4. Convex Sets and (Quasi-)Concave Functions

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Convex Sets

Definition 4.1

$A \subset \mathbb{R}^N$ is *convex* if

$$(1-\alpha)x + \alpha x' \in A$$

whenever $x, x' \in A$ and $\alpha \in [0,1]$.

Convex Combinations

- For $\alpha_1, \ldots, \alpha_M \geq 0$, $\sum_{m=1}^{M} \alpha_m = 1$, $\alpha_1 x^1 + \cdots + \alpha_M x^M$ is called a *convex combination* of $x^1, \ldots, x^M$.

Proposition 4.1

If $A \subset \mathbb{R}^N$ is convex, then
any convex combination of elements in A is contained in A.

Proof

By induction.

Convex Hull

Proposition 4.2

For any index set Λ*,*
if $C_\lambda \subset \mathbb{R}^N$ *is convex for all* $\lambda \in \Lambda$*, then* $\bigcap_{\lambda \in \Lambda} C_\lambda$ *is convex.*
(The intersection of any family of convex sets is convex.)

Definition 4.2

For $A \subset \mathbb{R}^N$, the *convex hull* of A, denoted $\mathrm{Co}\, A$, is the intersection of all convex sets that contain A (or, the smallest convex set that contains A).

Proposition 4.3

For $A \subset \mathbb{R}^N$*,* $\operatorname{Co} A$ *equals the set of all convex combinations of elements in* A*, i.e.,*

$$\operatorname{Co} A = \left\{ x \in \mathbb{R}^N \;\middle|\; x = \sum_{m=1}^{M} \alpha_m x^m \text{ for some } M \in \mathbb{N},\ x^1, \ldots, x^M \in A, \text{ and } \alpha_1, \ldots, \alpha_M \geq 0 \text{ with } \sum_{m=1}^{M} \alpha_m = 1 \right\}.$$

Proposition 4.4

Let $A, B \subset \mathbb{R}^N$*.*

1. $A \subset \operatorname{Co} A$*.*
2. *If* $A \subset B$*, then* $\operatorname{Co} A \subset \operatorname{Co} B$*.*
3. $\operatorname{Co} \operatorname{Co} A = \operatorname{Co} A$*.*

Algebra of Convex Sets

Proposition 4.5

1. *If* $A, B \subset \mathbb{R}^N$ *are convex, then* $A + B = \{x \in \mathbb{R}^N \mid x = a + b$ *for some* $a \in A$ *and* $b \in B\}$ *is convex.*

2. *If* $A \subset \mathbb{R}^N$ *is convex, then for* $t \in \mathbb{R}$*,* $tA = \{x \in \mathbb{R}^N \mid x = ta$ *for some* $a \in A\}$ *is convex.*

Proposition 4.6

For $A_1, \ldots, A_M \subset \mathbb{R}^N$, $\mathrm{Co} \sum_{m=1}^{M} A_m = \sum_{m=1}^{M} \mathrm{Co}\, A_m.$

Proof

- (LHS) $\subset$ (RHS): Exercise.
- (LHS) $\supset$ (RHS): Sufficient to show for $M = 2$:

  If $x \in \mathrm{Co}\, A_1 + \mathrm{Co}\, A_2$, then for some $y^1, \ldots, y^I \in A_1$ and $z^1, \ldots, z^J \in A_2$, we have

  $$x = \textstyle\sum_i \alpha_i y^i + \sum_j \beta_j z^j = \sum_i \alpha_i \sum_j \beta_j (y^i + z^j),$$

  where $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_i \alpha_i = \sum_j \beta_j = 1$.

  This implies that $x \in \mathrm{Co}\, \mathrm{Co}(A_1 + A_2) = \mathrm{Co}(A_1 + A_2)$.

Convex Cones

Definition 4.3

- $A \subset \mathbb{R}^N$ is a *cone* if

  $$x \in A \Rightarrow \alpha x \in A$$

  for any $\alpha \geq 0$.
- $A \subset \mathbb{R}^N$ is a *convex cone* if

  $$x, y \in A \Rightarrow \alpha x + \beta y \in A$$

  for any $\alpha, \beta \geq 0$.

(Some textbooks define with "for any $\alpha > 0$" and "for any $\alpha, \beta > 0$".)

Carathéodory's Theorem

- For $\alpha_1, \ldots, \alpha_M \geq 0$, $\alpha_1 x^1 + \cdots + \alpha_M x^M$ is called a *conic combination* of $x^1, \ldots, x^M$.
- For $A \subset \mathbb{R}^N$, the *conic hull* of A, denoted $\mathrm{Cone}\, A$, is the set of all conic combinations of elements of A;
- or equivalently, the smallest convex cone that contains A.

Proposition 4.7 (Carathéodory's Theorem)

1. *For $A \subset \mathbb{R}^N$, $A \neq \{0\}$, each $x \in \mathrm{Cone}\, A$ is written as a conic combination of linearly independent elements of A.*
2. *For $A \subset \mathbb{R}^N$, each $x \in \mathrm{Co}\, A$ is written as a convex combination of at most $N + 1$ elements in A.*

Proof

1.

- Let $x \in \operatorname{Cone} A$.

  Let M be the smallest integer such that x is written in the form of

  $$x = \alpha_1 x^1 + \cdots + \alpha_M x^M, \tag{1}$$

  where $x^1, \ldots, x^M \in A$ and $\alpha_1, \ldots, \alpha_M > 0$.

- Suppose that $x^1, \ldots, x^M$ are linearly dependent, so that

  $$c_1 x^1 + \cdots + c_M x^M = 0 \tag{2}$$

  for some $(c_1, \ldots, c_M) \neq (0, \ldots, 0)$.

  Assume that $c_m > 0$ for some m
  (if $c_m \leq 0$ for all m, then multiply both sides by -1).

Proof

- Let $\mu = \min\left\{\frac{\alpha_m}{c_m} \mid c_m > 0\right\} > 0$.
- By (1) and (2) we have

$$x = (\alpha_1 - \mu c_1)x^1 + \cdots + (\alpha_M - \mu c_M)x^M,$$

  where

  - $\alpha_m - \mu c_m \geq 0$ for all m, and
  - $\alpha_m - \mu c_m = 0$ for some m.

- Thus x has been written as a conic combination of $M - 1$ (or fewer) elements of A.

  This contradicts the minimality of M.

Proof

2.

- For $A \subset \mathbb{R}^N$, let $x \in \operatorname{Co} A$.

  Then we have $x = \alpha_1 x^1 + \ldots + \alpha_J x^J$
  for some $x^1, \ldots, x^J \in A$ and some $\alpha_1, \ldots, \alpha_J \geq 0$ with
  $\alpha_1 + \cdots + \alpha_J = 1$.

- Consider $B = \{(x^1, 1), \ldots, (x^J, 1)\} \subset \mathbb{R}^{N+1}$.

  Then $(x, 1) \in \operatorname{Cone} B$.

- By part 1, there are linearly independent elements
  $\{(x^{j_1}, 1), \ldots, (x^{j_K}, 1)\}$ from B such that
  $(x, 1) = \beta_1 (x^{j_1}, 1) + \cdots + \beta_K (x^{j_K}, 1)$, where $\beta_k \geq 0$ and
  $K \leq N + 1$.

- From the 1st through Nth coordinates we have
  $x = \beta_1 x^{j_1} + \cdots + \beta_K x^{j_K}$, while from the $(N+1)$st
  coordinate we have $\beta_1 + \cdots + \beta_K = 1$.

A General Form of Carathéodory's Theorem

Lemma 4.8

Let $A_1, \ldots, A_I \subset \mathbb{R}^N$*.*
If $x \in \mathrm{Co} \sum_{i=1}^{I} A_i$*, then there exist* $x^{ij} \in A_i$*,* $i = 1, \ldots, I$*,* $j = 1, \ldots, K_i$*, where* $K_i \geq 1$*, such that*

$$x \in \sum_{i=1}^{I} \mathrm{Co}\{x^{i1}, \ldots, x^{iK_i}\}$$

and

$$\sum_{i=1}^{I} K_i \leq N + I.$$

Proof

- Let $A_1, \dots, A_I \subset \mathbb{R}^N$, and let $x \in \mathrm{Co} \sum_{i=1}^I A_i$.

  Since $\mathrm{Co} \sum_{i=1}^I A_i = \sum_{i=1}^I \mathrm{Co}\, A_i$ by Proposition 4.6, x is written as $x = \sum_{i=1}^I y^i$ for some $y^i \in A_i$, $i = 1, \dots, I$, where each y^i is written as $y^i = \sum_{j=1}^{J_i} \alpha_{ij} y^{ij}$ for some $y^{ij} \in A_i$ and $\alpha_{ij} \geq 0$, $j = 1, \dots, J_i$, with $\sum_{j=1}^{J_i} \alpha_{ij} = 1$.
- Consider the following vectors in $\mathbb{R}^{N+I}$:

$$
\begin{aligned}
z &= (x, 1, 1, \dots, 1, 1), \\
z^{1j} &= (y^{1j}, 1, 0, \dots, 0, 0),\ j = 1, \dots, J_1, \\
z^{2j} &= (y^{2j}, 0, 1, \dots, 0, 0),\ j = 1, \dots, J_2, \\
&\vdots \\
z^{Ij} &= (y^{Ij}, 0, 0, \dots, 0, 1),\ j = 1, \dots, J_I.
\end{aligned}
$$

- By construction, z is written as a conic combination of z^{ij}'s: $z = \sum_{i=1}^I \sum_{j=1}^{J_i} \alpha_{ij} z^{ij}$.

Proof

- By the cone version of Carathéodory's Theorem (Proposition 4.7(1)), there are at most $N + I$ linearly independent elements of $\{z^{ij},\ i = 1, \dots, I,\ j = 1, \dots, J_i\}$ such that z is written as a conic combination of them:

  i.e., there exist $\beta_{ij} \geq 0$, $i = 1, \dots, I$, $j = 1, \dots, J_i$, such that $z = \sum_{i=1}^{I} \sum_{j=1}^{J_i} \beta_{ij} z^{ij}$ and $\sum_{i=1}^{I} |\{j = 1, \dots, J_i \mid \beta_{ij} > 0\}| \leq N + I$.

- From the 1st through Nth coordinates we have $x = \sum_{i=1}^{I} \sum_{j=1}^{J_i} \beta_{ij} y^{ij}$.

- From the $(N + 1)$st through $(N + I)$th coordinates we have $\sum_{j=1}^{J_i} \beta_{ij} = 1$, $i = 1, \dots, I$, where $\beta_{ij} > 0$ for at least one j.

Shapley-Folkman Theorem

Proposition 4.9

Let $A_1, \ldots, A_I \subset \mathbb{R}^N$*.*
If $x \in \mathrm{Co} \sum_{i=1}^{I} A_i$*, then*

$$x \in \sum_{i \in \mathcal{I}'} A_i + \sum_{i \in \{1,\ldots,I\} \setminus \mathcal{I}'} \mathrm{Co}\, A_i$$

for some $\mathcal{I}' \subset \{1, \ldots, I\}$ *with* $|\mathcal{I}'| \geq I - N$*.*

(See Kreps, Chapter 13 for an application of this theorem.)

Proof

- Let $x \in \mathrm{Co} \sum_{i=1}^{I} A_i$.
- Then by Lemma 4.8, there exist $x^{ij} \in A_i$, $i = 1, \ldots, I$, $j = 1, \ldots, K_i$, where $K_i \geq 1$, such that
  - $x \in \sum_{i=1}^{I} \mathrm{Co}\{x^{i1}, \ldots, x^{iK_i}\}$, and
  - $\sum_{i=1}^{I} K_i \leq N + I$.
- Let $\mathcal{I}' = \{i = 1, \ldots, I \mid K_i = 1\}$, and let $|\mathcal{I}'| = n$.
- Then $\sum_{i=1}^{I} K_i \geq n + 2(I - n) = 2I - n$.
- With $\sum_{i=1}^{I} K_i \leq N + I$, this implies that $n \geq I - N$.

Topological Properties of Convex Sets

Proposition 4.10

If $A \subset \mathbb{R}^N$ *is open, then* $\operatorname{Co} A$ *is open.*

Proposition 4.11

If $A \subset \mathbb{R}^N$ *is convex, then* $\operatorname{Int} A$ *is convex.*

Proposition 4.12

If $A \subset \mathbb{R}^N$ *is convex, then* $\operatorname{Cl} A$ *is convex.*

Proof

- $\operatorname{Cl} A = \bigcap_{\varepsilon>0}(A + B_\varepsilon(0))$.

Topological Properties of Convex Sets

Fact 1

Let $A \subset \mathbb{R}^N$ be a convex set.
If $\mathrm{Int}(\mathrm{Cl}\, A) \neq \emptyset$, then $\mathrm{Int}\, A \neq \emptyset$.

Proposition 4.13

Let $A \subset \mathbb{R}^N$ *be a convex set.*
Then $\mathrm{Int}(\mathrm{Cl}\, A) = \mathrm{Int}\, A$.

- See "Topological Properties of Convex Sets".

Topological Properties of Convex Sets

Proposition 4.14

If $A \subset \mathbb{R}^N$ *is bounded, then* $\mathrm{Cl}(\mathrm{Co}\, A) = \mathrm{Co}(\mathrm{Cl}\, A)$*.*
In particular, if A *is compact, then* $\mathrm{Co}\, A$ *is compact.*

Proof

- Since $\mathrm{Co}\, A \supset A$, we have $\mathrm{Cl}(\mathrm{Co}\, A) \supset \mathrm{Cl}\, A$.

  Since $\mathrm{Cl}(\mathrm{Co}\, A)$ is convex (Proposition 4.12), we have $\mathrm{Cl}(\mathrm{Co}\, A) \supset \mathrm{Co}(\mathrm{Cl}\, A)$.

- Since $A \subset \mathrm{Cl}\, A$, we have $\mathrm{Co}\, A \subset \mathrm{Co}(\mathrm{Cl}\, A)$.

  We want to show that $\mathrm{Co}(\mathrm{Cl}\, A)$ is closed if A is bounded.

- Let $\{x^m\} \subset \mathrm{Co}(\mathrm{Cl}\, A)$, and assume $x^m \to \bar{x}$.

- By Carathéodory's Theorem (Proposition 4.7(2)), each x^m is written as

$$x^m = \alpha_1^m x^{m,1} + \cdots + \alpha_{N+1}^m x^{m,N+1},$$

  where

  - $(\alpha_1^m, \ldots, \alpha_{N+1}^m) \in \Delta = \{\alpha \in \mathbb{R}^{N+1} \mid \alpha_n \geq 0,\ \sum_n \alpha_n = 1\}$,
  - $x^{m,1}, \ldots, x^{m,N+1} \in \mathrm{Cl}\, A$.

Proof

- Since Δ and $\mathrm{Cl}\,A$ are compact, there exists a sequence $\{m(k)\}$ such that the limits $\bar{\alpha}_n = \lim_{k\to\infty} \alpha_n^{m(k)}$ and $\bar{x}^n = \lim_{k\to\infty} x^{m(k),n}$ exist where $(\bar{\alpha}_1, \ldots, \bar{\alpha}_{N+1}) \in \Delta$ and $\bar{x}^1, \ldots, \bar{x}^{N+1} \in \mathrm{Cl}\,A$.
- Hence,

$$\bar{x} = \bar{\alpha}_1 \bar{x}^1 + \cdots + \bar{\alpha}_{N+1} \bar{x}^{N+1},$$

so that $\bar{x} \in \mathrm{Co}(\mathrm{Cl}\,A)$.

Concave Functions

Definition 4.4

Let $X \subset \mathbb{R}^N$ be a non-empty convex set.

- A function $f\colon X \to \mathbb{R}$ is *concave* if

$$f((1-\alpha)x + \alpha x') \geq (1-\alpha)f(x) + \alpha f(x')$$

  for all $x, x' \in X$ and all $\alpha \in [0,1]$.

- $f\colon X \to \mathbb{R}$ is *strictly concave* if

$$f((1-\alpha)x + \alpha x') > (1-\alpha)f(x) + \alpha f(x')$$

  for all $x, x' \in X$ with $x \neq x'$ and all $\alpha \in (0,1)$.

- $f\colon X \to \mathbb{R}$ is *convex* (*strictly convex*, resp.) if $-f$ is concave (strictly concave, resp.).

Hypograph and Epigraph

Let $X \subset \mathbb{R}^N$ be a non-empty set.

- The *hypograph* of a function $f\colon X \to \mathbb{R}$ is the set

$$\operatorname{hyp} f = \{(x, y) \in \mathbb{R}^N \times \mathbb{R} \mid x \in X,\ y \leq f(x)\}.$$

- The *epigraph* of a function $f\colon X \to \mathbb{R}$ is the set

$$\operatorname{epi} f = \{(x, y) \in \mathbb{R}^N \times \mathbb{R} \mid x \in X,\ y \geq f(x)\}.$$

Proposition 4.15

Let $X \subset \mathbb{R}^N$ be a nonempty convex set.
$f\colon X \to \mathbb{R}$ is a concave (convex, resp.) function if and only if $\operatorname{hyp} f$ ($\operatorname{epi} f$, resp.) is a convex set.

Jensen's Inequality

Proposition 4.16

Let $X \subset \mathbb{R}^N$ *be a nonempty convex set.*
If $f \colon X \to \mathbb{R}$ *is concave, then*

$$f(\alpha_1 x^1 + \cdots + \alpha_M x^M) \geq \alpha_1 f(x^1) + \cdots + \alpha_M f(x^M)$$

for any $x^1, \ldots, x^M \in X$ *and* $\alpha_1, \ldots, \alpha_M \geq 0$ *with* $\sum_{m=1}^{M} \alpha_m = 1$.

Proposition 4.17

Let $I \subset \mathbb{R}$ *be a nonempty closed interval.*
If $f \colon I \to \mathbb{R}$ *is concave, then*

$$f\left(\int x \, dF(x)\right) \geq \int f(x) \, dF(x)$$

for any distribution function F *on* I.

Properties of Concave Functions

Let $X \subset \mathbb{R}^N$ be a non-empty convex set.

Lemma 4.18

$f\colon X \to \mathbb{R}$ is (strictly) concave if and only if for any $x \in X$ and any $z \in \mathbb{R}^N$ with $x + z \in X$, for $t \in (0, 1]$,

$$\frac{f(x + tz) - f(x)}{t}$$

is nonincreasing (strictly decreasing) in t.

Proof

If $t' < t$ with $t' = \alpha t$, $\alpha \in (0, 1)$, then we have

$$f(x + t'z) \geq (1 - \alpha) f(x) + \alpha f(x + tz)$$
$$\iff \frac{f(x + t'z) - f(x)}{\alpha t} \geq \frac{f(x + tz) - f(x)}{t}.$$

Continuity of Concave Functions

Let $X \subset \mathbb{R}^N$ be a non-empty convex set.

Lemma 4.19

Let $f\colon X \to \mathbb{R}$ be a concave function. If $\bar{x} \in \operatorname{Int} X$, then there exist $\varepsilon > 0$ and M such that $|f(x)| \leq M$ for all $x \in B_\varepsilon(\bar{x})$.

Proposition 4.20

A concave function $f\colon X \to \mathbb{R}$ is continuous on $\operatorname{Int} X$.

Proof of Lemma 4.19

- Let $\bar{x} \in \operatorname{Int} X$.

  Let $\delta > 0$ be such that $\bar{B}_\delta(\bar{x}) \subset X$, and let $\varepsilon = \delta/\sqrt{N}$.
- Let $S = \{x \in \mathbb{R}^N \mid \|x - \bar{x}\|_\infty \leq \varepsilon\} \subset \bar{B}_\delta(\bar{x})$.

  Let $v^1, \dots, v^m$ be the $m = 2^N$ vertices of S (so that $S = \operatorname{Co}\{v^1, \dots, v^m\}$).
- Let $L = \min\{f(v^1), \dots, f(v^m)\}$.

  Then $f(x) \geq L$ for all $x \in S$ by the concavity of f.
- Take any $x \in B_\varepsilon(\bar{x})$, and let $y \in B_\varepsilon(\bar{x})$ be such that $\bar{x} = \frac{1}{2}x + \frac{1}{2}y$.
- Since $f(\bar{x}) \geq \frac{1}{2}f(x) + \frac{1}{2}f(y)$, we have $f(x) \leq 2f(\bar{x}) - f(y) \leq 2f(\bar{x}) - L$.
- Finally, let $M = \max\{|L|, |2f(\bar{x}) - L|\}$.

Proof of Proposition 4.20

- Let $\bar{x} \in \operatorname{Int} X$.

  By Lemma 4.19, we can take $r > 0$ and M such that $|f(x)| \leq M$ for all $x \in B_{2r}(\bar{x})$.

- Take any $x, y \in B_r(\bar{x})$.

  We want to show that $|f(y) - f(x)| \leq \frac{2M}{r}\|y - x\|$.

- Let $z = x + \frac{\|y-x\|+r}{\|y-x\|}(y - x)$.

  Then $z \in B_{2r}(\bar{x})$.

- Then we have

$$
\begin{aligned}
f(y) - f(x) &\geq \frac{\|y-x\|}{\|y-x\|+r}(f(z) - f(x)) \quad \text{(by Lemma 4.18)} \\
&\geq -\frac{\|y-x\|}{\|y-x\|+r}|f(z) - f(x)| \\
&\geq -\frac{\|y-x\|}{r}|f(z) - f(x)| \\
&\geq -\frac{\|y-x\|}{r}(|f(z)| + |f(x)|) \\
&\geq -\frac{\|y-x\|}{r} \times 2M.
\end{aligned}
$$

- By a symmetric argument, we have

$$
f(x) - f(y) \geq -\frac{\|x-y\|}{r} \times 2M.
$$

Extended Real Valued Functions

Let $X \subset \mathbb{R}^N$ be a nonempty convex set.

Definition 4.5

A function $f\colon X \to (-\infty, \infty]$ is defined to be convex if

$$f((1-\alpha)x + \alpha x') \leq (1-\alpha)f(x) + \alpha f(x')$$

for all $x, x' \in X$ and all $\alpha \in [0,1]$, where

- $\alpha \times \infty = \infty$ if $\alpha > 0$,
- $0 \times \infty = 0$,
- $\infty + y = y + \infty = \infty$ for $y \in (-\infty, \infty]$, and
- $y \leq \infty$ for $y \in (-\infty, \infty]$.

(Concavity of a function $f\colon X \to [-\infty, \infty)$ is defined analogously.)

Extended Real Valued Functions

Let $X \subset \mathbb{R}^N$ be a nonempty convex set.

Proposition 4.21

A function $f\colon X \to (-\infty, \infty]$ *is convex if and only if* $\operatorname{epi} f$ *is a convex set.*

- Any convex function $f\colon X \to \mathbb{R}$ can be extended to $\mathbb{R}^N$ keeping convexity, by assigning ∞ to $x \notin X$.

Convex Optimal Value Functions

Let $X \subset \mathbb{R}^N$ be a nonempty set, and let $P \subset \mathbb{R}^M$ be a nonempty convex set.

Proposition 4.22

Consider a function $f\colon X \times P \to \mathbb{R}$.
If for all $x \in X$, $f(x,p)$ is convex in p,
then the function $v\colon P \to (-\infty, \infty]$ defined by

$$v(p) = \sup_{x \in X} f(x,p)$$

is convex.

Proof

Show that $\operatorname{epi} v$ is a convex set. ($\to$ Homework)

Support Functions

- For a nonempty set $A \subset \mathbb{R}^N$,
  the function $\phi_A \colon \mathbb{R}^N \to (-\infty, \infty]$ defined by

  $$\phi_A(p) = \sup_{x \in A} p \cdot x$$

  is called the *support function* of A.
- The *profit function* is the support function of the production set (but only defined for nonnegative/positive price vectors).
- The *cost function* is the "concave support function" of the input requirement set (Section 5.C), which is defined with "inf" in place of "sup".
- The *expenditure function* is the "concave support function" of the upper utility level set (Section 3.E).

Support Functions

Proposition 4.23

The support function $\phi_A \colon \mathbb{R}^N \to (-\infty, \infty]$ *is*

1. *convex and*
2. *homogeneous of degree one,*
   i.e., for all $p \in \mathbb{R}^N$, $\phi_A(tp) = t\phi_A(p)$ *for all* $t > 0$.

Proof

1. By Proposition 4.22.
2.
   - For all $x \in A$, $(tp) \cdot x \leq t \sup_{x' \in A} p \cdot x'$, so $\sup_{x \in A} (tp) \cdot x \leq t \sup_{x' \in A} p \cdot x'$.
   - For all $x \in A$, $\sup_{x' \in A} (tp) \cdot x' \geq t(p \cdot x)$, so $(1/t) \sup_{x' \in A} (tp) \cdot x' \geq \sup_{x \in A} p \cdot x$.

Quasi-Concave Functions

Definition 4.6

Let $X \subset \mathbb{R}^N$ be a non-empty convex set.

- $f: X \to \mathbb{R}$ is *quasi-concave* if
  $f((1-\alpha)x + \alpha x') \geq f(x)$
  for all $x, x' \in X$ such that $f(x') \geq f(x)$ and all $\alpha \in [0, 1]$.

- $f: X \to \mathbb{R}$ is *strictly quasi-concave* if
  $f((1-\alpha)x + \alpha x') > f(x)$
  for all $x, x' \in X$ with $x \neq x'$ such that $f(x') \geq f(x)$ and all $\alpha \in (0, 1)$.

- $f: X \to \mathbb{R}$ is *semi-strictly quasi-concave* if
  $f((1-\alpha)x + \alpha x') > f(x)$
  for all $x, x' \in X$ such that $f(x') > f(x)$ and all $\alpha \in (0, 1)$.

- f is quasi-/strictly quasi-/semi-strictly quasi-convex if $-f$ is quasi-/strictly quasi-/semi–strictly quasi-concave.

Equivalent Definition

Proposition 4.24

$f\colon X \to \mathbb{R}$ is quasi-concave if and only if $\{x \in X \mid f(x) \geq t\}$ is convex for all $t \in \mathbb{R}$.

Properties of Quasi-Concave Functions

Let $X \subset \mathbb{R}^N$ be a non-empty convex set.

Proposition 4.25

If $f\colon X \to \mathbb{R}$ is quasi-concave (strictly quasi-concave) and $h\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing (strictly increasing), then $h \circ f$ is quasi-concave (strictly quasi-concave).

Properties of Quasi-Concave Functions

Let $X \subset \mathbb{R}^N$ be a non-empty convex set.

For $f \colon X \to \mathbb{R}$, write $X^* = \{x \in X \mid f(x) = \sup_{x' \in X} f(x')\}$.

Proposition 4.26

1. *If f is quasi-concave, then X^* is a convex set.*
2. *If f is strictly quasi-concave, then X^* is either empty or a singleton set.*