\varepsilon(x)$, $\varepsilon > 0$, is an open set.

Relative Openness and Closedness

- In Consumer Theory, for example, we usually work with $\mathbb{R}^N_+$ (set of nonnegative consumption bundles) rather than $\mathbb{R}^N$.
- We want to say

  $$\{x \in \mathbb{R}^2 \mid x_1 + x_2 < 1,\ x_1 \geq 0,\ x_2 \geq 0\} \quad (= \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 < 1\})$$

  is an open set in the world of $\mathbb{R}^2_+$.

Definition 2.3

For $X \subset \mathbb{R}^N$,

- $A \subset X$ is an open set relative to X if for any $x \in A$, there exists $\varepsilon > 0$ such that $(B_\varepsilon(x) \cap X) \subset A$.
- $A \subset X$ is a closed set relative to X if $X \setminus A$ is an open set relative to X.

- Open sets, closed sets, and other concepts relative to X are defined with
  - X in place of $\mathbb{R}^N$, and
  - $B_\varepsilon(x) \cap X$ in place of $B_\varepsilon(x)$.
- $A \subset X$ is an open set relative to X if and only if $A = B \cap X$ for some open set $B \subset \mathbb{R}^N$ (relative to $\mathbb{R}^N$).

Properties of Open Sets

Proposition 2.3

Let $X \subset \mathbb{R}^N$.

1. $\emptyset$ *and* X *are open sets relative to* X.

2. *For any index set* Λ,
   if O_λ *is an open set relative to* X *for all* $\lambda \in \Lambda$,
   then $\bigcup_{\lambda \in \Lambda} O_\lambda$ *is an open set relative to* X.

   (The union of any family of open sets is open.)

3. *For any* $M \in \mathbb{N}$,
   if O_m *is an open set relative to* X *for all* $m = 1, \ldots, M$,
   then $\bigcap_{m=1}^{M} O_m$ *is an open set relative to* X.

   (The intersection of any finite family of open sets is open.)

Properties of Closed Sets

Proposition 2.4

Let $X \subset \mathbb{R}^N$.

1. $\emptyset$ *and* X *are closed sets relative to* X*.*

2. *For any index set* Λ*,*
   if C_λ *is a closed set relative to* X *for all* $\lambda \in \Lambda$*,*
   then $\bigcap_{\lambda \in \Lambda} C_\lambda$ *is a closed relative to* X*.*

   (The intersection of any family of closed sets is closed.)

3. *For any* $M \in \mathbb{N}$*,*
   if C_m *is a closed set relative to* X *for all* $m = 1, \ldots, M$*,*
   then $\bigcup_{m=1}^{M} C_m$ *is a closed set relative to* X*.*

   (The union of any finite family of closed sets is closed.)

Properties of Closed Sets

Proposition 2.5

Let $X \subset \mathbb{R}^N$.
$A \subset X$ is a closed set relative to X
$\Longleftrightarrow$ for any convergent sequence $\{x^m\}_{m=1}^{\infty} \subset A$ with $x^m \to \bar{x} \in X$, we have $\bar{x} \in A$.
(A closed set is closed with respect to convergence.)

Proof

- By definition,
  $A \subset X$ is a closed set relative to X
  $\iff \forall\, x \in X \setminus A\ \exists\, \varepsilon > 0 : B_\varepsilon(x) \cap A = \emptyset$.
- Therefore, if A is closed,
  then $\forall\, x \in X \setminus A$, any sequence in A cannot converge to x.
- Conversely, if A is not closed,
  then $\exists\, \bar{x} \in X \setminus A\ \forall\, \varepsilon > 0 : B_\varepsilon(\bar{x}) \cap A \neq \emptyset$.

  Then construct a sequence $\{x^m\}_{m=1}^{\infty} \subset A$ by

  $$x^m \in B_{\frac{1}{m}}(\bar{x}) \cap A \quad (m = 1, 2, \ldots).$$

  By construction, $x^m \to \bar{x} \notin A$.

Interior, Closure, and Boundary

Definition 2.4

For $X \subset \mathbb{R}^N$ and $A \subset X$,

- the interior of A relative to X:

$$\operatorname{Int}_X A = \{x \in A \mid (B_\varepsilon(x) \cap X) \subset A \text{ for some } \varepsilon > 0\};$$

- the closure of A relative to X: $\operatorname{Cl}_X A = X \setminus \operatorname{Int}_X(X \setminus A)$;
- the boundary of A relative to X: $\operatorname{Bdry}_X A = \operatorname{Cl}_X A \setminus \operatorname{Int}_X A$.

(We write $\operatorname{Int}_{\mathbb{R}^N} = \operatorname{Int}$, $\operatorname{Cl}_{\mathbb{R}^N} = \operatorname{Cl}$, and $\operatorname{Bdry}_{\mathbb{R}^N} = \operatorname{Bdry}$.)

Characterization of Interior

Proposition 2.6

Let $X \subset \mathbb{R}^N$ *and* $A \subset X$*.*

1. $\operatorname{Int}_X A \subset A$*.*
2. $\operatorname{Int}_X A$ *is an open set relative to* X*.*
3. *If* $B \subset A$ *and if* B *is open relative to* X*, then* $B \subset \operatorname{Int}_X A$*.*

Hence,

$$\operatorname{Int}_X A = \bigcup \{B \subset X \mid B \subset A \text{ and } B \text{ is open relative to } X\},$$

i.e., $\operatorname{Int}_X A$ is the largest open set (relative to X) contained in A.

Proof

1. By definition.

2.
   - Take any $x \in \operatorname{Int}_X A$.

     By definition, $(B_\varepsilon(x) \cap X) \subset A$ for some $\varepsilon > 0$.

     We want to show that $(B_\varepsilon(x) \cap X) \subset \operatorname{Int}_X A$.
   - Take any $y \in B_\varepsilon(x) \cap X$.

     Let $\varepsilon' = \varepsilon - \|y - x\| > 0$.

     Then $B_{\varepsilon'}(y) \subset B_\varepsilon(x)$.
   - Hence, $(B_{\varepsilon'}(y) \cap X) \subset (B_\varepsilon(x) \cap X) \subset A$, which implies that $y \in \operatorname{Int}_X A$.

3. Take any $x \in B$.

   By the openness of B, $(B_\varepsilon(x) \cap X) \subset B$ for some $\varepsilon > 0$.

   By $B \subset A$, $(B_\varepsilon(x) \cap X) \subset A$.

   Therefore, $x \in \operatorname{Int}_X A$.

Characterization of Closure

Proposition 2.7

Let $X \subset \mathbb{R}^N$ *and* $A \subset X$*.*

1. $A \subset \mathrm{Cl}_X A$*.*
2. $\mathrm{Cl}_X A$ *is a closed set relative to* X*.*
3. *If* $A \subset B$ *and if* B *is closed relative to* X*, then* $\mathrm{Cl}_X A \subset B$*.*

Hence,

$$\mathrm{Cl}_X A = \bigcap \{B \subset X \mid B \supset A \text{ and } B \text{ is closed relative to } X\},$$

i.e., $\mathrm{Cl}_X A$ is the smallest closed set (relative to X) containing A.

Proof

- By Proposition 2.6.

Examples

- For $X = \mathbb{R}$,

$$\begin{aligned}
&\mathrm{Int}[0,1) = (0,1), \\
&\mathrm{Cl}[0,1) = [0,1], \\
&\mathrm{Bdry}[0,1) = \{0,1\}.
\end{aligned}$$

- What are the interior, closure, and boundary of $\mathbb{Q} \cap [0,1]$? $\rightarrow$ Homework

- For $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1,\ x_2 = 0\}$,

  $\mathrm{Int}\, A\ (= \mathrm{Int}_{\mathbb{R}^2} A) = \emptyset$, while $\mathrm{Int}_{\mathbb{R}}\, A = (0,1)$.

  Remark
  There is an abuse of notation in "$\mathrm{Int}_{\mathbb{R}}\, A = (0,1)$":
  To be precise, one should write
  $\mathrm{Int}_{\{x \in \mathbb{R}^2 \mid x_1 \in \mathbb{R},\ x_2 = 0\}}\, A = \{x \in \mathbb{R}^2 \mid x_1 \in (0,1),\ x_2 = 0\}$.

Characterizations of Open/Closed Sets by Interior/Closure

Proposition 2.8

Let $X \subset \mathbb{R}^N$ *and* $A \subset X$*.*

1. A *is open relative to* $X \iff \operatorname{Int}_X A = A$*.*
2. A *is closed relative to* $X \iff \operatorname{Cl}_X A = A$*.*

Characterizations of Closure

Proposition 2.9

Let $X \subset \mathbb{R}^N$ *and* $A \subset X$.

1. $\mathrm{Cl}_X A = \{x \in X \mid B_\varepsilon(x) \cap A \neq \emptyset \text{ for all } \varepsilon > 0\}$
   $= \bigcap_{\varepsilon > 0} B_\varepsilon(A) \cap X$,

   where $B_\varepsilon(A) = \{x \in \mathbb{R}^N \mid \|x - a\| < \varepsilon \text{ for some } a \in A\}$.
2. $\mathrm{Cl}_X A = \{x \in X \mid x^m \to x \text{ for some } \{x^m\} \subset A\}$.

- Thus, the definition in MWG and that in Debreu are equivalent.

Proof of Proposition 2.9

Recall the definition: $\mathrm{Cl}_X A = X \setminus \mathrm{Int}_X(X \setminus A)$.

1. For $x \in X$, we have
   $$\begin{aligned} & x \in \mathrm{Cl}_X A \\ \iff & x \notin \mathrm{Int}_X(X \setminus A) \\ \iff & \forall\, \varepsilon > 0 : (B_\varepsilon(x) \cap X) \not\subset (X \setminus A) \\ \iff & \forall\, \varepsilon > 0 : B_\varepsilon(x) \cap A \neq \emptyset \\ \iff & \forall\, \varepsilon > 0 : x \in B_\varepsilon(A). \end{aligned}$$

2. If $x \in \mathrm{Cl}_X A$, construct $\{x^m\} \subset A$ by $x^m \in B_{\frac{1}{m}}(x) \cap A$, where $B_{\frac{1}{m}}(x) \cap A \neq \emptyset$ by part 1.

   Then $x^m \to x$.

   Conversely, let $\{x^m\} \subset A$ and $x^m \to x \in X$.

   For any $\varepsilon > 0$, there exists M such that $x^M \in B_\varepsilon(x)$, so that $B_\varepsilon(x) \cap A \neq \emptyset$.

   Hence, $x \in \mathrm{Cl}_X A$ by part 1.

Dense Sets

Definition 2.5

For $X \subset \mathbb{R}^N$,
$A \subset X$ is dense in X if $\mathrm{Cl}_X A = X$.

Proposition 2.10

For $X \subset \mathbb{R}^N$ and $A \subset X$, the following statements are equivalent:

1. *A is dense in X.*
2. *$\mathrm{Int}_X(X \setminus A) = \emptyset$.*
3. *$O \cap A \neq \emptyset$ for every nonempty open set $O \subset X$ relative to X.*

Proof

- By the definitions of interior and closure.

Compact Sets

- $A \subset \mathbb{R}^N$ is *bounded* if there exists $r \in \mathbb{R}$ such that $\|x\| < r$ for all $x \in A$.

Definition 2.6

$A \subset \mathbb{R}^N$ is compact if it is bounded and closed (relative to $\mathbb{R}^N$).

Examples:

- $[0, 1] \subset \mathbb{R}$ is compact.
- $[0, \infty) \subset \mathbb{R}$ is not compact.
- $(0, 1] \subset \mathbb{R}$ is not compact.

Sequential Compactness

Proposition 2.11

For $A \subset \mathbb{R}^N$, the following statements are equivalent:

1. *A is compact.*
2. *For every sequence $\{x^m\} \subset A$,*
   there exist a subsequence $\{x^{m(k)}\}$ of $\{x^m\}$ and a point $x \in A$
   such that $x^{m(k)} \to x$.

Proof

$2 \Rightarrow 1$ If A is not bounded, then for all $m \in \mathbb{N}$, there exists $x^m \in A$ such that $\|x^m\| \geq m$.

No subsequence of the sequence $\{x^m\} \subset A$ can be convergent ($\because \forall\, x \in \mathbb{R}^N \exists\, M \in \mathbb{N} : \|x^m - x\| \geq \|x^m\| - \|x\| \geq 1 \;\forall\, m \geq M$).

If A is not closed, then there exists $\bar{x} \notin A$ such that for all $m \in \mathbb{N}$, there exists $x^m \in B_{\frac{1}{m}}(\bar{x}) \cap A$.

The sequence $\{x^m\} \subset A$, and any subsequence, converges to $\bar{x} \notin A$.

Proof

$1 \Rightarrow 2$ Take any sequence $\{x^m\} \subset A$.

Suppose that A is bounded.

- Since $\{x_1^m\} \subset \mathbb{R}$ is bounded,
  there is a subsequence $\{x^{m_1(k)}\}$ of $\{x^m\}$ such that
  $\{x_1^{m_1(k)}\}$ is convergent.
- Since $\{x_2^{m_1(k)}\} \subset \mathbb{R}$ is bounded,
  there is a subsequence $\{x^{m_2(k)}\}$ of $\{x^{m_1(k)}\}$ such that
  $\{x_2^{m_2(k)}\}$ is convergent.
- $\dots$
- Since $\{x_N^{m_{N-1}(k)}\} \subset \mathbb{R}$ is bounded,
  there is a subsequence $\{x^{m_N(k)}\}$ of $\{x^{m_{N-1}(k)}\}$ such that
  $\{x_N^{m_N(k)}\}$ is convergent.

Thus, we have a convergent subsequence $\{x^{m_N(k)}\}$.

If in addition, A is closed, then its limit is contained in A.

Open Covers and Finite Intersections

Proposition 2.12

For $A \subset \mathbb{R}^N$, the following statements are equivalent:

1. A *is compact.*

2. *Any family $\mathcal{O}$ of open sets such that $A \subset \bigcup \mathcal{O}$ has a finite subset $\mathcal{O}'$ such that $A \subset \bigcup \mathcal{O}'$.*

   (I.e., Any open cover of A has a finite subcover.)

3. *For any family $\mathcal{C}$ of closed subsets of A that has the property that $\bigcap \mathcal{C}' \neq \emptyset$ for any finite subset $\mathcal{C}'$ of $\mathcal{C}$, we have $\bigcap \mathcal{C} \neq \emptyset$.*

(The property in 3 is called the *finite intersection property*.)

sup and inf

Proposition 2.13

Let A be a nonempty subset of $\mathbb{R}$.

- *If A is bounded, then $\sup A \in \operatorname{Cl} A$ and $\inf A \in \operatorname{Cl} A$.*
- *If in addition, A is closed, then $\sup A \in A$ and $\inf A \in A$.*

Thus, a nonempty compact subset of $\mathbb{R}$ has a maximum and a minimum.

Continuous Functions

Let X be a nonempty subset of $\mathbb{R}^N$.

Definition 2.7

- A function $f\colon X \to \mathbb{R}^K$ is continuous at $\bar{x} \in X$ if for any sequence $\{x^m\} \subset X$ such that $x^m \to \bar{x}$ as $m \to \infty$, we have $f(x^m) \to f(\bar{x})$ as $m \to \infty$

  (i.e., $\lim_{m\to\infty} f(x^m) = f(\lim_{m\to\infty} x^m)$).
- For $A \subset X$, $f\colon X \to \mathbb{R}^K$ is continuous on A if it is continuous at all $\bar{x} \in A$.
- $f\colon X \to \mathbb{R}^K$ is continuous if it is continuous on X.

Note:

- A function $f\colon X \to \mathbb{R}^K$ is continuous at $\bar{x} \in X$ if and only if each coordinate function f_k is continuous at $\bar{x}$.
  ($f\colon x \mapsto f(x) = (f_1(x), \dots, f_K(x)) \in \mathbb{R}^K$.)

Equivalent Definitions of Continuity

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.14

A function $f\colon X \to \mathbb{R}^K$ is continuous at $\bar{x} \in X$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - \bar{x}\| < \delta,\ x \in X \Longrightarrow \|f(x) - f(\bar{x})\| < \varepsilon.$$

The Limit of a Function

Let X be a nonempty subset of $\mathbb{R}^N$.

Definition 2.8

For a function $f\colon X \to \mathbb{R}^K$ and for $\bar{x} \in \operatorname{Cl} X$ and $\hat{y} \in \mathbb{R}^K$,
we write

$$\lim_{x \to \bar{x}} f(x) = \hat{y} \qquad \text{or} \qquad f(x) \to \hat{y} \text{ as } x \to \bar{x}$$

if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|x - \bar{x}\| < \delta,\ x \in X \implies \|f(x) - \hat{y}\| < \varepsilon.$$

Equivalent Definitions of Continuity

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.15

A function $f\colon X \to \mathbb{R}^K$ is continuous at $\bar{x} \in X$ if and only if

$$\lim_{x \to \bar{x}} f(x) = f(\bar{x}).$$

Equivalent Definitions of Continuity

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.16

A function $f\colon X \to \mathbb{R}^K$ is continuous at $\bar{x} \in X$ if and only if for any open neighborhood V of $f(\bar{x})$, there exists an open neighborhood U of $\bar{x}$ relative to X such that $f(U) \subset V$.

- $U \subset X$ is an *open neighborhood* of $\bar{x}$ relative to X if it is an open set relative to X such that $\bar{x} \in U$.
- For $A \subset X$,
  $f(A) = \{y \in \mathbb{R}^K \mid y = f(x) \text{ for some } x \in A\}$

  $\cdots$ the *image* of A under f.

Equivalent Definitions of Continuity

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.17

For a function $f: X \to \mathbb{R}^K$, the following statements are equivalent:

1. *f is continuous.*
2. *For any open set $O \subset \mathbb{R}^K$, $f^{-1}(O)$ is open relative to X.*
3. *For any closed set $C \subset \mathbb{R}^K$, $f^{-1}(C)$ is closed relative to X.*

- For $B \subset \mathbb{R}^K$,
  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$

  $\cdots$ the *inverse image* of B under f.

Equivalent Definitions of Continuity: $\mathbb{R}$-Valued Functions

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.18

For a function $f\colon X \to \mathbb{R}$, the following statements are equivalent:

1. *f is continuous.*
2. *For any open interval $I \subset \mathbb{R}$,*
   $\{x \in X \mid f(x) \in I\}$ is open relative to X.
3. *For all $c \in \mathbb{R}$,*
   $\{x \in X \mid f(x) > c\}$ and $\{x \in X \mid f(x) < c\}$ are open relative to X.
4. *For all $c \in \mathbb{R}$,*
   $\{x \in X \mid f(x) \geq c\}$ and $\{x \in X \mid f(x) \leq c\}$ are closed relative to X.

Intermediate Value Theorem

Proposition 2.19

Let $X \subset \mathbb{R}$ be a nonempty subset of $\mathbb{R}$,
and suppose that $[a, b] \subset X$, where $a < b$.
If a function $f\colon X \to \mathbb{R}$ is continuous on $[a, b]$ and $f(a) < f(b)$,
then for any $M \in (f(a), f(b))$, there exists $c \in (a, b)$ such that
$f(c) = M$.

Proof (1/2)

- Let $M \in (f(a), f(b))$, and let $A = \{x \in [a,b] \mid f(x) < M\}$.
- $A \neq \emptyset$ since $a \in A$. A is bounded above by b.
- Hence, $\sup A$ exists (where $\sup A \in \mathrm{Cl}\, A \subset [a,b]$).
  Let us denote it by c.

Proof (2/2)

- Since c is an upper bound of A,
  for all $m \in \mathbb{N}$, $c + \frac{1}{m} \notin A$, or $f(c + \frac{1}{m}) \geq M$.

  By continuity,
  $f(c) = f\left(\lim_{m\to\infty} c + \frac{1}{m}\right) = \lim_{m\to\infty} f(c + \frac{1}{m}) \geq M$.

- Since c is the least upper bound of A,
  for each $m \in \mathbb{N}$, there is some x^m such that
  $c - \frac{1}{m} < x^m \leq c$ ($\Rightarrow \lim_{m\to\infty} x^m = c$) and
  $x^m \in A$, or $f(x^m) < M$.

  By continuity,
  $f(c) = f\left(\lim_{m\to\infty} x^m\right) = \lim_{m\to\infty} f(x^m) \leq M$.

- Hence, $f(c) = M$.

Image of a Compact Set under a Continuous Function

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.20

Let $f\colon X \to \mathbb{R}^K$ be a continuous function.
If $A \subset X$ is compact, then $f(A)$ is compact.

- $f(A) = \{y \in \mathbb{R}^K \mid y = f(x) \text{ for some } x \in A\}$

  $\cdots$ the *image* of A under f.

Proof

- Take any sequence $\{y^m\} \subset f(A)$.

  We want to show that it has a convergent subsequence with a limit in $f(A)$.

- For each $m \in \mathbb{N}$, take an $x^m \in A$ such that $y^m = f(x^m)$.

- Since A is compact, $\{x^m\}$ has a convergent subsequence $\{x^{m(k)}\}$ with a limit $x \in A$.

- By the continuity of f,
  $\lim_{k\to\infty} f(x^{m(k)}) = f(\lim_{k\to\infty} x^{m(k)}) = f(x)$.

  That is, $y^{m(k)} \to f(x) \in f(A)$.

Extreme Value Theorem

Proposition 2.21

If $X \subset \mathbb{R}^N$ is a nonempty compact set and
$f : X \to \mathbb{R}$ is a continuous function,
then f has a maximizer and a minimizer, i.e.,
there exist $x^*, x^{**} \in X$ such that $f(x^{**}) \leq f(x) \leq f(x^*)\ \forall\, x \in X$.

Proof

- By the previous proposition, $f(X) \subset \mathbb{R}$ is compact.
- $\Rightarrow \sup f(X)$ and $\inf f(X)$ exist, and $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$ by Proposition 2.13.
- That is, there exist $x^*, x^{**} \in X$ such that $f(x^*) = \sup f(X)$ and $f(x^{**}) = \inf f(X)$.

$\limsup$ and $\liminf$

Definition 2.9

For a sequence $\{x^m\}$ in $\mathbb{R}$:

$$\limsup_{m\to\infty} x^m = \lim_{m\to\infty} \sup_{n\geq m} x^n, \qquad \liminf_{m\to\infty} x^m = \lim_{m\to\infty} \inf_{n\geq m} x^n.$$

- $\limsup \cdots$ "limit supremum" or "limit superior"
  $\liminf \cdots$ "limit infimum" or "limit inferior"
- $\limsup_{m\to\infty} x^m$ and $\liminf_{m\to\infty} x^m$ always exist,
  if we allow a limit to be ∞ or $-\infty$.
- $\lim_{m\to\infty} x^m$ exists if and only if
  $\liminf_{m\to\infty} x^m = \limsup_{m\to\infty} x^m$,
  in which case the three terms coincide.

Semi-Continuous Functions

Let X be a nonempty subset of $\mathbb{R}^N$.

Definition 2.10

- A function $f\colon X \to \mathbb{R}$ is upper semi-continuous at $\bar{x} \in X$ if for any sequence $\{x^m\} \subset X$ such that $x^m \to \bar{x}$ as $m \to \infty$, we have $\limsup_{m\to\infty} f(x^m) \leq f(\bar{x})$.
- A function $f\colon X \to \mathbb{R}$ is lower semi-continuous at $\bar{x} \in X$ if for any sequence $\{x^m\} \subset X$ such that $x^m \to \bar{x}$ as $m \to \infty$, we have $\liminf_{m\to\infty} f(x^m) \geq f(\bar{x})$.
- For $A \subset X$, f is upper (lower) semi-continuous on A if it is upper (lower) semi-continuous at all $\bar{x} \in A$.
- f is upper (lower) semi-continuous if it is upper (lower) semi-continuous on X.

Semi-Continuous Functions

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.22

$f\colon X \to \mathbb{R}$ is continuous at $\bar{x}$
if and only if it is upper and lower semi-continuous at $\bar{x}$.

Semi-Continuous Functions

- A continuous function has no jump.
- An upper semi-continuous function
  - may have a downward jump, but
  - may not have an upward jump.
- A lower semi-continuous function
  - may have an upward jump, but
  - may not have a downward jump.

Equivalent Definitions of Upper Semi-Continuity

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.23

For a function $f: X \to \mathbb{R}$, the following statements are equivalent:

1. *f is upper semi-continuous at $\bar{x} \in X$.*
2. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|x - \bar{x}\| < \delta,\ x \in X \Longrightarrow f(\bar{x}) > f(x) - \varepsilon.$$

Proposition 2.24

For a function $f: X \to \mathbb{R}$, the following statements are equivalent:

1. *f is upper semi-continuous.*
2. *For all $c \in \mathbb{R}$, $\{x \in X \mid f(x) < c\}$ is open relative to X.*
3. *For all $c \in \mathbb{R}$, $\{x \in X \mid f(x) \geq c\}$ is closed relative to X.*

Equivalent Definitions of Lower Semi-Continuity

Let X be a nonempty subset of $\mathbb{R}^N$.

Proposition 2.25

For a function $f: X \to \mathbb{R}$, the following statements are equivalent:

1. *f is lower semi-continuous at $\bar{x} \in X$.*
2. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|x - \bar{x}\| < \delta,\ x \in X \Longrightarrow f(\bar{x}) < f(x) + \varepsilon.$$

Proposition 2.26

For a function $f: X \to \mathbb{R}$, the following statements are equivalent:

1. *f is lower semi-continuous.*
2. *For all $c \in \mathbb{R}$, $\{x \in X \mid f(x) > c\}$ is open relative to X.*
3. *For all $c \in \mathbb{R}$, $\{x \in X \mid f(x) \leq c\}$ is closed relative to X.*

Extreme Value Theorem for Semi-Continuous Functions

Proposition 2.27

Let $X \subset \mathbb{R}^N$ be a nonempty compact set, and let $f \colon X \to \mathbb{R}$.

1. *If f is upper semi-continuous, then f has a maximizer.*
2. *If f is lower semi-continuous, then f has a minimizer.*