Topological Properties of Convex Sets

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This document collects some useful topological properties of convex sets. Proofs are mostly from Rockafellar (1970).

Proposition 1. *For any convex set $C \subset \mathbb{R}^N$, if $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$, then $(1-\lambda)x + \lambda y \in \operatorname{int} C$ for any $\lambda \in [0, 1)$.*

Proof. Let $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$. Fix any $\lambda \in [0, 1)$. We want to show that there exists $\bar{\varepsilon} > 0$ such that $(1-\lambda)x + \lambda y + \bar{\varepsilon} u \in C$ for all $u \in B$, where B is the unit open ball in $\mathbb{R}^N$ around 0.

Since $x \in \operatorname{int} C$, we can take an $\varepsilon_0 > 0$ such that $x + \varepsilon_0 u \in C$ for all $u \in B$. Given this $\varepsilon_0 > 0$, let $\bar{\varepsilon} = \varepsilon_0(1-\lambda)/(1+\lambda) > 0$. Since $y \in \operatorname{cl} C$, we can take $u_0 \in B$ such that $y - \bar{\varepsilon} u_0 \in C$.

Fix any $u \in B$. Let $u' = [\lambda/(1+\lambda)]u_0 + [1/(1+\lambda)]u$, where $u' \in B$ by the convexity of B. Then, $(1-\lambda)x + \lambda y + \bar{\varepsilon} u$ can be expressed as

$$\begin{aligned}
(1-\lambda)x + \lambda y + \bar{\varepsilon} u &= (1-\lambda)x + \lambda(y - \bar{\varepsilon} u_0) + \lambda \bar{\varepsilon} u_0 + \bar{\varepsilon} u \\
&= (1-\lambda)\left(x + \frac{\lambda \bar{\varepsilon}}{1-\lambda} u_0 + \frac{\bar{\varepsilon}}{1-\lambda} u\right) + \lambda(y - \bar{\varepsilon} u_0) \\
&= (1-\lambda)(x + \varepsilon_0 u') + \lambda(y - \bar{\varepsilon} u_0)
\end{aligned}$$

(note that $\bar{\varepsilon}\lambda/(1-\lambda) = \varepsilon_0 \lambda/(1+\lambda)$, $\bar{\varepsilon}/(1-\lambda) = \varepsilon_0/(1+\lambda)$ by the definition of $\bar{\varepsilon}$). Since $x + \varepsilon_0 u' \in C$ by the choice of ε_0, we have $(1-\lambda)(x + \varepsilon_0 u') + \lambda(y - \bar{\varepsilon} u_0) \in C$ by the convexity of C. ∎

Proposition 2. *For any convex set $C \subset \mathbb{R}^N$, $\operatorname{cl} C$ and $\operatorname{int} C$ are convex.*

Proof. The convexity of $\operatorname{cl} C$ follows from the formula $\operatorname{cl} C = \bigcap_{\varepsilon > 0}(C + \varepsilon B)$ (where B is the unit ball in $\mathbb{R}^N$ around 0, which is convex). The convexity of $\operatorname{int} C$ follows from Proposition 1. ∎

For $x^0, \ldots, x^m \in \mathbb{R}^N$, write

$$\operatorname{aff}(\{x^0, \ldots, x^m\}) = \{x^0 + \textstyle\sum_{i=1}^m \lambda_i (x^i - x^0) \mid \lambda_1, \ldots, \lambda_m \in \mathbb{R}\},$$

which is called the affine space spanned by $x^0, \ldots, x^m$, or the affine hull of $x^0, \ldots, x^m$.

Proposition 3. *For $x^0, \dots, x^m \in \mathbb{R}^N$, $\operatorname{aff}(\{x^0, \dots, x^m\})$ is a closed set.*

Proof. Let $A \in \mathbb{R}^{N \times \ell}$ consist of a maximal linearly independent subset of $\{x^1 - x^0, \dots, x^m - x^0\}$ (as columns). Take any sequence $\{x^k\}$ in $\operatorname{aff}(\{x^0, \dots, x^m\})$, where for each k, $x^k = x^0 + A\lambda^k$ for some $\lambda^k \in \mathbb{R}^\ell$, and assume that $x^k \to x^*$. We have $A^{\mathrm{T}}(x^k - x^0) = A^{\mathrm{T}} A \lambda^k$, where $A^{\mathrm{T}} A$ is non-singular. To see this, let $A^{\mathrm{T}} A z = 0$. Then we have $z^{\mathrm{T}} A^{\mathrm{T}} A z = 0$, or $\|Az\|^2 = 0$, which holds only if $Az = 0$. Since the columns of A are linearly independent, this holds only if $z = 0$.

Thus, we have $\lambda^k = (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}}(x^k - x^0)$. Then letting $k \to \infty$, we have $\lambda^k \to \lambda^* = (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}}(x^* - x^0)$. Hence, we have $x^* = \lim_{k \to \infty}(x^0 + A\lambda^k) = x^0 + A\lambda^* \in \operatorname{aff}(\{x^0, \dots, x^m\})$. This proves that $\operatorname{aff}(\{x^0, \dots, x^m\})$ is closed. ▮

A finite set $\{x^0, \dots, x^m\} \subset \mathbb{R}^N$ is *affinely independent* if $\{x^1 - x^0, \dots, x^m - x^0\}$ is linearly independent. The *dimension* of $C \subset \mathbb{R}^N$, denoted $\dim C$, is the largest number m such that C contains some affinely independent set $\{x^0, \dots, x^m\}$. If $\{x^0, \dots, x^m\}$ is affinely independent, then $\dim(\operatorname{aff}(\{x^0, \dots, x^m\})) = m$.

Proposition 4. *Let $C \subset \mathbb{R}^N$.*

(1) $\dim(\operatorname{cl} C) = \dim C$.

(2) *If* $\operatorname{int} C \neq \emptyset$, *then* $\dim C = N$.

(3) *Suppose that C is convex. If* $\dim C = N$, *then* $\operatorname{int} C \neq \emptyset$.

Proof. (1) Let $\{x^0, \dots, x^m\}$ be a maximal affinely independent subset of C. By maximality, $C \subset \operatorname{aff}(\{x^0, \dots, x^m\})$, and by the closedness of $\operatorname{aff}(\{x^0, \dots, x^m\}$, $\operatorname{cl} C \subset \operatorname{aff}(\{x^0, \dots, x^m\})$. This implies that $\dim(\operatorname{cl} C) \leq m = \dim C$. The converse inequality holds obviously.

(2) Suppose that $\operatorname{int} C \neq \emptyset$, and let $x^0 \in \operatorname{int} C$. We can take an $\varepsilon > 0$ such that $x^0 + \varepsilon e^i \in C$ for all $i = 1, \dots, N$, where e^i is the ith unit vector of $\mathbb{R}^N$, and $\{x^0, x^0 + \varepsilon e^1, \dots, x^0 + \varepsilon e^N\}$ is affinely independent. Hence, $\dim C = N$.

(3) Suppose that C is convex and $\dim C = N$. Let $\{x^0, \dots, x^N\} \subset C$ be affinely independent, and let $A \in \mathbb{R}^{N \times N}$ consist of $x^1 - x^0, \dots, x^N - x^0$ (as columns), where A is invertible. Denote $S = \{x \in \mathbb{R}^N \mid x_i > 0,\ \sum_{i=1}^N x_i < 1\}$, which is nonempty and open. Then $\{x^0 + \sum_{i=1}^N \lambda_i (x^i - x^0) \mid \lambda_i > 0,\ \sum_{i=1}^N \lambda_i < 1\} = \{x^0\} + A(S)$ is nonempty and open and is contained in C. Hence, $\operatorname{int} C \neq \emptyset$. ▮

Proposition 5. *For any convex set $C \subset \mathbb{R}^N$, if* $\operatorname{int}(\operatorname{cl} C) \neq \emptyset$, *then* $\operatorname{int} C \neq \emptyset$.

Proof. By Proposition 4,

$$\begin{aligned} \operatorname{int}(\operatorname{cl} C) \neq \emptyset &\implies \dim(\operatorname{cl} C) = N \\ &\implies \dim C = N \\ &\implies \operatorname{int} C \neq \emptyset, \end{aligned}$$

as claimed. ▮

Proposition 6. *For any convex set $C \subset \mathbb{R}^N$, $\operatorname{int}(\operatorname{cl} C) = \operatorname{int} C$.*

Proof. It suffices to show that $\operatorname{int}(\operatorname{cl} C) \subset \operatorname{int} C$. Suppose that $\operatorname{int}(\operatorname{cl} C) \neq \emptyset$ (otherwise the conclusion holds trivially). Then, by the convexity of C, it follows from Proposition 5 that $\operatorname{int} C \neq \emptyset$. Let $z \in \operatorname{int}(\operatorname{cl} C)$. We want to show that $z \in \operatorname{int} C$. Take any $x \in \operatorname{int} C$ ($\neq \emptyset$). Suppose that $x \neq z$ (otherwise $z \in \operatorname{int} C$ holds trivially). For $\varepsilon > 0$, let

$$y = z - \varepsilon(x - z),$$

and let ε be sufficiently small so that $y \in \operatorname{cl} C$. Then z can be written as $z = [1 - 1/(1+\varepsilon)]x + [1/(1+\varepsilon)]y$, where $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$. Therefore, by Proposition 1 it follows that $z \in \operatorname{int} C$. ▮

Remark 1. The property stated in Proposition 6 is used when reducing the weak version of the separating hyperplane theorem to the strict version, in a step that shows that if C is a convex set and $b \notin \operatorname{int} C$, then $b \notin \operatorname{int}(\operatorname{cl} C)$, so that there exists a sequence $\{b^k\}$ such that $b^k \notin \operatorname{cl} C$ and $b^k \to b$, a statement which sounds intuitive, but is not obvious at all. In fact, Propositions 5 and 6 do not hold if one drops the convexity of C. For example, let $N = 1$ and $C = \mathbb{Q} \cap [0, 1]$. Then $\operatorname{cl} C = [0, 1]$ and therefore $\operatorname{int}(\operatorname{cl} C) = (0, 1)$ and $\operatorname{bd}(\operatorname{cl} C) = \{0, 1\}$, whereas $\operatorname{int} C = \emptyset$ and $\operatorname{bd} C = [0, 1]$.

References

Rockafellar, R. T. (1970). *Convex Analysis*, Princeton University Press, Princeton.