6. Negative (Semi-)Definite Matrices

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Some Facts from Linear Algebra

Let $M \in \mathbb{R}^{N \times N}$.

- M is said to be *nonsingular* if there exists $A \in \mathbb{R}^{N \times N}$ such that $MA = AM = I$.

  In this case, A is called the *inverse matrix* of M and denoted by M^{-1}.

- The following are equivalent:
  - M is nonsingular.
  - $\operatorname{rank} M = N$.
  - $|M| \neq 0$.
  - $\{z \in \mathbb{R}^N \mid Mz = 0\} = \{0\}$.
  - 0 is not a characteristic root of M.

Some Facts from Linear Algebra

Let $M \in \mathbb{R}^{N \times N}$.

- The equation in λ,

  $$|M - \lambda I| = 0,$$

  is called the *characteristic equation* of M.

- The characteristic equation of M has N solutions in $\mathbb{C}$ (counted with multiplicity).

- The solutions to the characteristic equation of M are called the *characteristic roots* of M.

- If $\lambda_1, \ldots, \lambda_N$ are the characteristic roots of M, then $|M| = \prod_{n=1}^{N} \lambda_n$.

- If M is nonsingular and $\lambda_1, \ldots, \lambda_N$ are its characteristic roots, then $\lambda_1^{-1}, \ldots, \lambda_N^{-1}$ are the characteristic roots of M^{-1}.

Some Facts from Linear Algebra

Let $M \in \mathbb{R}^{N \times N}$.

- $\lambda \in \mathbb{C}$ is an *eigenvalue* of M if there exists $z \in \mathbb{C}^N$ with $z \neq 0$ such that

  $$Mz = \lambda z.$$

  In this case, z is called an *eigenvector* of M that corresponds (or belongs) to λ.

- λ is an eigenvalue of M if and only if it is a characteristic root of M.

Some Facts from Linear Algebra

Let $M \in \mathbb{R}^{N \times N}$ be a symmetric matrix.

- All the eigenvalues (hence characteristic roots) of M are real.
- Each eigenvalue of M has real eigenvectors.
- $\exists\, U \in \mathbb{R}^{N \times N}$ orthogonal (i.e., $U^{\mathrm{T}}U = UU^{\mathrm{T}} = I$) such that

$$U^{\mathrm{T}} M U = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_N \end{pmatrix} \quad (= \mathrm{diag}(\lambda_1, \ldots, \lambda_N)),$$

  where $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ are the eigenvalues of M.
- If M is nonsingular, then M^{-1} is symmetric.

Negative (Semi-)Definite Matrices

Definition 6.1

- $M \in \mathbb{R}^{N \times N}$ is *negative semi-definite* if

  $$z \cdot Mz \leq 0$$

  for all $z \in \mathbb{R}^N$.

- $M \in \mathbb{R}^{N \times N}$ is *negative definite* if

  $$z \cdot Mz < 0$$

  for all $z \in \mathbb{R}^N$ with $z \neq 0$.

- $M \in \mathbb{R}^{N \times N}$ is *positive definite* (*positive semi-definite*, resp.) if $-M$ is negative definite (negative semi-definite, resp.).

Remark

- In many math books,
  negative definiteness is defined only for symmetric matrices, or for quadratic forms $\sum_{i,j=1}^{N} a_{ij} z_i z_j$.

  (Any quadratic form is written as $z \cdot Mz$ for some symmetric M.)

- Sometimes, matrices (not necessarily symmetric) that are negative definite in our sense are called negative quasi-definite.

Example: Negative (Semi-)Definiteness of Jacobi Matrices

Let $X \subset \mathbb{R}^N$ be a non-empty open convex set.

Suppose that $f\colon X \to \mathbb{R}^N$ is differentiable.

1. $(y - x) \cdot (f(y) - f(x)) \leq 0$ for all $x, y \in X$ if and only if $Df(x)$ is negative semi-definite for all $x \in X$.
2. If $Df(x)$ is negative definite for all $x \in X$, then $(y - x) \cdot (f(y) - f(x)) < 0$ for all $x, y \in X$, $x \neq y$.

- For $N = 1$, "$(y - x) \cdot (f(y) - f(x)) \leq 0$ (< 0) for all $x, y \in X$" implies that f is nonincreasing (strictly decreasing).
- Cf. Proposition 5.20.

Example: Negative (Semi-)Definiteness of Hesse Matrices

Let $X \subset \mathbb{R}^N$ be a non-empty open convex set.

Suppose that $f\colon X \to \mathbb{R}$ is differentiable and ∇f is differentiable.

1. f is concave if and only if
   $D^2 f(x)$ is negative semi-definite for all $x \in X$.
2. If $D^2 f(x)$ is negative definite for all $x \in X$, then
   f is strictly concave.

- Proposition 5.21.

Characterizations of Negative (Semi-)Definiteness

Proposition 6.1

Let $M \in \mathbb{R}^{N \times N}$*.*

1. M *is negative definite*
   $\iff M + M^{\mathrm{T}}$ *is negative definite.*
2. *Suppose that* M *is symmetric.*
   M *is negative definite*
   $\iff$ *all the characteristic roots of* M *are negative.*
3. M *is negative definite*
   $\implies M$ *is nonsingular and* M^{-1} *is negative definite.*

Proof

1. For any $z \in \mathbb{R}^N$, $z^\mathrm{T}(M + M^\mathrm{T})z = 2z^\mathrm{T}Mz$.
2. Since $M = U^\mathrm{T}\operatorname{diag}(\lambda_1, \dots, \lambda_N)U$ for some U orthogonal (hence nonsingular),

$$\begin{aligned}
&z^\mathrm{T}Mz < 0 \text{ for all } z \in \mathbb{R}^N \setminus \{0\} \\
&\iff (Uz)^\mathrm{T}\operatorname{diag}(\lambda_1, \dots, \lambda_N)(Uz) < 0 \text{ for all } z \in \mathbb{R}^N \setminus \{0\} \\
&\iff \textstyle\sum_{n=1}^N \lambda_n (y_n)^2 = y^\mathrm{T}\operatorname{diag}(\lambda_1, \dots, \lambda_N)y < 0 \\
&\qquad \text{for all } y \in \{Uz \mid z \in \mathbb{R}^N \setminus \{0\}\} = \mathbb{R}^N \setminus \{0\} \\
&\iff \lambda_1, \dots, \lambda_N < 0.
\end{aligned}$$

3. Suppose $Mz = 0$. Then $z^{\mathrm{T}}(M + M^{\mathrm{T}})z = 0$.
   Thus, if M is negative definite (and so is $M + M^{\mathrm{T}}$), we must have $z = 0$.

   Take any $z \in \mathbb{R}^N$, $z \neq 0$.

   Let $x = M^{-1}z$ ($\neq 0$). Then $z = Mx$.

   Then we have

$$\begin{aligned} z^{\mathrm{T}}M^{-1}z &= (Mx)^{\mathrm{T}}M^{-1}(Mx) \\ &= x^{\mathrm{T}}M^{\mathrm{T}}x = x^{\mathrm{T}}Mx < 0. \end{aligned}$$

Characterizations of Negative (Semi-)Definiteness

Proposition 6.2

Let $M \in \mathbb{R}^{N \times N}$ *be symmetric.*

1. M *is negative semi-definite*
   $\iff \exists\, B \in \mathbb{R}^{N \times N}$ *such that* $M = -B^{\mathrm{T}} B$.
2. M *is negative definite*
   $\iff \exists\, B \in \mathbb{R}^{N \times N}$ *nonsingular such that* $M = -B^{\mathrm{T}} B$.

Proof

- The "if" part:

  Suppose that $M = -B^{\mathrm{T}}B$. Then for any $z \in \mathbb{R}^N$,

  $$z^{\mathrm{T}}Mz = -z^{\mathrm{T}}B^{\mathrm{T}}Bz = -\|Bz\|^2 \leq 0.$$

- If B is nonsingular and $z \neq 0$, then $\|Bz\| \neq 0$.

Proof

- The "only if" part:

  Since M is symmetric, we have $U^{\mathrm{T}}MU = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_N \end{pmatrix}$

  for some U orthogonal (hence nonsingular).

  If M is negative semi-definite, then $\lambda_1, \ldots, \lambda_N \leq 0$.

- Let $B = \begin{pmatrix} \sqrt{-\lambda_1} & & O \\ & \ddots & \\ O & & \sqrt{-\lambda_N} \end{pmatrix} U^{\mathrm{T}}$.

  Then $-B^{\mathrm{T}}B = U \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_N \end{pmatrix} U^{\mathrm{T}} = M$.

- If M is negative definite, then $\lambda_1, \ldots, \lambda_N < 0$, so that B is nonsingular.

Characterizations of Negative (Semi-)Definiteness

Proposition 6.3

Let $M \in \mathbb{R}^{N \times N}$ *be symmetric.*
M *is negative definite*
$\iff (-1)^r |{}_r M_r| > 0$ *for all* $r = 1, \ldots, N$.

- ${}_r M_r \in \mathbb{R}^{r \times r}$ is the $r \times r$ submatrix of M obtained by deleting the last $N - r$ columns and rows of M,
  which is called the *leading principal submatrix* of order r of M.
- $|{}_r M_r|$ is called the *leading principal minor* of order r of M.
- ${}_r M \in \mathbb{R}^{r \times N}$ will denote the $r \times N$ submatrix of M obtained by deleting the last $N - r$ rows of M.

Proof

- The "only if" part:

  If M is negative definite, then ${}_rM_r$ is negative definite and its characteristic roots $\lambda_1, \ldots, \lambda_r$ are all negative, and thus,

  $$(-1)^r |{}_rM_r| = (-\lambda_1) \times \cdots \times (-\lambda_r) > 0.$$

- The "if" part: by induction:

  Trivial for $N = 1$.

- Assume that the assertion holds for $N - 1$.

  Suppose that $(-1)^r |{}_rM_r| > 0$ for all $r = 1, \ldots, N$.
  Then $L = {}_{N-1}M_{N-1}$ is negative definite by the induction hypothesis.

  Hence,

  - L is nonsingular, and
  - $L = -\tilde{B}^{\mathrm{T}} \tilde{B}$ for some nonsingular $\tilde{B}$.

Proof

- Write $M = \begin{pmatrix} L & b \\ b^{\mathrm{T}} & a_{NN} \end{pmatrix}$, where $b \in \mathbb{R}^{(N-1)\times 1}$.
- Let $U = \begin{pmatrix} I_{N-1} & L^{-1}b \\ 0^{\mathrm{T}} & 1 \end{pmatrix}$.

  Then one can verify that $M = U^{\mathrm{T}} \begin{pmatrix} L & 0 \\ 0^{\mathrm{T}} & c \end{pmatrix} U$,
  where $c = a_{NN} - b^{\mathrm{T}} L^{-1} b$.
- Thus, $|M| = c|L|$.

  But by assumption, $(-1)^N |M| > 0$ and $(-1)^{N-1}|L| > 0$, so that $c < 0$.
- Let $B = \begin{pmatrix} \tilde{B} & 0 \\ 0^{\mathrm{T}} & \sqrt{-c} \end{pmatrix} U$, which is nonsingular,
  where $L = -\tilde{B}^{\mathrm{T}}\tilde{B}$.

  Then $M = -B^{\mathrm{T}}B$. Hence, M is negative definite.

Note

- "$(-1)^r |_r M_r| \geq 0$ for all $r = 1, \ldots, N$" does not imply that M is negative semi-definite.
- For example,

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies this condition ($(-1)|_1 M_1| = (-1)^2 |M| = 0$), but is not negative semi-definite.

Characterizations of Negative (Semi-)Definiteness

Proposition 6.4

Let $M \in \mathbb{R}^{N \times N}$.

1. *Suppose that* M *is symmetric.*

   M *is negative semi-definite*
   $\iff (-1)^r |_r M_r^\pi| \geq 0$ *for all* $r = 1, \ldots, N$ *and*
   for all permutations π *of* $\{1, \ldots, N\}$.

2. *If (not necessarily symmetric)* M *is negative semi-definite,*
   then $(-1)^r |_r M_r^\pi| \geq 0$ *for all* $r = 1, \ldots, N$ *and*
   for all permutations π *of* $\{1, \ldots, N\}$.

Application to Concave Functions

Denote $f_{ij}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$.

- $f(x_1, x_2)$ is strictly concave
  $\Longleftarrow D^2 f(x_1, x_2)$ is negative definite $\quad \forall (x_1, x_2)$
  $\Longleftrightarrow (-1) f_{11} > 0$ and $(-1)^2 \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0 \quad \forall (x_1, x_2)$
  $\Longleftrightarrow f_{11} < 0$ and $f_{11} f_{22} - (f_{12})^2 > 0 \quad \forall (x_1, x_2)$

- $f(x_1, x_2)$ is concave
  $\Longleftrightarrow D^2 f(x_1, x_2)$ is negative semi-definite $\quad \forall (x_1, x_2)$
  $\Longleftrightarrow (-1) f_{11} \geq 0$, $(-1)^2 \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \geq 0$,
  $(-1) f_{22} \geq 0$, and $(-1)^2 \begin{vmatrix} f_{22} & f_{21} \\ f_{12} & f_{11} \end{vmatrix} \geq 0 \quad \forall (x_1, x_2)$
  $\Longleftrightarrow f_{11} \leq 0$, $f_{22} \leq 0$, and $f_{11} f_{22} - (f_{12})^2 \geq 0 \quad \forall (x_1, x_2)$

Characterizations of Negative (Semi-)Definiteness

Proposition 6.5

Let $M \in \mathbb{R}^{N\times N}$ be symmetric, and $B \in \mathbb{R}^{N\times S}$ with $S \leq N$ be such that $\operatorname{rank} B = S$. Let $W = \{z \in \mathbb{R}^N \mid B^T z = 0\}$.

1. *M is negative definite on W if and only if*

$$(-1)^r \begin{vmatrix} {}_rM_r & {}_rB \\ ({}_rB)^{\mathrm{T}} & 0 \end{vmatrix} > 0$$

for all $r = S+1, \ldots, N$.

2. *M is negative semi-definite on W if and only if*

$$(-1)^r \begin{vmatrix} {}_rM_r^{\pi} & {}_rB^{\pi} \\ ({}_rB^{\pi})^{\mathrm{T}} & 0 \end{vmatrix} \geq 0$$

for all $r = S+1, \ldots, N$ and for all permutations π of $\{1, \ldots, N\}$.

Application to Quasi-Concave Functions

Denote $f_i(x) = \frac{\partial f}{\partial x_i}(x)$ and $f_{ij}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$.

- $f(x_1, x_2)$ is strictly quasi-concave
  $\Longleftarrow D^2 f(x_1, x_2)$ is negative definite on $T_{\nabla f(x_1,x_2)}$ $\forall\,(x_1, x_2)$
  $$\Longleftrightarrow (-1)^2 \begin{vmatrix} f_{11} & f_{12} & f_1 \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix} > 0 \quad \forall\,(x_1, x_2)$$
  $$\Longleftrightarrow 2 f_1 f_2 f_{12} - (f_1)^2 f_{22} - (f_2)^2 f_{11} > 0 \quad \forall\,(x_1, x_2)$$

  where $T_{\nabla f(x_1,x_2)} = \{z \in \mathbb{R}^N \mid \nabla f(x_1, x_2) \cdot z = 0\}$.

Characterizations of Negative (Semi-)Definiteness

For $p \in \mathbb{R}^N$, we denote $T_p = \{z \in \mathbb{R}^N \mid p \cdot z = 0\}$.

Proposition 6.6

Let $M \in \mathbb{R}^{N \times N}$,
and suppose that $p \gg 0$, $Mp = 0$, and $M^{\mathrm{T}} p = 0$.
Let $\hat{M} \in \mathbb{R}^{(N-1)\times(N-1)}$ be the matrix obtained by deleting the nth row and column for some n.

1. *If $\operatorname{rank} M = N - 1$, then $\operatorname{rank} \hat{M} = N - 1$.*
2. *If M is negative definite on T_p, then M is negative definite on $\mathbb{R}^N \setminus \{z \in \mathbb{R}^N \mid z = \lambda p$ for some $\lambda \in \mathbb{R}\}$.*
3. *M is negative definite on T_p if and only if $\hat{M}$ is negative definite.*

Stable Matrices

Definition 6.2

$M \in \mathbb{R}^{N \times N}$ is *stable* if all of its characteristic roots have a negative real part.

Proposition 6.7

For $M \in \mathbb{R}^{N \times N}$ and $K \in \mathbb{R}^{N \times N}$,
suppose that M is negative definite and K is symmetric.
Then KM is stable if and only if K is positive definite.

Some Other Results

Definition 6.3

$M = (a_{ij}) \in \mathbb{R}^{N \times N}$ has a *dominant diagonal* if there exists $p \gg 0$ such that $|p_i a_{ii}| > \sum_{j \neq i} |p_j a_{ij}|$ for all $i = 1, \ldots, N$.

Definition 6.4

- $M = (a_{ij}) \in \mathbb{R}^{N \times N}$ has the *gross substitute sign pattern* if $a_{ij} > 0$ for all i, j with $i \neq j$.
- $M = (a_{ij}) \in \mathbb{R}^{N \times N}$ is a *Metzler matrix* if $a_{ij} \geq 0$ for all i, j with $i \neq j$.
- M is a *Z-matrix* if $-M$ is a Metzler matrix.

- Obviously, if M has the gross substitute sign pattern, then it is a Metzler matrix.

Some Other Results

Proposition 6.8

Let $M \in \mathbb{R}^{N \times N}$.

1. *If M has a dominant diagonal, then it is nonsingular.*
2. *Suppose that M is symmetric.*
   If M has a negative and dominant diagonal,
   then it is negative definite.
3. *If M is a Metzler matrix and if $Mp \ll 0$ and $M^{\mathrm{T}}p \ll 0$ for some $p \gg 0$, then M is negative definite.*
4. *If M has the gross substitute sign pattern and*
   if $Mp = 0$ and $M^{\mathrm{T}}p = 0$ for some $p \gg 0$,
   then $\hat{M}$ is negative definite,
   where $\hat{M} \in \mathbb{R}^{(N-1)\times(N-1)}$ is the matrix obtained by deleting the nth row and column for some n.

Proof

1. Suppose that $Mz = 0$. We want to show that $z = 0$.

   Let $p \gg 0$ be as in the definition of diagonal dominance.

   Let $y_i = z_i/p_i$, and let i be such that $|y_i| \geq |y_j|$ for all j.

   Since $a_{ii}(p_i y_i) = -\sum_{j \neq i} a_{ij}(p_j y_j)$, we have

$$|p_i a_{ii}||y_i| = \left| \sum_{j \neq i} p_j a_{ij} y_j \right| \leq \sum_{j \neq i} |p_j a_{ij}||y_j| \leq \sum_{j \neq i} |p_j a_{ij}||y_i|,$$

   and hence $\left(|p_i a_{ii}| - \sum_{j \neq i} |p_j a_{ij}| \right) |y_i| \leq 0$.

   Since $|p_i a_{ii}| - \sum_{j \neq i} |p_j a_{ij}| > 0$ by the dominant diagonal, it follows that $|y_i| = 0$, which implies that $z = 0$.

2. We show that all the eigenvalues of M are negative.

Let $\lambda \in \mathbb{R}$ be any eigenvalue of M, and let $z \in \mathbb{R}^N$, $z \neq 0$, be a corresponding eigenvector, i.e., we have $Mz = \lambda z$.

Let $y_i = z_i/p_i$, and let i be such that $|y_i| \geq |y_j|$ for all j, where $|y_i| \neq 0$.

Since $(a_{ii} - \lambda)(p_i z_i) = -\sum_{j \neq i} a_{ij}(p_j z_j)$, we have

$$|p_i a_{ii} - p_i \lambda||y_i| = \left|\sum_{j \neq i} p_j a_{ij} y_j\right| \leq \sum_{j \neq i} |p_j a_{ij}||y_j| \leq \sum_{j \neq i} |p_j a_{ij}||y_i| < |p_i a_{ii}||y_i|$$

by the dominant diagonal, and hence $|a_{ii} - \lambda| < |a_{ii}|$.

By $a_{ii} < 0$, this holds if and only if $2a_{ii} < \lambda < 0$, in particular only if $\lambda < 0$.

3. We show that $M + M^{\mathrm{T}}$ is a negative and dominant diagonal, which implies that $M + M^{\mathrm{T}}$ is negative definite by 2.

   By $Mp \ll 0$ and $M^{\mathrm{T}}p \ll 0$ where $p \gg 0$, we have $p_i(2a_{ii}) < -\sum_{j \neq i} p_j(a_{ij} + a_{ji})$ for all i.

   By $a_{ij} \geq 0$ for all $i \neq j$, we have $2a_{ii} < 0$ and $|p_i(2a_{ii})| = -p_i(2a_{ii}) > \sum_{j \neq i} p_j(a_{ij} + a_{ji}) = \sum_{j \neq i} |p_j(a_{ij} + a_{ji})|$ for all i.

4. Take any $n = 1, \ldots, N$, and let $\hat{M}$ be the $(N-1) \times (N-1)$ matrix obtained by deleting the nth row and column.

   By the assumptions, $\hat{M}$ is a Metzler matrix, and for all $i \neq n$, $\sum_{j \neq n} p_j a_{ij} = -p_n a_{in} < 0$ and $\sum_{j \neq n} p_j a_{ji} = -p_n a_{ni} < 0$, so that $\hat{M}p \ll 0$ and $\hat{M}^{\mathrm{T}}p \ll 0$.

   Hence, by 3, $\hat{M}$ is negative definite.

Some Results on Nonnegative Matrices

- $M = (a_{ij}) \in \mathbb{R}^{N \times N}$ is called a *nonnegative* (*positive*) *matrix* if $a_{ij} \geq 0$ ($a_{ij} > 0$) for all $i, j = 1, \ldots, N$.

Some Results on Nonnegative Matrices I

Proposition 6.9

For a nonnegative matrix $M \in \mathbb{R}^{N \times N}$,
the following conditions are equivalent:

1. *For every $c \geq 0$, there exists $z \geq 0$ such that $Mz + c = z$.*
2. *There exists $z \geq 0$ such that $Mz \ll z$.*
3. *There exists $z \gg 0$ such that $Mz \ll z$.*
4. $|_r(I - M)_r| > 0$ *for all* $r = 1, \ldots, N$ *("Hawkins-Simon condition").*
5. *There exist lower and upper triangular matrices L and U with positive diagonals and nonpositive off-diagonals such that $I - M = LU$.*
6. *$I - M$ is nonsingular, and $(I - M)^{-1} \geq 0$.*

Some Results on Nonnegative Matrices II

Proposition 6.9

7. $|\lambda_i| < 1$ *for all* $i = 1, \ldots, N$,
   where $\lambda_1, \ldots, \lambda_N$ *are the characteristic roots of* M*.*

8. $\lim_{k\to\infty} \sum_{\ell=0}^{k} M^\ell$ *exists (which is equal to* $(I - M)^{-1}$*).*

9. $\lim_{k\to\infty} M^k = O$*.*

Spectral Radius

- For $M \in \mathbb{R}^{N \times N}$, let

  $$\lambda(M) = \max\{|\lambda_1|, \ldots, |\lambda_N|\},$$

  where $\lambda_1, \ldots, \lambda_N$ are the characteristic roots of M.

- $\lambda(M)$ is called the *spectral radius* of M.

Some Results on Nonnegative Matrices

Proposition 6.10 (Perron-Frobenius Theorem)

1. *Let $M \in \mathbb{R}^{N \times N}$ be a positive matrix.*
   - *$\lambda(M) > 0$, $\lambda(M)$ is an eigenvalue of M, and there exists a positive eigenvector that belongs to $\lambda(M)$.*
   - *$\lambda(M)$ is a simple root of the characteristic equation.*
   - *An eigenvector that belongs to $\lambda(M)$ is unique (up to multiplication).*
   - *If $Mz = \mu z$, $\mu \geq 0$, for some $z \geq 0$, $z \neq 0$, then $\mu = \lambda(M)$.*
   - *If $M \geq L \geq O$ and $M \neq L$, then $\lambda(M) > \lambda(L)$.*
2. *Let $M \in \mathbb{R}^{N \times N}$ be a nonnegative matrix.*
   - *$\lambda(M)$ is an eigenvalue of M, and there exists a nonnegative eigenvector that belongs to $\lambda(M)$.*
   - *If $M \geq L \geq O$, then $\lambda(M) \geq \lambda(L)$.*