Lecture Notes on Set-Valued Dynamical Systems

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In these notes, we study the theory of set-valued dynamical systems, with applications in mind to population game dynamics, especially to the *best response dynamics* (Gilboa and Matsui (1991) and Hofbauer (1995)) and the *sampling best response dynamics* (Oyama et al. (2009, OST henceforth)). These classes of dynamics are defined by differential inclusions, and they naturally induce set-valued dynamical systems. In Section 1, we consider a specific class of differential inclusions relevant for our applications and present the existence and key properties of their solutions. In Sections 2–6, we review stability concepts defined through a set-valued dynamical system. These sections are based on Benaïm et al. (2005, BHS henceforth), while here we only allow for systems with forward time. Section 7 and Section 8 present theorems that play important roles in the study of the sampling best response dynamics in OST. In Section 7, we prove the transitivity property of asymptotic stability for nested sets, where we extend the result by Conley (1978) to set-valued dynamical systems that are defined only for forward time. Section 8 contains a result on the partial order of solutions to dynamical systems satisfying a certain monotonicity property that corresponds to the supermodularity in games. Definitions and theorems about correspondences and function spaces that are needed along our study are available in the Appendix.

1 Differential Inclusions

A *differential inclusion* is an expression of the form

$$\dot{x} \in F(x),$$

where F is a correspondence (set-valued map). Given our applications to population game dynamics, we restrict our attention to the following class of differential inclusions on a nonempty convex compact set $X \subset \mathbb{R}^n$:

$$\dot{x} \in G(x) - x, \tag{DI}$$

where $G\colon X \to X$ is a correspondence that satisfies the following properties:

(i) G is nonempty-valued;

(ii) G is convex-valued;

(iii) G has a closed graph.

Since X is compact, property (iii) is equivalent to the following:

(iii′) G is compact-valued and upper semi-continuous.

A solution to (DI) with initial condition $\xi \in X$ is a Lipschitz function $x\colon [0,\infty) \to X$ such that $x(0) = \xi$ and

$$\dot{x}(t) + x(t) \in G(x(t))$$

for almost all $t \geq 0$. We show the existence of solutions for this particular class, and study their properties.

Let $L = \max\{|a - b| \mid a, b \in X\}$, and let C^L be the set of Lipschitz continuos functions from $[0,\infty)$ to X with Lipschitz constant L. Solutions to (DI) are contained in C^L. Denote by C the space of bounded continuous functions from $[0,\infty)$ to $\mathbb{R}^n$ equipped with the topology of uniform convergence on compact intervals. This space is a complete normed space (i.e., a Banach space) with the norm

$$\|x\| = \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{t\in[0,k]} |x(t)|,$$

or

$$\|x\| = \sup_{t\in[0,\infty)} e^{-rt}|x(t)|, \quad r > 0.$$

Then C^L is a compact subset of C; see Appendix C.

For the differential inclusion (DI), we have the following.

Theorem 1.1. *For each $\xi \in X$, there exists a solution to* (DI) *with initial condition ξ.*

The set of solutions to (DI) *is compact in C.*

Proof. See Appendix E. ∎

Given the solutions to (DI), define the correspondence $\Phi\colon [0,\infty) \times X \to X$ by

$$\Phi_t(\xi) = \{x(t) \in X \mid x \in C^L \text{ is a solution to (DI) with } x(0) = \xi\}. \tag{1.1}$$

Proposition 1.2. Φ *satisfies the following properties.*

(0) $\Phi_t(\xi) \neq \emptyset$ *for all $t \geq 0$ and all $\xi \in X$;*

(1) $\Phi_0(\xi) = \{\xi\}$ *for all $\xi \in X$;*

(2) $\Phi_t(\Phi_s(\xi)) = \Phi_{t+s}(\xi)$ *for all $t, s \geq 0$ and all $\xi \in X$; and*

(3) Φ *is compact-valued and upper semi-continuous.*

2 Set-Valued Dynamical Systems

Let X be a nonempty compact subset of $\mathbb{R}^n$. A *set-valued dynamical system* is a correspondence $\Phi\colon [0,\infty) \times X \to X$ such that

(0) $\Phi_t(\xi) \neq \emptyset$ for all $t \geq 0$ and all $\xi \in X$;

(1) $\Phi_0(\xi) = \{\xi\}$ for all $\xi \in X$;

(2) $\Phi_t(\Phi_s(\xi)) = \Phi_{t+s}(\xi)$ for all $t, s \geq 0$ and all $\xi \in X$; and

(3) Φ is compact-valued and upper semi-continuous.

For $I \subset [0,\infty)$ and $A \subset X$, we write

$$\Phi_t(A) = \bigcup_{\xi \in A} \Phi_t(\xi), \;\; \Phi_I(\xi) = \bigcup_{t \in I} \Phi_t(\xi), \;\text{ and }\; \Phi_I(A) = \bigcup_{t \in I} \bigcup_{\xi \in A} \Phi_t(\xi).$$

As we have seen in the previous section, the differential inclusion (DI) induces a set-valued dynamical system by (1.1). In what follows, we study a general set-valued dynamical system, which may or may not be induced by a differential inclusion; all the results are derived from the key properties (0)–(3).

Lemma 2.1. *For any compact interval $I \subset [0,\infty)$, $\xi \mapsto \Phi_I(\xi)$ is compact-valued and upper semi-continuous.*

For any compact set $A \subset X$, $t \mapsto \Phi_t(A)$ is compact-valued and upper semi-continuous.

Let $\Phi^{-1}\colon [0,\infty) \times X \to X$ be the correspondence defined by

$$\Phi_t^{-1}(z) = \{\xi \in X \mid z \in \Phi_t(\xi)\}.$$

Note that $\Phi_t^{-1}(z)$ may be empty in general, while $\Phi_t^{-1}(z) \neq \emptyset$ whenever $z \in \Phi_{t'}(\xi)$ for some $t' > t$ and $\xi \in X$, as there must be some $y \in \Phi_{t'-t}(\xi)$ such that $z \in \Phi_t(y)$, or $y \in \Phi_t^{-1}(z)$.

Proposition 2.2. *Φ^{-1} is compact-valued and upper semi-continuous.*

Proof. Φ^{-1} has a closed graph, and its codomain X is compact. ∎

We will in particular use the following.

Corollary 2.3. *$\Phi_t^{-1}(z)$ is upper semi-continuous in t at $t = 0$.*

While this is a corollary to the upper semi-continuity of Φ^{-1}, one may directly prove it as follows: Let V be an open neighborhood of $\Phi_0^{-1}(z) = \{z\}$. Then $X \setminus \{z\}$ is a neighborhood of $X \setminus V = \Phi_0(X \setminus V)$. By the upper semi-continuity of $\Phi_t(X \setminus V)$ in t at $t = 0$, there exists $\delta > 0$ such that $\Phi_{[0,\delta)}(X \setminus V) \subset X \setminus \{z\}$. Then for any $s \in [0,\delta)$, if $\xi \in \Phi_s^{-1}(z)$, or $z \in \Phi_s(\xi)$, then we must have $\xi \notin X \setminus V$, i.e., $\xi \in V$.

Corollary 2.4. *Suppose that $z \in \Phi_{t_0}(\xi)$ for $t_0 > 0$. Then for any neighborhood V of z, there exists $\delta \in (0, t_0]$ such that for any $s \in (0,\delta)$ and for any $y \in \Phi_{t_0-s}(\xi)$ such that $z \in \Phi_s(y)$, we have $y \in V$.*

Proof. Since V is a neighborhood of $\{z\} = \Phi_0^{-1}(z)$, by the upper semi-continuity of $\Phi_t^{-1}(z)$ in t at $t = 0$ there exists $\delta \in (0, t_0]$ such that $\Phi_s^{-1}(z) \subset V$ for all $s \in (0, \delta)$. Then for any $s \in (0, \delta)$ and for any $y \in \Phi_{t_0-s}(\xi)$ such that $z \in \Phi_s(y)$, we have $y \in V$. ∎

3 Invariant Sets and Limit Sets

Definition 3.1. A set $A \subset X$ is *strongly positively invariant* if $\Phi_t(A) \subset A$ for all $t \geq 0$.

Remark 3.1. In BHS, where Φ is induced by a differential inclusion, A is said to be *invariant* if for every $\xi \in A$, there exists a complete trajectory x (i.e., a solution defined for positive and negative time) of the differential inclusion such that $x(0) = \xi$ and $x(\mathbb{R}) \subset A$. Note, in contrast, that in our setting, Φ is the primitive, and it is defined only for nonnegative time.

For $\xi \in X$, define

$$\omega(\xi) = \bigcap_{t \geq 0} \overline{\Phi_{[t,\infty)}(\xi)},$$

and for $A \subset X$,

$$\omega(A) = \bigcap_{t \geq 0} \overline{\Phi_{[t,\infty)}(A)}.$$

Since X is compact, $\omega(A) \neq \emptyset$ whenever $A \neq \emptyset$ (Exercise 3.1). Note that $z \in \omega(A)$ if and only if there exist $\{t_n\} \subset [0, \infty)$, $\{z_n\} \subset X$, and $\{\xi_n\} \subset A$ such that $z_n \in \Phi_{t_n}(\xi_n)$, and $t_n \to \infty$ and $z_n \to z$ as $n \to \infty$. Clearly, $\omega(A) \subset \omega(B)$ if $A \subset B$, but in general $\bigcup_{\xi \in A} \omega(\xi) \neq \omega(A)$.

Example 3.1. Let $X = [0, 1]$, and consider $\dot{x} = x(1 - x)$.[1] For any $0 < a \leq b$, $\omega([a, b]) = \{1\}$ and $\omega((0, b]) = [0, 1]$, while $\bigcup_{\xi \in (0,1]} \omega(\xi) = \{1\}$.

Exercise 3.1. Let X be a compact topological space, and $\{C_\lambda\}_{\lambda \in \Lambda}$ a family of closed subsets of X. If for any finite subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ, $\bigcap_{i=1}^n C_{\lambda_i} \neq \emptyset$ (in which case $\{C_\lambda\}_{\lambda \in \Lambda}$ to said to have the *finite intersection property*), then $\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset$. Find a counter-example when the compactness of X is dropped.

Lemma 3.1. *Let $A, B \subset X$.*

(1) $\omega(A) \subset \Phi_t(\omega(A))$ *for all* $t \geq 0$.

(2) $\omega(A) \subset \omega(\omega(A))$.

(3) *If* $\omega(A) \subset B$, *then* $\omega(A) \subset \omega(B)$.

[1] We have $\Phi_t(\xi) = \{0\}$ if $\xi = 0$ and $\Phi_t(\xi) = \{1/[1 + ((1 - \xi)/\xi)e^{-t}]\}$ if $\xi > 0$ (consider differentiating $\log[x/(1 - x)]$).

Proof. Let $z \in \omega(A)$, and fix any $t \geq 0$. By definition, there exist $\{t_n\} \to \infty$ and $\{z_n\}$ such that $z_n \in \Phi_{t_n}(A)$ and $z_n \to z$. For each n such that $t_n \geq t$, since $z_n \in \Phi_t(\Phi_{t_n - t}(A))$, there exists $\xi_n \in X$ such that

$$\xi_n \in \Phi_{t_n - t}(A) \tag{3.1}$$

and

$$z_n \in \Phi_t(\xi_n). \tag{3.2}$$

By the compactness of X, there exists a convergent subsequence with limit $\xi \in X$. From (3.1) we have $\xi \in \omega(A)$, and from (3.2) we have $z \in \Phi_t(\xi)$ by the upper semi-continuity of Φ_t. Thus, we have $z \in \Phi_t(\omega(A))$ as desired. ▮

In general, $\Phi_t(\omega(A)) \not\subset \omega(A)$, i.e., $\omega(A)$ may not be strongly positively invariant; see Example 3.7 in BHS.

Exercise 3.2. For any $z \in \omega(A)$, $\Phi_t(z) \cap \omega(A) \neq \emptyset$ for all $t \geq 0$.

Thus, if Φ is single-valued, then $\omega(A) = \Phi_t(\omega(A))$ for all $t \geq 0$, and hence $\omega(\omega(A)) = \omega(A)$.

Exercise 3.3. If $A \subset B$ and $\omega(B) = A$, then $\omega(A) = A$.

If $\omega(A) = A$, then $\Phi_t(A) = A$ for all $t \geq 0$.

4 Attracting Sets

For $A \subset X$, denote

$$N^\varepsilon(A) = \{\xi \in X \mid |\xi - a| < \varepsilon \text{ for some } a \in A\}.$$

Definition 4.1. A set $A \subset X$ is an *attracting set* if it is a compact set and there exists a neighborhood U of A such that for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\Phi_{[t_\varepsilon, \infty)}(U) \subset N^\varepsilon(A). \tag{4.1}$$

Such a set U is called a *fundamental neighborhood* of A.[2]

Exercise 4.1. If attracting set A is a singleton, then it is strongly positively invariant. (This follows from Proposition 4.1 and Exercise 3.3, but show it from the definition.)

Example 4.1. Let $X = [0,1]$, and consider $\dot{x} = x(1-x)$. For any $0 < a \leq b$, $\omega([a,b]) = \{1\}$ and $\omega((0,b]) = [0,1]$, while $\bigcup_{x \in (0,1]} \omega(x) = \{1\}$.

Any compact set that contains 1 is an attracting set, while any $[a,1]$ is asymptotically stable; recall that we require only forward invariance for asymptotic stability. When $a > 0$, a fundamental neighborhood must be a proper subset of $(0,1]$.

[2] In BHS, A is said to be an *attractor* if in addition A is invariant, whereas we will not use this concept.

Proposition 4.1. *Let A be a nonempty compact subset of X, and U a neighborhood of A. Then A is an attracting set with fundamental neighborhood U if and only if $\omega(U) \subset A$.*

The compactness condition is indispensable. Let $X = (0, 1]$ and consider $\dot{x} = -x$. Then, for any $x \in (0, 1]$ and any neighborhood U of x, we have $\omega(U) = \emptyset$. For another example, let $X = [0, \infty)$ and consider $\dot{x} = x(x - 1)$. Consider $A = \{0\} \cup \{2\}$. For any neighborhood U of A that does not intersect a neighborhood of 1, we have $\omega(U) = \{0\}$, but A is not an attracting set.

Proof. If A is an attracting set with fundamental neighborhood U, then

$$\omega(U) \subset \bigcap_{\varepsilon>0} \overline{N^{\varepsilon}(A)} = \overline{A} = A.$$

Conversely, for $t \geq 0$, $V_t = \overline{\Phi_{[t,\infty)}(U)}$ defines a decreasing family of closed subsets of a compact set X such that $\bigcap_{t\geq 0} V_t = \omega(U)$. Hence, $\bigcap_{t\geq 0} V_t \subset A$ implies that for any $\varepsilon > 0$, there exists t_ε such that $V_{t_\varepsilon} \subset N^{\varepsilon}(A)$ (Exercise 4.2). ▮

Remark 4.1. By the characterization given by BHS (Proposition 3.10(ii)), one may define an attractor to be a compact set A such that $\omega(U) = A$ for some neighborhood U of A.

Exercise 4.2. Let $\{V_t\}_{t\geq 0}$ be a family of closed subsets of a compact topological space X such that $V_t \supset V_s$ if $t < s$. Let U be an open set such that $\bigcap_{t\geq 0} V_t \subset U$. Then there exists t' such that $V_{t'} \subset U$.

5 Asymptotic Stability

Definition 5.1. A set $A \subset X$ is *Lyapunov stable* if for any neighborhood V of A, there exists a neighborhood U of A such that

$$\Phi_{[0,\infty)}(U) \subset V. \tag{5.1}$$

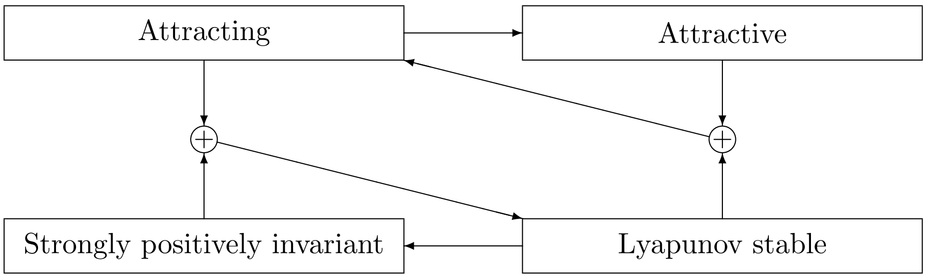
A is *attractive* if there exists a neighborhood U of A such that

$$\bigcup_{x\in U} \omega(x) \subset A. \tag{5.2}$$

If $U = X$, then A is said to be *globally attractive*. A is *asymptotically stable* if it is attractive and Lyapunov stable.[3]

Theorem 5.1. *A compact set A is attracting and strongly positively invariant if and only if it is asymptotically stable.*

If a compact set A is asymptotically stable and if U is a neighborhood of A such that $\bigcup_{x\in U} \omega(x) \subset A$, then any compact neighborhood of A contained in U is a fundamental neighborhood of A.



This theorem follows from the following lemmas.

Lemma 5.2. *If a set A is attracting, then it is attractive.*

Proof. Follows from Proposition 4.1. ▌

Lemma 5.3. *If a closed set A is Lyapunov stable, then it is strongly positively invariant.*

Proof. For each $\varepsilon > 0$, there is a neighborhood of A, V_ε, such that $\Phi_{[0,\infty)}(V_\varepsilon) \subset N^\varepsilon(A)$, so that $\Phi_{[0,\infty)}(A) \subset N^\varepsilon(A)$. By the closedness of A, we have $\Phi_{[0,\infty)}(A) \subset \bigcap_{\varepsilon>0} N^\varepsilon(A) = \overline{A} = A$. ▌

Lemma 5.4. *If a compact set A is asymptotically stable and if U is a neighborhood of A such that $\bigcup_{x\in U} \omega(x) \subset A$, then A is an attracting set of which any compact neighborhood contained in U is a fundamental neighborhood.*

Proof. Let B be any compact neighborhood of A contained in U. Let W be any neighborhood of A. We want to show that there exists T such that $\Phi_{[T,\infty)}(B) \subset W$.

By the Lyapunov stability of A, we can take an open neighborhood V of A such that $\Phi_{[0,\infty)}(V) \subset W$. By the attractiveness of A, for each $x \in U$ there exists t_x such that $\Phi_{t_x}(x) \in V$, and hence $\Phi_{[t_x,\infty)}(x) \subset W$ by the Lyapunov stability. By the upper semi-continuity of $\Phi_{t_x}(\cdot)$, each $x \in B$ has an open neighborhood N_x such that $\Phi_{t_x}(N_x) \subset V$. By the compactness of B, there are finitely many $x^1, \ldots, x^m$ such that $N_{x^1} \cup \cdots \cup N_{x^m} \supset B$. Let $T = \max\{t_{x^1}, \ldots, t_{x^m}\}$. Then for $i = 1, \ldots, m$, we have $\Phi_{[T,\infty)}(N_{x^i}) \subset \Phi_{[t_{x^i},\infty)}(N_{x^i}) = \Phi_{[0,\infty)}(\Phi_{t_{x^i}}(N_{x^i})) \subset \Phi_{[0,\infty)}(V) \subset W$, as desired. ▌

Lemma 5.5. *If a set A is attracting and strongly positively invariant, then it is Lyapunov stable.*

Proof. Let V be any neighborhood of A. We want to show that there exists a neighborhood W of A such that $\Phi_{[0,\infty)}(W) \subset V$.

Let U be a fundamental neighborhood of A. Since A is compact, there exists $\varepsilon > 0$ such that $N^\varepsilon(A) \subset V$ (Exercise 5.1). By the attraction of A, there exists T such that $\Phi_{[T,\infty)}(U) \subset N^\varepsilon(A) \subset V$. By the strong positive invariance of A, we have $\Phi_{[0,T]}(A) \subset A$. By the upper semi-continuity of $\Phi_{[0,T]}$, there exists a neighborhood W of A such that

[3]BHS require that A be also invariant to be asymptotically stable, whereas we do not.

$\Phi_{[0,T]}(W) \subset V$ and $W \subset U$. By the choice of T, we have $\Phi_{[T,\infty)}(W) \subset V$. Hence, we have $\Phi_{[0,\infty)}(W) \subset V$, as desired. ∎

Exercise 5.1. Let A be a compact subset of a metric space, and V an open set containing A. Then there exists $\varepsilon > 0$ such that $N^{\varepsilon}(A) \subset V$. Find a counter example when the compactness is dropped.

6 Lyapunov Functions

Theorem 6.1. *Let $\Lambda \subset X$ be a compact set, U an open neighborhood of Λ, and $V : \overline{U} \to \mathbb{R}$ a continuous function. Assume that the following conditions hold:*

(i) *U is strongly positively invariant;*

(ii) $\arg\max_{x \in \overline{U}} V(x) = \Lambda$*; and*

(iii) *$V(y) > V(x)$ for all $x \in U \setminus \Lambda$, $y \in \Phi_t(x)$, and $t > 0$.*

Then Λ is asymptotically stable.

Proof. In light of Theorem 5.1, it is sufficient to show that Λ is Lyapunov stable and attracting. Let $M = \max_{x \in \overline{U}} V(x)$, and for $r > 0$, let $U_r = \{x \in U \mid V(x) > M - r\}$. Then $\{\overline{U}_r\}_{r>0}$ defines a decreasing family of closed subsets of a compact set $\overline{U}$ such that $\bigcap_{r>0} \overline{U}_r = \Lambda$, so that there is some $\bar{r} > 0$ such that $\overline{U}_r \subset U$ for all $r < \bar{r}$. We first show that for any $r < \bar{r}$, we have $\Phi_t(\overline{U}_r) \subset \overline{U}_r$ for all $t \geq 0$.

Let $r < \bar{r}$, and fix any $t > 0$. Let $\alpha \in \mathbb{R}$, $z \in U$, and $t' \in [0, t]$ be such that $\alpha = \min_{x \in \Phi_{[0,t]}(\overline{U}_r)} V(x)$, $V(z) = \alpha$, and $z \in \Phi_{t'}(\overline{U}_r)$, which are well defined by the compactness of $\Phi_{[0,t]}(\overline{U}_r)$ and the continuity of V. If $\alpha < M - r$, and hence $z \in U \setminus \overline{U}_r \subset U \setminus \Lambda$ and $t' > 0$, then by Corollary 2.4, there exist $s \in (0, t')$ and $y \in \Phi_{t'-s}(\overline{U}_r) \cap (U \setminus \Lambda)$ such that $z \in \Phi_s(y)$, which implies that $V(y) < V(z)$, contradicting the assumption that z minimizes V on $\Phi_{[0,t]}(\overline{U}_r)$. Thus, $\alpha \geq M - r$ and hence $\Phi_{[0,t]}(\overline{U}_r) \subset \overline{U}_r$.

It thus follows that for any $r < \bar{r}$, $\Phi_{[0,\infty)}(\overline{U}_r) \subset \overline{U}_r$, which implies that Λ is Lyapunov stable, since for any neighborhood W of Λ, there is some $r < \bar{r}$ such that $\overline{U}_r \subset W$, for which we have $\Phi_{[0,\infty)}(\overline{U}_r) \subset \overline{U}_r \subset W$.

It remains to show that Λ is attracting. Fix any $r < \bar{r}$, and let $A = \omega(\overline{U}_r) \subset \overline{U}_r \subset U$. Let $\beta = \min_{x \in A} V(x)$ be reached at $z \in A \subset U$. If $\beta < M$ and hence $z \in U \setminus \Lambda$, then by the upper semi-continuity of $\Phi_t^{-1}(z)$ in t at $t = 0$, there exists some $s > 0$ such that $\Phi_s^{-1}(z) \subset U \setminus \Lambda$. By Lemma 3.1, $z \in \Phi_s(y)$ for some $y \in A$, but for such a y we have $y \in U \setminus \Lambda$ and therefore $V(y) < V(z)$, which contradicts the assumption that z minimizes V on A. Thus, $\beta = M$ and hence $A = \omega(\overline{U}_r) \subset \Lambda$, which implies that Λ is attracting by Proposition 4.1. ∎

7 Transitivity Theorem

For $Y, Z \subset X$ with $Z \subset Y$, we say that Z is *asymptotically stable in* Y if Y is strongly positively invariant and Z is asymptotically stable with respect to $\Phi|_Y$, the restriction of Φ to Y. We say that $Y \subset X$ is *strongly negatively invariant* if $\Phi_t^{-1}(Y) \subset Y$ for all $t \geq 0$, where $\Phi_t^{-1}(Y)$ is the upper inverse image of Y under Φ_t, i.e.,

$$\Phi_t^{-1}(Y) = \{\xi \in X \mid \Phi_t(\xi) \subset Y\}.$$

Theorem 7.1 (Transitivity Theorem)**.** *Let $Y, Z \subset X$ be compact sets with $Z \subset Y$. If Y is strongly negatively invariant and asymptotically stable (in X) and if Z is asymptotically stable in Y, then Z is asymptotically stable (in X).*

The proof is a modification of that of 5.3.D in Conley (1978) to the case of set-valued dynamical systems defined only for forward time.

Proof. Let U_Y be a fundamental neighborhood of Y in X (so that $\omega(U_Y) \subset Y$ by Proposition 4.1 and Theorem 5.1), and U_Z a fundamental neighborhood of Z in Y (so that $\omega(U_Y) \subset Z$). Let U be a compact neighborhood of Z in X such that $U \subset U_Y$ and $U \cap Y \subset U_Z$.

Claim 1. *For all $z \in \partial U$, there exists $t_z > 0$ such that $\Phi_t^{-1}(z) \cap U = \emptyset$ for all $t \geq t_z$.*

Proof. Suppose not. Then there exist $z \in \partial U$ and a sequence $\{t_n\} \to \infty$ such that $\Phi_{t_n}^{-1}(z) \cap U \neq \emptyset$. Let $\{\xi_n\}$ be such that, for each n, $\xi_n \in \Phi_{t_n}^{-1}(z) \cap U$, i.e., $z \in \Phi_{t_n}(\xi_n)$ and $\xi_n \in U$. If $z \in \partial U \setminus Y$, then, since $\{\xi_n\} \subset U \subset U_Y$, we have $z \in \omega(U_Y)$, which contradicts $\omega(U_Y) \subset Y$. If $z \in \partial U \cap Y$, then, since $\{\xi_n\} \subset Y$ by the strong negative invariance of Y and hence $\{\xi_n\} \subset U \cap Y \subset U_Z$, we have $z \in \omega(U_Z)$, which contradicts $\omega(U_Z) \subset Z$. ▮

Since Φ_t^{-1} is upper semi-continuous for any t, for each $z \in \partial U$ there exists an open neighborhood V_z of z such that $\Phi_{t_z}^{-1}(z') \subset U^{\mathrm{c}}$ for all $z' \in V_z$, where t_z is as in Claim 1. Since ∂U is compact, there are $z^1, \ldots, z^N \in \partial U$ such that $\bigcup_i V_{z^i} \supset \partial U$. Thus, if $z \in \partial U$, then for some z^i, $z \in V_{z^i}$ and $\Phi_{t_z^i}^{-1}(z) \subset U^{\mathrm{c}}$. Let $T = \max_i t_{z^i}$.

Claim 2. *For all $\xi \in U$, if $\Phi_{[0,T]}(\xi) \subset U$, then $\Phi_{[0,\infty)}(\xi) \subset U$.*

Proof. Suppose that $\Phi_{[0,T]}(\xi) \subset U$. Let $t' = \sup\{t \mid \Phi_{[0,t]}(\xi) \subset U\}$ $(\geq T)$. Assume that $t' < \infty$. Then we have $\Phi_{t'}(\xi) \subset \overline{U} = U$ by the lower semi-continuity of $\Phi_t(\xi)$ in t (see Corollary 2.4) and the closedness of U, and thus $\Phi_{[t'-T,t']}(\xi) \subset U$. If $\Phi_{t'}(\xi) \cap \partial U \neq \emptyset$, then for any $z \in \Phi_{t'}(\xi) \cap \partial U$, we have $\Phi_{t_z^i}^{-1}(z) \subset U^{\mathrm{c}}$ (where z^i is such that $z \in V_{z^i}$) and hence $\Phi_{t-t_z^i}(\xi) \cap U^{\mathrm{c}} \neq \emptyset$, which is a contradiction. Therefore $\Phi_{t'}(\xi) \cap \partial U = \emptyset$, which implies that $\Phi_{t'+\varepsilon}(\xi) \subset U$ for some $\varepsilon > 0$ by the upper semi-continuity, but then t' could not be the supremum as defined. Hence, $t' = \infty$. ▮

Let U^0 be the upper inverse image of U under the correspondence $\Phi_{[0,T]}(\cdot)$, i.e., $U^0 = \{\xi \in X \mid \Phi_{[0,T]}(\xi) \subset U\}$. Note that $U^0 \subset U$. Since Z is strongly positively invariant, we have $\Phi_{[0,T]}(Z) \subset Z \subset U$. Therefore, by the upper semi-continuity of $\Phi_{[0,T]}(\cdot)$, U^0 is a neighborhood of Z in X.

Now, by Claim 2, we have $\Phi_{[0,\infty)}(U^0) \subset \overline{U} = U$, and hence, $\omega(U^0) \subset U$ by the closedness of U. Since $U^0 \subset U_Y$, $\omega(U^0) \subset \omega(U_Y) \subset Y$ by the attraction of Y. Thus, $\omega(U^0) \subset U \cap Y \subset U_Z$. Therefore, we have $\omega(U^0) \subset \omega(U_Z) \subset Z$ by Lemma 3.1 and the attraction of Z in Y, and hence Z is an attracting set in X by Lemma 4.1. Since Z is strongly positively invariant, it follows from Lemma 5.1 that Z is asymptotically stable in X. ▮

8 Comparison Theorem

Let $X \subset \mathbb{R}^n$ be partially ordered by the usual vector order $\leq$. The following is from Walter (1970).

Theorem 8.1 (Comparison Theorem)**.** *Let $f\colon X \to \mathbb{R}^n$ be a Lipschitz continuous function such that $f_i(x)$ is nondecreasing in x_j for all $j \neq i$. If absolutely continuous functions $y, z\colon [0,\infty) \to X$ satisfy $y(0) \ll z(0)$, $\dot{y}(t) \leq f(y(t))$, and $\dot{z}(t) \geq f(z(t))$ for almost all $t \geq 0$, then $y(t) \ll z(t)$ for all $t \geq 0$.*

Proof. Let $T = \sup\{t \geq 0 \mid y(s) \ll z(s) \text{ for all } s \in [0,t)\}$, where $T > 0$ by the continuity of y and z. Suppose by way of contradiction that $T < \infty$. Then $z(t) - y(t) \geq 0$ for all $t \in [0,T]$ and $z_i(T) - y_i(T) = 0$ for some i. For any such i,

$$
\begin{aligned}
\frac{d}{dt}(z_i(t) - y_i(t)) &\geq f_i(z(t)) - f_i(y(t)) \\
&\geq f_i(y_1(t), \ldots, y_{i-1}(t), z_i(t), y_{i+1}(t), \ldots, y_n(t)) - f_i(y(t)) \\
&\geq -K(z_i(t) - y_i(t))
\end{aligned}
$$

for almost all $t \in [0,T]$, where the second inequality follows from the assumption that $\partial f_i/\partial x_j \geq 0 \geq 0$ for all $j \neq i$, and the third from the Lipschitz continuity of f with Lipschitz constant $K > 0$. Then we have

$$\frac{d}{dt}e^{Kt}(z_i(t) - y_i(t)) \geq 0$$

for almost all $t \in [0,T]$, and then integrate both sides to obtain

$$z_i(T) - y_i(T) \geq e^{-KT}(z_i(0) - y_i(0)).$$

But since the right hand side of this inequality is positive, we have a contradiction to the definition of T. ▮

This theorem has the following implication. Let $\Phi\colon X \to X$ and $\varphi\colon X \to X$ be the set-valued and point-valued dynamical systems induced by a differential inclusion

$\dot{x} \in F(x)$ and a differential equation $\dot{x} = f(x)$, respectively, where we assume that f is Lipschitz continuous and that Φ satisfies properties (0)–(3). To be concrete, let $X = \{\xi \in \mathbb{R}^n \mid 0 \leq \xi_n \leq \cdots \leq \xi_1 \leq 1\}$ as in the application in OST. Note that $0 = \min X$.

Proposition 8.2. *Suppose that $y \leq f(x)$ for all $x \in X$ and all $y \in F(x)$ and that $f_i(x)$ is nondecreasing in x_j for all $j \neq i$. If 0 is asymptotically stable under φ, then it is asymptotically stable also under Φ.*

Proof. In light of Theorem 5.1 and Exercise 4.1, it suffices to show that the singleton set $\{0\}$ is attracting. Let $\xi \in X$ be such that $\xi \gg 0$ and $\varphi_t(\xi) \to \min X$ as $t \to \infty$. Let $U = \{z \in X \mid z \ll \xi\}$, which is a neighborhood of 0 (relative to X). By Theorem 8.1, for all $z \in \Phi_t(U)$, $z \leq \varphi_t(\xi)$ for all $t \geq 0$. Given any $\varepsilon > 0$, let t_ε be such that $\phi_t(\xi) \in N^\varepsilon(0)$ for all $t \geq t_\varepsilon$. Then we have $\Phi_{[t_\varepsilon,\infty)}(U) \subset N^\varepsilon(0)$ as desired. ∎

9 Sampling Best Response Dynamics

See OST.

Appendix

A Compact Sets in Metric Spaces

Proposition A.1. *A compact subset of a metric space is closed and bounded.*

Proposition A.2. *Let K be a subset of a metric space. The following conditions are equivalent:*

(1) *K is compact.*

(2) *K is sequentially compact.*

(3) *K is totally bounded and complete.*

Proposition A.3. *A closed subset of a complete metric space is complete.*

Exercise A.1. Let (X, d) be a compact metric space, and let $X^\infty = \prod_{k=1}^\infty X$ be endowed with the distance $D(\mathbf{x}, \mathbf{y}) = \sup_k \frac{1}{2^k} d(x^k, y^k)$, where $\mathbf{x} = (x^1, x^2, \dots), \mathbf{y} = (y^1, y^2, \dots) \in X^\infty$. Then (X^∞, D) is a compact metric space. (Show this without using the Tychonoff theorem.)

B Continuity of Correspondences

Let X and Y be topological spaces. A *correspondence* $F\colon X \to Y$ is a mapping that associates with each $x \in X$ a subset $F(x) \subset Y$. Recall that a set U is a neighborhood of a set A if there exists an open set V such that $A \subset V \subset U$.

Definition B.1. A correspondence $F\colon X \to Y$ is *upper semi-continuous* at $x \in X$ if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(U) \subset V$. F is upper semi-continuous if F is upper semi-continuous at all $x \in X$.

A correspondence $F\colon X \to Y$ is *lower semi-continuous* at $x \in X$ if for any open set V such that $F(x) \cap V \neq \emptyset$, there exists a neighborhood U of x such that $F(z) \cap V \neq \emptyset$ for all $z \in U$. F is lower semi-continuous if F is lower semi-continuous at all $x \in X$.

For $V \subset Y$, we denote

$$F^{-1}(V) = \{x \in X \mid F(x) \subset V\},$$
$$F_{-1}(V) = \{x \in X \mid F(x) \cap V \neq \emptyset\}.$$

$F^{-1}(V)$ is called the *upper inverse image* (or *strong inverse image*) of V under F, while $F_{-1}(V)$ is called the *lower inverse image* (or *weak inverse image*) of V under F. Verify that $F^{-1}(V) = X \setminus F_{-1}(Y \setminus V)$ and $F_{-1}(V) = X \setminus F^{-1}(Y \setminus V)$, and $F^{-1}(\bigcap_\lambda V_\lambda) = \bigcap_\lambda F^{-1}(V_\lambda)$ and $F_{-1}(\bigcup_\lambda V_\lambda) = \bigcup_\lambda F_{-1}(V_\lambda)$.

Proposition B.1. *Let X and Y be topological spaces. A correspondence $F\colon X \to Y$ is upper (lower, resp.) semi-continuous if and only if $F^{-1}(V)$ is open (closed, resp.) for any open (closed, resp.) subset V of Y.*

A correspondence $F\colon X \to Y$ is upper (lower, resp.) semi-continuous if and only if $F_{-1}(V)$ is closed (open, resp.) for any closed (open, resp.) subset V of Y.

Proposition B.2. *Let X and Y be topological spaces, and $F\colon T \times X \to Y$ a compact-valued upper semi-continuous correspondence. Then for any compact set $A \subset X$, $F(A)$ is compact in Y.*

Proposition B.3. *Let X, Y, and Z be topological spaces. If correspondences $F\colon X \to Y$ and $G\colon Y \to Z$ are upper semi-continuous, then the correspondence $G \circ F\colon X \to Z$ defined by $(G \circ F)(x) = \bigcup_{y \in F(x)} G(y)$ is upper semi-continuous. If, in addition, F and G are compact-valued, then $G \circ F$ is compact-valued.*

Proposition B.4. *Let X and Y be topological spaces, and T a compact topological space. If a correspondence $F\colon T \times X \to Y$ is upper semi-continuous, then the correspondence $F_T\colon X \to Y$ defined by $F_T(x) = \bigcup_{t \in T} F(t, x)$ is upper semi-continuous. If, in addition, F is compact-valued, then F_T is compact-valued.*

Proof. Let V be a neighborhood of $F_T(x)$. By the upper semi-continuity of F, for each $t \in T$ there exist open neighborhoods $W_t \subset T$ and $U_t \subset X$ of t and x, respectively, such that $F(W_t \times U_t) \subset V$. By the compactness of T, there exist $t_1, \ldots, t_K \subset T$ such that $\bigcup_{k=1}^K W_{t_k} = T$. Let $U = \bigcap_{k=1}^K U_{t_k}$, which is a neighborhood of x. Then for any $x' \in U$ and any $t \in T$, we have $F(t, x') \subset F(W_{t_k} \times U_{t_k}) \subset V$ for some t_k. ∎

The graph of F is the set

$$\{(x, y) \in X \times Y \mid y \in F(x)\}.$$

F has a closed graph, or is closed, if its graph is closed in $X \times Y$.

Proposition B.5. *Let X and Y be metric spaces. A correspondence $F\colon X \to Y$ is upper semi-continuous at x and $F(x)$ is compact if and only if for any sequence $\{x_n\} \subset X$ such that $x_n \to x$, any sequence $\{y_n\} \subset Y$ such that $y_n \in F(x_n)$ has a convergent subsequence with a limit in $F(x)$.*

Proof. The "only if" part: Almost same as the proof of the theorem that in metric spaces, compactness implies sequential compactness.

The "if" part: If F is not upper semi-continuous at x, there exists a neighborhood V of $F(x)$ such that for each n, there exist x_n and y_n such that $x_n \in B_{1/n}(x)$, $y_n \in F(x_n)$, and $y_n \notin V$. Clearly, $\{y_n\}$ has no limit point in $F(x)$. ∎

Corollary B.6. *Let X and Y be metric spaces. If a correspondence $F\colon X \to Y$ is upper semi-continuous and compact-valued, then it has a closed graph.*

Suppose that Y is compact. Then if a correspondence $F\colon X \to Y$ has a closed graph, then it is upper semi-continuous and closed- (hence compact-)valued.

The compactness of Y is indispensable. For example, let $X = Y = \mathbb{R}$, and $F(x) = \{1/x\}$ if $x \neq 0$ and $F(0) = \mathbb{R}$ (this has a closed graph, but is not compact-valued), or $F(0) = \{0\}$ (this has a closed graph, but is not upper semi-continuous).

Proposition B.7. *Let X and Y be metric spaces. A correspondence $F\colon X \to Y$ is lower semi-continuous at x if and only if for any sequence $\{x_n\} \subset X$ such that $x_n \to x$ and any $y \in F(x)$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $y_k \in F(x_{n_k})$ for each k such that $y_k \to y$.*

Proof. The "only if" part: For each k, let U_k be a neighborhood of x such that $F(z) \cap N^{1/k}(y) \neq \emptyset$ for all $z \in U_k$. Let n_k be such that $x_{n_k} \in U_k$, and let $y_k \in F(x_{n_k}) \cap N^{1/k}(y)$. Then $y_k \to y$.

The "if" part: If F is not lower semi-continuous at x, there exists an open set V with $F(x) \subset V \neq \emptyset$ such that for any neighborhood U of x, $F(z) \cap V = \emptyset$ for some $z \in U$. For each n, let $x_n \in N^{1/n}(x)$ be such that $F(x_n) \cap V = \emptyset$. Then any sequence $\{y_n\}$ such that $y_n \in F(x_n)$ has no limit point in $F(x) \cap V$. ∎

A subset of a topological space is called an F_σ set if it is written as a countable union of closed sets.[4] The following is from Aubin and Cellina (1984, Proposition 1.1.4, p.43)

Proposition B.8. *Let X be a topological space, and Y a metric space. If a correspondence $F\colon X \to Y$ is upper semi-continuous, then the lower inverse image of any open set is an F_σ set.*

Proof. Let $V \subset Y$ be open. Let $K_n = \{y \in Y \mid d(y, z) \geq 1/n \text{ for all } z \notin V\}$, which is closed. Then, $V = \bigcup_{n=1}^{\infty} K_n$. By the upper semi-continuity of F, $F_{-1}(K_n)$ is closed. Therefore, $F_{-1}(V) = \bigcup_{n=1}^{\infty} F_{-1}(K_n)$ is an F_σ set. ∎

[4] A set that is written as a countable intersection of open sets is called a G_δ set.

Exercise B.1. Let X be a topological space, Y a convex compact subset of $\mathbb{R}^n$, and for each $k = 1, 2, \ldots$, $F^k \colon X \to Y$ a compact-valued upper semi-continuous correspondence. Let $\{\lambda_k\}_{k=1}^{\infty}$ be such that $\lambda_k \geq 0$ and $\sum_k \lambda_k = 1$. Then the correspondence $F \colon X \to Y$ defined by $F(x) = \sum_k \lambda_k F^k(x)$ is a compact-valued upper semi-continuous correspondence.

C Some Facts from Functional Analysis

Definition C.1. $x \colon [0, T] \to \mathbb{R}^n$ is absolutely continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any countable collection of disjoint subintervals $[a_k, b_k]$ of $[0, T]$ such that

$$\sum (b_k - a_k) < \delta,$$

we have

$$\sum |x(b_k) - x(a_k)| < \varepsilon.$$

$x \colon [0, \infty) \to \mathbb{R}^n$ is absolutely continuous if it is absolutely continuous on $[0, T]$ for all $T > 0$.

Definition C.2. $x \colon [0, \infty) \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant $L > 0$ if

$$|x(t) - x(s)| \leq L|t - s|$$

for all $t, s \geq 0$.

Proposition C.1. *An absolutely continuous function is continuous. A Lipschitz continuous function is absolutely continuous.*

Proposition C.2. *An absolutely continuous function is differentiable almost everywhere.*

Proposition C.3. $x \colon [0, T] \to \mathbb{R}^n$ *is absolutely continuous if and only if there exists an integrable function* $v \colon [0, T] \to \mathbb{R}^n$ *such that*

$$x(t) = x(0) + \int_0^t v(s)\, ds$$

for all $t \in [0, T]$. *In this case,* $\dot{x} = v$ *almost everywhere.*

Thus, absolutely continuous functions are precisely the functions that are represented as

$$x(t) = x(0) + \int_0^t \dot{x}(s)\, ds.$$

Denote by $C([0,T],\mathbb{R}^n)$ the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$ equipped with the norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)|.$$

Proposition C.4. *$C([0,T],\mathbb{R}^n)$ is a Banach space, i.e., a complete normed vector space.*

$K \subset C([0,T],\mathbb{R}^n)$ is *uniformly bounded* if there exists $M > 0$ such that for all $x \in K$, $|x(t)| \leq M$ for all $t \in [0,T]$. K is *equicontinuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in K$, $|x(t) - x(s)| < \varepsilon$ for all $t, s \in [0,T]$ such that $|t - s| < \delta$.

Proposition C.5 (Ascoli-Arzelà theorem)**.** *$K \subset C([0,T],\mathbb{R}^n)$ is totally bounded if and only if it is uniformly bounded and equicontinuous.*

Denote by $C([0,\infty),\mathbb{R}^n)$ the space of bounded continuous functions from $[0,\infty)$ to $\mathbb{R}^n$ equipped with the topology of uniform convergence on compact intervals. This space is a Banach space with the norm

$$\|x\| = \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{t\in[0,k]} |x(t)|,$$

or

$$\|x\| = \sup_{t\in[0,\infty)} e^{-rt}|x(t)|, \quad r > 0.$$

Proposition C.6. *If $K \subset C([0,\infty),\mathbb{R}^n)$ is uniformly bounded and equicontinuous, then it is totally bounded.*

D Some Facts about Correspondences

The following statement of Kakutani's Fixed Point Theorem is from Aubin and Cellina (1984, Corollary 1.12.1, p.85).

Theorem D.1 (Kakutani's Fixed Point Theorem)**.** *Let K be a convex compact subset of a Banach space, and $F\colon K \to K$ a nonempty-, convex-, and compact-valued upper semi-continuous correspondence. Then F has a fixed point, i.e., there exists $x^* \in K$ such that $x^* \in F(x^*)$.*

Let (X,Σ) be a measurable space, and Y a topological space. A correspondence $F\colon X \to Y$ is *weakly measurable* if $F_{-1}(V) \in \Sigma$ for any open set $V \subset Y$. The following is due to Kuratowski and Ryll-Nardzewski (see, e.g., Aubin and Cellina (1984, Corollary 1.12.1, p.85) or Aliprantis and Border (2006, Theorem 18.13)).

Theorem D.2. *Let (X,Σ) be a measurable space, and Y a complete separable metric space. If a correspondence $F\colon X \to Y$ is nonempty- and closed-valued and weakly measurable, then it has a Σ-measurable selection, i.e., there exists a Σ-measurable function $f\colon X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

Given Proposition B.8, we have the following as a corollary to the above theorem.

Corollary D.3 (Measurable Selection Theorem)**.** *Let X be a topological space with its Borel algebra $\mathcal{B}_X$, and Y a complete separable metric space. If a correspondence $F\colon X \to Y$ is nonempty- and closed-valued and upper semi-continuous, then it has a $\mathcal{B}_X$-measurable selection.*

E Existence of Solutions to Differential Inclusions

Let X be a nonempty convex compact subset of $\mathbb{R}^n$, and let $G\colon X \to X$ be a nonempty-, convex-, and compact-valued upper semi-continuous correspondence. We want to prove the existence of solutions to a differential inclusion of the form

$$\dot{x} \in G(x) - x. \tag{DI}$$

A solution to (DI) with initial condition $\xi \in X$ is defined to be an *absolutely continuous* function $x\colon [0, \infty) \to X$ that satisfies $x(0) = \xi$ and

$$\dot{x}(t) + x(t) \in G(x(t)) \tag{E.1}$$

for almost all $t \geq 0$. In fact, any solution x to (DI) is Lipschitz continuous with Lipschitz constant L, where $L = \max\{|a - b| \mid a, b \in X\}$, since $|\dot{x}(t)| \leq L$ for almost all $t \geq 0$, so that $|x(t') - x(t)| \leq \int_t^{t'} |\dot{x}(s)|\, ds \leq L|t' - t|$. Hence, there is no loss of generality to define a solution to (DI) with initial condition $\xi \in X$ to be a *Lipschitz continuous* function $x\colon [0, \infty) \to X$ with Lipschitz constant L that satisfies $x(0) = \xi$ and (E.1) for almost all $t \geq 0$.

Observation E.1. *Any absolutely continuous function that satisfies* (E.1) *for almost all $t \geq 0$ is Lipschitz continuous with Lipschitz constant L.*

There are several ways to prove the existence of solution to a differential inclusion; see, e.g., Aubin and Cellina (1984) or Smirnov (2002). We economists or game theorists are familiar with the idea of associating a "solution", whatever it is, with a fixed point of some map, so here we follow such an approach.

Denote by $C^L([0, \infty), X)$ the set of Lipschitz continuos functions from $[0, \infty)$ to X with Lipschitz constant L, where $L = \max\{|a - b| \mid a, b \in X\}$, and for $\xi \in X$, denote $C^L_\xi([0, \infty), X) = \{x \in C^L([0, \infty), X) \mid x(0) = \xi\}$.

Lemma E.2. *$C^L([0, \infty), X)$ is convex and compact in $C([0, \infty), \mathbb{R}^n)$.*

Denote by $AC([0, \infty), \mathbb{R}^n)$ the set of absolutely continuous functions from $[0, \infty)$ to $\mathbb{R}^n$. For $x \in C([0, \infty), X)$, where $C([0, \infty), X)$ denotes the set of continuous functions from $[0, \infty)$ to $\mathbb{R}^n$, define

$$\begin{aligned} \beta(x) = \{y \in AC([0, \infty), \mathbb{R}^n) \mid {} & y(0) = x(0) \text{ and} \\ & \dot{y}(t) + y(t) \in G(x(t)) \text{ for almost all } t \geq 0\}. \end{aligned} \tag{E.2}$$

Denote by S_ξ the set of solutions to (DI) with initial condition $\xi \in X$, and denote $S = \bigcup_{\xi \in X} S_\xi$.

Observation E.3. *$x \in S$ if and only if x is a fixed point of β, i.e., $x \in \beta(x)$.*

To apply Kakutani's Fixed Point Theorem, we show that β is a nonempty-, convex-, and compact-valued upper semi-continuous correspondence that maps $C([0,\infty), X)$ into $C^L([0,\infty), X)$.

Lemma E.4. *For any $x \in C([0,\infty), X)$, $\beta(x) \neq \emptyset$ and $\beta(x) \subset C^L([0,\infty), X)$.*

We will use the following (see, e.g., Aubin and Cellina (1984, Theorem 0.6.3, p.21)).

Lemma E.5. *Let $A \subset \mathbb{R}^n$ be a closed convex set. Let $f\colon [0,T] \to \mathbb{R}_+$ be an integrable function such that $\int_0^T f(s)\,ds = 1$. If $v\colon [0,T] \to \mathbb{R}^n$ is an integrable function such that $v(t) \in A$ for almost all $t \in [0,T]$, then*

$$\int_0^T v(s) f(s)\,ds \in A.$$

Proof of Lemma E.4. $t \mapsto G(x(t))$ is a nonempty- and compact-valued upper semi-continuous correspondence. Thus by the Measurable Selection Theorem, there exists a measurable function $v\colon [0,\infty) \to X$ such that $v(t) \in G(x(t))$ for all $t \geq 0$. Then define $y \in AC([0,\infty)$ by $y(0) = \xi$, and

$$y(t) = e^{-t} y(0) + (1 - e^{-t}) \int_0^t v(s) \frac{e^{s-t}}{1 - e^{-t}}\,ds.$$

Then, $y(t) \in X$ for all $t \geq 0$ by Lemma E.5, y is L-Lipschitz, and $\dot{y}(t) = v(t) - y(t) \in G(x(t)) - y(t)$ for almost all $t \geq 0$. This implies that $\beta(x) \neq \emptyset$.

If $y \in \beta(x)$, then $|\dot{y}(t)| \leq L$ for almost all $t \geq 0$. Therefore, $|y(t') - y(t)| \leq \int_t^{t'} |\dot{y}(s)|\,ds \leq L|t' - t|$. Hence $y \in C^L$. ▮

Lemma E.6. *β is convex- and compact-valued and upper semi-continuous.*

Proof. Let $y, z \in \beta(x)$. For any $t \geq 0$ at which y and z are differentiable, $((1-\alpha)y + \alpha z)'(t) + ((1-\alpha)y + \alpha z)(t) = (1-\alpha)(\dot{y}(t) + y(t)) + \alpha(\dot{z}(t) + z(t)) \in G(x(t))$ by the convexity of $G(x(t))$.

Since the values are contained in the compact set C^L, it suffices to show that β has a closed graph. Let $\{x^k\}_{k=1}^\infty$ and $\{x^k\}_{k=1}^\infty$ be such that $y^k \in \beta(x^k)$, and assume that $x^k \to x \in C^L$ and $y^k \to y \in C^L$ as $k \to \infty$. Take any t_0 at which y is differentiable. We want to show that $\dot{y}(t_0) \in G(x(t_0)) - y(t_0)$.

Fix any $\varepsilon > 0$. By the upper semi-continuity of G, there exists $\delta > 0$ such that if $|\xi - x(t_0)| < \delta$ and $|\zeta - y(t_0)| < \delta$ then $G(\xi) \subset G(x(t_0)) + (\varepsilon/2)B$ and $\zeta \in y(t_0) + (\varepsilon/2)B$ so that

$$G(\xi) - \zeta \subset G(x(t_0)) - y(t_0) + \varepsilon B,$$

where B is the closed unit ball in $\mathbb{R}^n$. By the continuity of x and y, we can take an $\eta > 0$ such that

$$|x(t) - x(t_0)| < \frac{\delta}{2}, \qquad |y(t) - y(t_0)| < \frac{\delta}{2}$$

for all $t \in (t_0 - \eta, t_0 + \eta)$. Since x^k and y^k uniformly converge to x and y, respectively, there exists K such that for all $k \geq K$,

$$|x^k(t) - x(t)| < \frac{\delta}{2}, \qquad |y^k(t) - y(t)| < \frac{\delta}{2}$$

for all $t \in (t_0 - \eta, t_0 + \eta)$. Therefore, for $k \geq K$ and for almost all $t \in (t_0 - \eta, t_0 + \eta)$, we have $|x^k(t) - x(t_0)| < \delta$ and $|y^k(t) - y(t_0)| < \delta$, and hence,

$$\dot{y}^k(t) \in G(x^k(t)) - y^k(t) \subset G(x(t_0)) - y(t_0) + \varepsilon B$$

by the choice of δ. Since $\dot{y}_k \in L^1$ and $G(x(t_0)) - y(t_0) + \varepsilon B$ is closed and convex, we have

$$\frac{y^k(t_0 + h) - y^k(t_0)}{h} = \frac{1}{h} \int_{t_0}^{t_0+h} \dot{y}_k(s)\, ds \in G(x(t_0)) - y(t_0) + \varepsilon B$$

for all $k \geq K$ and all small $h > 0$ by Lemma E.5. By letting $k \to \infty$, we have

$$\frac{y(t_0 + h) - y(t_0)}{h} \in G(x(t_0)) - y(t_0) + \varepsilon B.$$

Finally, by letting $h \to 0$, we have

$$\dot{y}(t_0) \in G(x(t_0)) - y(t_0) + \varepsilon B.$$

Since $\varepsilon > 0$ has been taken arbitrarily and $G(x(t_0)) - y(t_0)$ is closed, we have $\dot{y}(t_0) \in G(x(t_0)) - y(t_0)$ as desired. ▮

Theorem E.7. *$S_\xi \neq \emptyset$ for all $\xi \in X$. S_ξ and S are compact in $C([0, \infty), \mathbb{R}^n)$.*

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