On the Relationship between Robustness to Incomplete Information and Noise-Independent Selection in Global Games*

Daisuke Oyama

Faculty of Economics, University of Tokyo
Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
oyama@e.u-tokyo.ac.jp

and

Satoru Takahashi

Department of Economics, Princeton University
Princeton, NJ 08544, USA
satorut@princeton.edu

April 28, 2011; revised August 21, 2011

Abstract

This note demonstrates that a symmetric 3×3 supermodular game may fail to have any equilibrium robust to incomplete information. Since the global game solution in symmetric 3×3 supermodular games is known to be independent of the noise structure, this result implies that a noise-independent selection in global games may not be a robust equilibrium. Our proof reveals that the assumption in global games that the noise errors are independent of the state imposes a non-trivial restriction on incomplete information perturbations. *Journal of Economic Literature* Classification Numbers: C72, D82.

Keywords: equilibrium selection; supermodular game; incomplete information; robustness; contagion; global game.

*We would like to thank Christian Basteck, Stephen Morris, Olivier Tercieux, and Takashi Ui, and especially Atsushi Kajii for helpful comments and discussions. D. Oyama acknowledges financial support by JSPS Grant-in-Aid for Scientific Research (S).

Web page: www.oyama.e.u-tokyo.ac.jp/papers/3x3rbst-nonexistence.html.

1 Introduction

Suppose that an analyst plans to model some strategic situation with a complete information game $\mathbf{g}$ and has a Nash equilibrium a^* of $\mathbf{g}$ in hand as his prediction of the situation. While he believes that the complete information game $\mathbf{g}$ correctly describes the situation with high probability, he is also aware that there is some uncertainty about the payoffs, so that the players may play some incomplete information game close to $\mathbf{g}$. Is his prediction a^* still valid even in the presence of a small amount of incomplete information? Kajii and Morris (1997, KM henceforth) formalize this robustness question as follows: Nash equilibrium a^* of complete information game $\mathbf{g}$ is *robust to incomplete information* if *every* incomplete information game in which the payoffs are given by $\mathbf{g}$ with high probability has a Bayesian Nash equilibrium such that a^* is played with high probability. This notion allows for a very rich structure of correlated types in incomplete information perturbations, making the robustness test very stringent. Indeed, even strict Nash equilibria may fail to be robust[1] and there are games that have no robust equilibrium,[2] whereas KM and subsequent studies have obtained several sufficient conditions for an equilibrium to be robust.[3]

In this note, we demonstrate that there is a non-empty open set of symmetric 3×3 supermodular games that have no robust equilibrium. For each game in this set, we construct a sequence of dominance-solvable incomplete information perturbations in which one action profile is played everywhere and another sequence of dominance-solvable perturbations in which another action profile is played everywhere.[4] This has an important implication regarding the relationship between robust equilibrium and noise-independent selection in global games.

Global games, first developed by Carlsson and van Damme (1993) for 2×2 games and subsequently used in various economic applications,[5] offer a natural way of introducing incomplete information perturbations that gives rise to equilibrium uniqueness through a "contagion" effect, where correlation in beliefs is generated by noisy signals of the true payoff state with noise errors independent of the state. For general supermodular games, Frankel

[1]See the earlier 2×2 example by Rubinstein (1989).

[2]KM construct a $3 \times 3 \times 3$ (non-supermodular) game whose unique (strict) Nash equilibrium is not robust. Morris (1999) demonstrates non-existence of robust equilibrium in a symmetric 4×4 supermodular game.

[3]KM show that a p-dominant equilibrium with p sufficiently small is robust, while Ui (2001) shows that in potential games, a potential maximizer is robust. Morris and Ui (2005) introduce a generalized notion of potential that unifies and generalizes the p-dominance and the potential maximization conditions and show that a generalized potential maximizer is robust. See Oyama and Tercieux (2009) and Uno (2011) for further developments.

[4]The conditions that define this set of games have been found by Honda (2010) to show that these games have no monotone potential maximizer.

[5]See the survey by Morris and Shin (2003).

et al. (2003, FMP henceforth) show, with a setting with one-dimensional signals, that as the signal noise vanishes, the global games always have a unique equilibrium that survives iterative dominance, while the surviving equilibrium may depend on the noise distribution.[6] While the global game approach only considers a particular class of perturbations as opposed to KM's robustness to all elaborations,[7] in classes of games considered in the literature so far the equilibrium that is played in global games independently of the noise structure has turned out to be also robust to all elaborations.[8] This might lead one to conjecture that noise-independent selection would imply robustness in all supermodular games, or put differently, the global game perturbations would constitute a critical class of perturbations that determines whether or not an equilibrium is robust to incomplete information. Here, we say that a class of elaborations is *critical* if robustness to that class of elaborations implies robustness to all elaborations.

Our result in this note, combined with that by Basteck and Daniëls (2010), falsifies this "critical class" conjecture on global games. Basteck and Daniëls (2010) show in the setting of FMP that generic symmetric 3×3 supermodular games have a noise-independent selection in global games,[9] while the present note shows that some of these games have no robust equilibrium. Hence, the set of incomplete information perturbations that KM's concept of robustness allows is significantly richer than the set that global games generate. Moreover, the proof of our result reveals that what makes the difference is the assumption in global games that the noise errors are independent of the state. To be precise, denote by $t_i = \theta + \nu\eta_i$ the signal that each player i observes, where θ is the state of the world, η_i is the noise error which is assumed to be independent of θ, and ν is a scale parameter. This state-independence assumption implies that, conditional on a player's signal observation t_i, the posterior distribution over the difference between his signal and that of the opponent, $t_{-i}-t_i = \nu(\eta_{-i}-\eta_i)$, is (approximately) invariant in the own signal t_i when the noise parameter ν is sufficiently small. We show that this invariance property imposes a non-trivial restriction on the contagion argument. That is, in our symmetric 3×3 supermouldar games, the action that is played in global games independent of the noise distribution is never played in some incomplete information perturbations whose posterior beliefs are not necessarily invariant. These perturbations cannot be generated by one-dimensional global games as FMP consider and may in

[6]FMP provide a symmetric 4×4 example in which different equilibria survive under different noise distributions.

[7]In fact, Oury and Tercieux (2007) and Basteck et al. (2010) show that in supermodular games, a robust equilibrium is a noise-independent selection in global games.

[8]For example, the sufficient condition for noise-independent selection provided by FMP in terms of a generalized notion of potential is also sufficient for robustness (Morris and Ui (2005)).

[9]See also FMP (Section 5) for a heuristic argument with symmetric noise distributions.

effect be considered as "two-dimensional" perturbations, where types with different values in the first coordinate have very different posterior beliefs over the opponent's types.[10] We will elaborate on this point in Section 5.

2 Preliminaries

2.1 Complete Information Games

We focus on two-player games. The set of players is denoted by $\mathcal{I} = \{1, 2\}$, and for $i \in \mathcal{I}$ we write $-i$ for player $j \neq i$. Each player $i \in \mathcal{I}$ has a linearly ordered, finite set of actions $A_i = \{0, 1, \dots, n_i\}$. These action sets are fixed throughout the analysis. A complete information game is thus represented by a profile of payoff functions $\mathbf{g} = (g_i)_{i \in \mathcal{I}}$, where $g_i \colon A = \prod_{i \in \mathcal{I}} A_i \to \mathbb{R}$, $i \in \mathcal{I}$. Let $\Delta(S)$ denote the set of probability distributions over a set S. We denote by $br_i(\pi_i)$ the set of player i's pure best responses to $\pi_i \in \Delta(A_{-i})$:

$$br_i(\pi_i) = \arg\max_{a_i \in A_i} g_i(a_i, \pi_i),$$

where $g_i(a_i, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i, a_{-i})$.

Complete information game $\mathbf{g}$ is *supermodular* if for each $i \in \mathcal{I}$,

$$g_i(a_i', a_{-i}) - g_i(a_i, a_{-i}) \leq g_i(a_i', a_{-i}') - g_i(a_i, a_{-i}')$$

whenever $a_i < a_i'$ and $a_{-i} < a_{-i}'$. It is well known that the best response correspondence of a supermodular game is nondecreasing in the stochastic dominance order. For $\pi_i, \pi_i' \in \Delta(A_{-i})$, we write $\pi_i \precsim \pi_i'$ (and $\pi_i' \succsim \pi_i$) if π_i' stochastically dominates π_i, i.e., if $\sum_{a_{-i}' \geq a_{-i}} \pi_i(a_{-i}') \leq \sum_{a_{-i}' \geq a_{-i}} \pi_i'(a_{-i}')$ for all $a_{-i} \in A_{-i}$. If $\mathbf{g}$ is supermodular, then for each $i \in \mathcal{I}$,

$$\min br_i(\pi_i) \leq \min br_i(\pi_i')$$
$$\max br_i(\pi_i) \leq \max br_i(\pi_i')$$

whenever $\pi_i \precsim \pi_i'$.

2.2 ε-Elaborations and Robust Equilibria

Given the game $\mathbf{g}$, we consider the following class of incomplete information games. Each player $i \in \mathcal{I}$ has a countable set of types, denoted by T_i, and we write $T = \prod_{i \in \mathcal{I}} T_i$. The (common) prior probability distribution on T is

[10] Oury (2009) studies global games with a multi-dimensional state space while maintaining the assumption that the noise errors are independent of the state. Under this independence assumption, she shows that noise-independent selection in one-dimensional global games extends to multi-dimensional global games. Our notion of "multi-dimensionality" is different from Oury's, and in particular, our "two-dimensional" perturbations do not fall within her multi-dimensional framework.

given by P. We assume that P satisfies that $\sum_{t_{-i}\in T_{-i}} P(t_i, t_{-i}) > 0$ for all $i \in \mathcal{I}$ and $t_i \in T_i$. Under this assumption, the conditional probability of t_{-i} given t_i, $P(t_{-i}|t_i)$, is well defined by $P(t_{-i}|t_i) = P(t_i, t_{-i})/\sum_{t'_{-i}\in T_{-i}} P(t_i, t'_{-i})$. The payoff function for player $i \in \mathcal{I}$ is a bounded function $u_i \colon A \times T \to \mathbb{R}$. Denote $\mathbf{u} = (u_i)_{i\in\mathcal{I}}$. The tuple $(T, P, \mathbf{u})$ defines an incomplete information game.

A (behavioral) strategy for player i is a function $\sigma_i \colon T_i \to \Delta(A_i)$. Denote by Σ_i the set of strategies for player i, and write $\Sigma = \prod_{i\in\mathcal{I}} \Sigma_i$. For a strategy $\sigma_i \in \Sigma_i$, we denote by $\sigma_i(a_i|t_i)$ the probability that $a_i \in A_i$ is chosen at $t_i \in T_i$. For $\sigma \in \Sigma$, we write $\sigma_P \in \Delta(A)$ for the probability distribution over A generated by σ, i.e., $\sigma_P(a) = \sum_{t\in T} P(t) \prod_{i\in\mathcal{I}} \sigma_i(a_i|t_i)$ for $a = (a_i)_{i\in\mathcal{I}} \in A$.

The expected payoff to player i of type $t_i \in T_i$ playing $a_i \in A_i$ against the opponent's strategy $\sigma_{-i} \in \Sigma_{-i}$ is given by

$$U_i(a_i, \sigma_{-i}|t_i) = \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\, u_i\big((a_i, \sigma_{-i}(t_{-i})), (t_i, t_{-i})\big),$$

where $u_i((a_i, \sigma_{-i}(t_{-i})), t) = \sum_{a_{-i}\in A_{-i}} \sigma_{-i}(a_{-i}|t_{-i}) u_i((a_i, a_{-i}), t)$. Let $BR_i(\sigma_{-i}|t_i)$ denote the set of pure best responses of player i of type $t_i \in T_i$ against $\sigma_{-i} \in \Sigma_{-i}$:

$$BR_i(\sigma_{-i}|t_i) = \arg\max_{a_i\in A_i} U_i(a_i, \sigma_{-i}|t_i).$$

A strategy profile $\sigma \in \Sigma$ is a *Bayesian Nash equilibrium* of $(T, P, \mathbf{u})$ if for all $i \in \mathcal{I}$, all $a_i \in A_i$, and all $t_i \in T_i$,

$$\sigma_i(a_i|t_i) > 0 \Rightarrow a_i \in BR_i(\sigma_{-i}|t_i).$$

Given $\mathbf{g}$, let $T_i^{g_i}$ be the set of types t_i such that payoffs of player i of type t_i are given by g_i and he knows his payoffs:

$$\begin{aligned} T_i^{g_i} = \{t_i \in T_i \mid u_i(a, (t_i, t_{-i})) = g_i(a) \\ \text{for all } a \in A \text{ and all } t_{-i} \in T_{-i} \text{ with } P(t_i, t_{-i}) > 0\}. \end{aligned}$$

Denote $T^{\mathbf{g}} = \prod_{i\in\mathcal{I}} T_i^{g_i}$.

Definition 1. Let $\varepsilon \in [0, 1]$. An incomplete information game $(T, P, \mathbf{u})$ is an *ε-elaboration* of $\mathbf{g}$ if $P(T^{\mathbf{g}}) = 1 - \varepsilon$.

Following KM, we say that an action distribution $\mu \in \Delta(A)$ is robust if, for small $\varepsilon > 0$, every ε-elaboration of $\mathbf{g}$ has a Bayesian Nash equilibrium σ such that the action distribution it generates, σ_P, is close to μ.

Definition 2. Action distribution $\mu \in \Delta(A)$ is *robust to incomplete information* in $\mathbf{g}$ if for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \le \bar{\varepsilon}$, any ε-elaboration $(T, P, \mathbf{u})$ of $\mathbf{g}$ has a Bayesian Nash equilibrium σ such that $\max_{a\in A} |\mu(a) - \sigma_P(a)| \le \delta$.

If $\mu \in \Delta(A)$ is robust in $\mathbf{g}$, then it must be a correlated equilibrium of $\mathbf{g}$ (KM, Corollary 3.5). We say that an action profile $a \in A$ is robust in $\mathbf{g}$ if the degenerate action distribution on a (i.e., $\mu \in \Delta(A)$ such that $\mu(a) = 1$) is robust in $\mathbf{g}$.

Given $\sigma_{-i} \in \Sigma_{-i}$, let $\pi_i(\sigma_{-i}|t_i) \in \Delta(A_{-i})$ be the belief of player i of type t_i over the opponent's actions, i.e.,

$$\pi_i(\sigma_{-i}|t_i)(a_{-i}) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}(a_{-i}|t_{-i})$$

for $a_{-i} \in A_{-i}$. Observe that

$$BR_i(\sigma_{-i}|t_i) = br_i(\pi_i(\sigma_{-i}|t_i))$$

for all $t_i \in T_i^{g_i}$.

Several sufficient conditions for robustness to incomplete information have been obtained. In particular, Morris and Ui (2005) introduce generalized notions of potential and show, among others, that a *monotone potential maximizer* (*MP-maximizer*), a special form of their generalized potential maximizer concept, is robust in supermodular games (and in games that admit a monotone potential function that is supermodular). Their result unifies and generalizes the previous results by KM in terms of p-dominance and by Ui (2001) in terms of potential maximization. On the other hand, Morris (1999) presents an example of a symmetric 4×4 supermodular game that has no robust equilibrium.

3 Result

We now restrict our attention to 3×3 games, so that $A_1 = A_2 = \{0, 1, 2\}$, and assume that $\mathbf{g}$ is supermodular. The game $\mathbf{g}$ is symmetric if $g_1(h, k) = g_2(k, h)$ for all $h, k \in \{0, 1, 2\}$. We associate a symmetric 3×3 game with an element in $\mathbb{R}^9$. We prove the following:

Proposition 1. *There is a non-empty open set of symmetric 3×3 supermodular games that have no robust equilibrium.*

Here, openness is relative to $\mathbb{R}^9$.[11]

The proof proceeds as follows. In Lemma 1, we present a condition under which there is a sequence of ε-elaborations with a unique Bayesian Nash equilibrium where action 2 is played everywhere. It implies that if the game satisfies this condition, no action distribution other than (the degenerate distribution on) $(2, 2)$ is robust. In Lemma 2, we then present a condition under which there is a sequence of ε-elaborations with a unique

[11] In fact, the non-existence obtains in an open neighborhood of this set relative to $\mathbb{R}^{18}$, including some asymmetric 3×3 games.

Bayesian Nash equilibrium where action 0 is played everywhere. Thus, if the game satisfies this condition, no action distribution other than $(0,0)$ is robust. Proposition 1 follows from the fact that there is a non-empty open set of symmetric 3×3 supermodular games that satisfy the conditions in Lemmata 1 and 2 simultaneously.[12]

In fact, these conditions have been found (and shown to be satisfied by some symmetric games) by Honda (2010) as a sufficient condition for a 3×3 game to have no MP-maximizer. He shows by direct computation that these conditions imply non-existence of an MP-maximizer. Since, as shown by Morris and Ui (2005), an MP-maximizer is robust, our non-existence result of robust equilibrium gives an indirect, alternative proof of the non-existence of MP-maximizer.

4 Proof

Let, for $p \in (0, 1/2)$,

$$\pi^a = \left(\tfrac{1}{2}, p, \tfrac{1}{2} - p\right), \quad \pi^b = \left(\tfrac{1}{2} - p, p, \tfrac{1}{2}\right),$$

and for $q, r \in (0,1)$, $r \leq q$,

$$\pi^c = \left(\tfrac{1+q}{2}, 0, \tfrac{1-q}{2}\right), \quad \pi^d = \left(\tfrac{1-r}{2}, 0, \tfrac{1+r}{2}\right), \quad \pi^e = \left(0, \tfrac{q+r}{2q}, \tfrac{q-r}{2q}\right).$$

The conditions are stated in terms of best responses to these beliefs.

Lemma 1. *If there exists $p \in (0, 1/2)$ such that*

$$\min br_i(\pi^a) \geq 1, \quad \min br_i(\pi^b) = 2, \tag{1}$$

then for all $\varepsilon > 0$, there exists an ε-elaboration where the strategy profile σ^* such that $\sigma_i^*(2|t_i) = 1$ for all $t_i \in T_i$ is the unique Bayesian Nash equilibrium.

Lemma 2. *If there exist $q, r \in (0,1)$ with $r \leq q$ such that*

$$\max br_i(\pi^c) = 0, \quad \max br_i(\pi^d) \leq 1, \quad \max br_i(\pi^e) = 0, \tag{2}$$

then for all $\varepsilon > 0$, there exists an ε-elaboration where the strategy profile σ^* such that $\sigma_i^*(0|t_i) = 1$ for all $t_i \in T_i$ is the unique Bayesian Nash equilibrium.

Our constructions of the desirable elaborations exploit the subtle structure of best responses to the above beliefs (π^a and π^b in Lemma 1 and π^c through π^e in Lemma 2) and are more involved than, for example, the

[12] In these games, action profiles $(0,0)$ and $(2,2)$ are the only pure Nash equilibria. Oyama and Takahashi (2009) show that symmetric 3×3 supermodular *coordination* games, where the three symmetric action profiles are all Nash equilibria, generically have an MP-maximizer and hence a robust equilibrium by Morris and Ui (2005).

construction of KM (Lemma 5.5), in which they demonstrate contagion of a strict **p**-dominant equilibrium with $\mathbf{p} = (p_i)_{i\in\mathcal{I}} \in [0,1)^{\mathcal{I}}$ such that $\sum_{i\in\mathcal{I}} p_i \leq 1$.[13] As in the definition of **p**-dominance, KM's construction exploits the "coarse" property of the equilibrium action being the best response to all the beliefs that assign at least probability p_i to the opponent's equilibrium action, and thus along the contagion it suffices to confirm that each type of the players assigns at least p_i to the opponent playing the equilibrium action. Consequently, the contagion argument in KM effectively treats the game as a binary game, where each player plays either the equilibrium action or "the other actions", so that the **p**-dominance condition pins down the behavior of the players in one step for each type. In contrast, our contagion argument will proceed with two steps, where in each step, the player's belief over the opponent's actions will turn out to be larger (in the stochastic dominance order) than π^a or π^b in Lemma 1, or smaller than π^c, π^d, or π^e in Lemma 2, thereby the condition (1) or (2) will narrow down the behavior of the players to a single action. See the proofs below for details.

Proof of Lemma 1. Let $p \in (0, 1/2)$ be such that condition (1) is satisfied. We construct a sequence of elaborations $(T, P^\varepsilon, \mathbf{u})_{\varepsilon>0}$, where $P^\varepsilon(T^{\mathbf{g}}) \to 1$ as $\varepsilon \to 0$, as follows. Let $T_i = \mathbb{Z}_+$ for each $i \in \mathcal{I}$. Define $P^\varepsilon \in \Delta(T)$ by

$$\begin{aligned} P^\varepsilon(\tau+1,\tau) &= P^\varepsilon(\tau,\tau+1) = p\varepsilon(1-\varepsilon)^{\tau-1},\ \tau \geq 1, \\ P^\varepsilon(\tau+2,\tau) &= P^\varepsilon(\tau,\tau+2) = \left(\frac{1}{2} - p\right)\varepsilon(1-\varepsilon)^{\tau},\ \tau \geq 0, \end{aligned}$$

and $P^\varepsilon(t_1, t_2) = 0$ otherwise; see Table 1. Define $u_i \colon A \times T \to \mathbb{R}$ for each $i \in \mathcal{I}$ by

$$u_i(a,t) = \begin{cases} g_i(a) & \text{if } t_i \neq 0, 1, \\ 1 & \text{if } t_i = 0, 1 \text{ and } a_i = 2, \\ 0 & \text{if } t_i = 0, 1 \text{ and } a_i \neq 2. \end{cases}$$

That is, types 0 and 1 are "crazy types" for which action 2 is a dominant action, and $T_i^{g_i} = \mathbb{Z}_+ \setminus \{0, 1\}$. (The constructed elaboration is an $\varepsilon\{1 + (1-2p)(1-\varepsilon)\}$-elaboration.)

Observe that the posterior beliefs generated by P^ε are invariant in translation, i.e., $P^\varepsilon(t_{-i} = \tau' + 1 | t_i = \tau + 1) = P^\varepsilon(t_{-i} = \tau' | t_i = \tau)$ for all $\tau \geq 2$ and $\tau' \geq 0$. We will use the following relationships between posterior probabilities and beliefs π^a and π^b:

[13] An action profile a^* is a strict **p**-dominant equilibrium if for each $i \in \mathcal{I}$, $br_i(\pi_i) = \{a_i^*\}$ holds for all $\pi_i \in (A_{-i})$ such that $\pi_i(a_{-i}^*) > p_i$.

(a) $\pi_i \succsim \pi^a = (1/2, p, 1/2 - p)$ for all $\pi_i \in \Delta(A_{-i})$ such that

$$\pi_i(2) \geq P^\varepsilon(t_{-i} = \tau - 1 | t_i = \tau + 1) = \frac{1/2 - p}{1 - \varepsilon/2},$$
$$\pi_i(1) + \pi_i(2) \geq P^\varepsilon(t_{-i} = \tau - 1, \tau | t_i = \tau + 1) = \frac{1/2}{1 - \varepsilon/2};$$

(b) $\pi_i \succsim \pi^b = (1/2 - p, p, p, 1/2)$ for all $\pi_i \in \Delta(A_{-i})$ such that

$$\pi_i(2) \geq P^\varepsilon(t_{-i} = \tau - 2, \tau - 1 | t_i = \tau) = \frac{1/2}{1 - \varepsilon/2},$$
$$\pi_i(1) + \pi_i(2) \geq P^\varepsilon(t_{-i} = \tau - 2, \tau - 1, \tau + 1 | t_i = \tau) = \frac{1/2 + p - p\varepsilon}{1 - \varepsilon/2}.$$

We want to show that $(T, P^\varepsilon, \mathbf{u})$ has a unique Bayesian Nash equilibrium, which plays action 2 everywhere. Consider any Bayesian Nash equilibrium σ^* of $(T, P^\varepsilon, \mathbf{u})$. We show by induction that

$$\sigma_i^*(2|\tau - 2) = \sigma_i^*(2|\tau - 1) = 1 \text{ and } \sigma_i^*(0|\tau) = 0, \ i = 1, 2 \qquad (*_\tau)$$

for all $\tau \geq 2$. We note that by the assumption (1) and the supermodularity of g_i, for any $t_i \in T_i^{g_i}$,

$$\min BR_i(\sigma_{-i}^*|t_i) \geq 1 \text{ if } \pi_i(\sigma_{-i}^*|t_i) \succsim \pi^a, \qquad (3)$$
$$\min BR_i(\sigma_{-i}^*|t_i) = 2 \text{ if } \pi_i(\sigma_{-i}^*|t_i) \succsim \pi^b. \qquad (4)$$

We first show $(*_2)$. Indeed, $\sigma_i^*(2|0) = \sigma_i^*(2|1) = 1$ by construction, and therefore, type $t_i = 2$ assigns at least probability $P^\varepsilon(t_{-i} = 0, 1 | t_i = 2)$ to the opponent playing action 2, so that $\pi_i(\sigma_{-i}^*|2) \succsim \pi^a$. Thus, $\sigma_i^*(0|2) = 0$ by (3).

Assume $(*_\tau)$. Then, type $t_i = \tau + 1$ assigns at least probability $P^\varepsilon(t_{-i} = \tau - 1 | t_i = \tau + 1)$ to the opponent playing 2 and at least probability $P^\varepsilon(t_{-i} = \tau - 1, \tau | t_i = \tau + 1)$ to the opponent playing 1 or 2. Therefore, we have $\pi_i(\sigma_{-i}^*|\tau + 1) \succsim \pi^a$ (recall (a)), so that $\sigma_i^*(0|\tau + 1) = 0$ by (3). Given this, go back to type $t_i = \tau$. This type now assigns at least probability $P^\varepsilon(t_{-i} = \tau - 2, \tau - 1, \tau + 1 | t_i = \tau)$ to the opponent playing 1 or 2 (and at least probability $P^\varepsilon(t_{-i} = \tau - 2, \tau - 1 | t_i = \tau)$ to the opponent playing 2). Therefore, we have $\pi_i(\sigma_{-i}^*|\tau) \succsim \pi^b$ (recall (b)), so that $\sigma_i^*(2|\tau) = 1$ by (4). Thus, $(*_{\tau+1})$ holds. ∎

Proof of Lemma 2. Let $q, r \in (0, 1)$, $r \leq q$, be such that condition (2) is satisfied. We construct a sequence of elaborations $(T, P^\varepsilon, \mathbf{u})_{\varepsilon > 0}$, where $P^\varepsilon(T^{\mathbf{g}}) \to 1$ as $\varepsilon \to 0$, as follows. Let $T_i = \{\alpha, \beta\} \times \mathbb{Z}_+$ for each $i \in \mathcal{I}$.

Define $P^\varepsilon \in \Delta(T)$ by

$$P^\varepsilon((\alpha,\tau+1),(\alpha,\tau)) = P^\varepsilon((\alpha,\tau),(\alpha,\tau+1)) = \frac{1-q}{2(1+q)}\varepsilon(1-\varepsilon)^\tau,$$
$$P^\varepsilon((\alpha,\tau),(\beta,\tau)) = P^\varepsilon((\beta,\tau),(\alpha,\tau)) = \frac{q+r}{2(1+q)}\varepsilon(1-\varepsilon)^\tau,$$
$$P^\varepsilon((\alpha,\tau+1),(\beta,\tau)) = P^\varepsilon((\beta,\tau),(\alpha,\tau+1)) = \frac{q-r}{2(1+q)}\varepsilon(1-\varepsilon)^\tau,$$

and $P^\varepsilon(t_1,t_2)=0$ otherwise; see Table 2. Define $u_i\colon A\times T\to\mathbb{R}$ for each $i\in\mathcal{I}$ by

$$u_i(a,t) = \begin{cases} g_i(a) & \text{if } t_i \neq (\alpha,0), \\ 1 & \text{if } t_i = (\alpha,0) \text{ and } a_i = 0, \\ 0 & \text{if } t_i = (\alpha,0) \text{ and } a_i \neq 0. \end{cases}$$

That is, type $(\alpha,0)$ is a "crazy type" for which action 0 is a dominant action, and $T_i^{g_i} = (\{\alpha,\beta\}\times\mathbb{Z}_+)\setminus\{(\alpha,0)\}$. (The constructed elaboration is an $\varepsilon(1+r)/(1+q)$-elaboration.)

Observe that the posterior beliefs generated by P^ε are invariant in translation in the second coordinate, i.e., $P^\varepsilon(t_{-i}=(\cdot,\tau'+1)|t_i=(\alpha,\tau+1)) = P^\varepsilon(t_{-i}=(\cdot,\tau')|t_i=(\alpha,\tau))$ for all $\tau\geq 1$ and $\tau'\geq 0$ and $P^\varepsilon(t_{-i} = (\cdot,\tau'+1)|t_i=(\beta,\tau+1)) = P^\varepsilon(t_{-i}=(\cdot,\tau')|t_i=(\beta,\tau))$ for all $\tau,\tau'\geq 0$. We will use the following relationships between posterior probabilities and beliefs π^c, π^d, and π^e:

(c) $\pi_i \precsim \pi^c = ((1+q)/2, 0, (1-q)/2)$ for all $\pi_i\in\Delta(A_{-i})$ such that

$$\begin{aligned}\pi_i(0) &\geq P^\varepsilon(t_{-i}=(\alpha,\tau),(\beta,\tau),(\beta,\tau+1)|t_i=(\alpha,\tau+1)) \\ &= \frac{1+q-(q+r)\varepsilon}{2-(1+r)\varepsilon};\end{aligned}$$

(d) $\pi_i \precsim \pi^d = ((1-r)/2, 0, (1+r)/2)$ for all $\pi_i\in\Delta(A_{-i})$ such that

$$\pi_i(0) \geq P^\varepsilon(t_{-i}=(\alpha,\tau),(\beta,\tau)|t_i=(\alpha,\tau+1)) = \frac{1-r}{2-(1+r)\varepsilon};$$

(e) $\pi_i \precsim \pi^e = (0, (q+r)/(2q), (q-r)/(2q))$ for all $\pi_i\in\Delta(A_{-i})$ such that

$$\pi_i(0)+\pi_i(1) \geq P^\varepsilon(t_{-i}=(\alpha,\tau+1)|t_i=(\beta,\tau+1)) = \frac{q+r}{2q}.$$

We want to show that $(T,P^\varepsilon,\mathbf{u})$ has a unique Bayesian Nash equilibrium, which plays action 0 everywhere. Consider any Bayesian Nash equilibrium σ^* of $(T,P^\varepsilon,\mathbf{u})$. We show by induction that

$$\sigma_i^*(0|(\alpha,\tau)) = \sigma_i^*(0|(\beta,\tau)) = 1,\ i=1,2 \qquad (\star_\tau)$$

for all $\tau \geq 0$. We note that by the assumption (2) and the supermodularity of g_i, for any $t_i \in T_i^{g_i}$,

$$\max BR_i(\sigma^*_{-i}|t_i) = 0 \ \text{ if } \pi_i(\sigma^*_{-i}|t_i) \precsim \pi^c, \tag{5}$$

$$\max BR_i(\sigma^*_{-i}|t_i) \leq 1 \ \text{ if } \pi_i(\sigma^*_{-i}|t_i) \precsim \pi^d, \tag{6}$$

$$\max BR_i(\sigma^*_{-i}|t_i) = 0 \ \text{ if } \pi_i(\sigma^*_{-i}|t_i) \precsim \pi^e. \tag{7}$$

We first show ($\star_0$). Indeed, $\sigma_i^*(0|(\alpha, 0)) = 1$ by construction, and therefore, type $t_i = (\beta, 0)$ assigns at least probability $P^\varepsilon(t_{-i} = (\alpha, 0)|t_i = (\beta, 0))$ to the opponent playing action 0, so that $\pi_i(\sigma^*_{-i}|(\beta, 0)) \precsim \pi^e$. Thus, $\sigma_i^*(0|(\beta, 0)) = 1$ by (7).

Assume ($\star_\tau$). Then, type $t_i = (\alpha, \tau + 1)$ assigns at least probability $P^\varepsilon(t_{-i} = (\alpha, \tau), (\beta, \tau)|t_i = (\alpha, \tau + 1))$ to the opponent playing 0, and therefore, we have $\pi_i(\sigma^*_{-i}|(\alpha, \tau+1)) \precsim \pi^d$ (recall (d)), so that $\sigma_i^*(2|(\alpha, \tau+1)) = 0$ by (6). Then, type $t_i = (\beta, \tau + 1)$ assigns at least probability $P^\varepsilon(t_{-i} = (\alpha, \tau + 1)|t_i = (\beta, \tau + 1))$ to the opponent playing 0 or 1, and therefore, we have $\pi_i(\sigma^*_{-i}|(\beta, \tau + 1)) \precsim \pi^e$ (recall (e)), so that $\sigma_i^*(0|(\beta, \tau + 1)) = 1$ by (7). Therefore, going back to type $t_i = (\alpha, \tau + 1)$, since this type now assigns at least probability $P^\varepsilon(t_{-i} = (\alpha, \tau), (\beta, \tau), (\beta, \tau + 1)|t_i = (\alpha, \tau + 1))$ to the opponent playing 0, we have $\pi_i(\sigma^*_{-i}|(\alpha, \tau + 1)) \precsim \pi^c$ (recall (c)), so that $\sigma_i^*(0|(\alpha, \tau + 1)) = 1$ by (5). Thus, ($\star_{\tau+1}$) holds. ▮

We close the proof of Proposition 1 by presenting two examples that satisfy the hypotheses of Lemmata 1 and 2 simultaneously. Example 1 is taken from Honda (2010). Example 2 presents a game involving some economic context, the so-called "Bilingual Game" studied by Galesloot and Goyal (1997), Goyal and Janssen (1997), and Oyama and Takahashi (2010), among others. Clearly, the conditions will continue to be satisfied with small perturbations of the payoffs.

Example 1 (Honda (2010))**.** Let the game **g** be given by

	0	1	2
0	13, 13	3, 5	0, 0
1	5, 3	0, 0	13, 2
2	0, 0	2, 13	16, 16

where (0, 0) and (2, 2) are the only pure Nash equilibria. One can verify that conditions 1 and 2 in Lemmata 1 and 2 are satisfied (with equalities) for $p \in (1/7, 5/32)$ and for q and r such that $q > 5/21$, $r < 1/4$, and $(15/17)q < r \leq q$, respectively (Honda (2010, Example 1)). This game thus has no robust equilibrium.

Example 2 (Bilingual Game)**.** Two players are to choose between two computer programing languages, or two types of technologies in general, A and B. Assume that A is more efficient while B is less risky: if both players choose A, then they each receive a payoff of 11, while if both choose B, then they both receive 10; if they choose different options, then the A-player receives 0, while the B-player receives 3. Thus, (A, A) Pareto-dominates (B, B), while (B, B) pairwise risk-dominates (A, A). In this 2×2 coordination game, the risk-dominant, and Pareto-dominated, equilibrium (B, B) is robust to incomplete information.

Now suppose that a "bilingual option", or compatible technology, AB is available with some cost $e > 0$. An AB-player adopts A against an A-player to receive a (gross) payoff 11 and adopts B against a B-player to receive 10. If both players choose AB, then they use the efficient option A and receive 11. This situation is described by

	0	1	2
0	$11, 11$	$11, 11 - e$	$0, 3$
1	$11 - e, 11$	$11-e, 11-e$	$10 - e, 10$
2	$3, 0$	$10, 10 - e$	$10, 10$

where the actions A, AB, and B are denoted 0, 1, and 2, respectively, and with the order $0 < 1 < 2$ the game is supermodular. The profiles (A, A) and (B, B) are the only pure Nash equilibria of this game (against AB, A is the best response).

For this game, it is conceivable that if the cost e is large so that AB is too costly, then the game is strategically similar to the original 2×2 game, and thus the pairwise risk-dominant equilibrium (B, B) will be robust, while if e is small enough, then B will tend to be abandoned in the presence of the even less risky option AB, and thus the efficient option A will be robust. In fact, by Oyama and Takahashi (2010), it turns out that if $e > 40/19$, then (B, B) is an MP-maximizer and hence a robust equilibrium, while if $e < 5/3$, then (A, A) is an MP-maximizer and hence a robust equilibrium; in the middle case when $5/3 < e < 40/19$, the conditions in Lemmata 1 and 2 are simultaneously satisfied and therefore the game has no robust equilibrium.

5 Discussion

Let us discuss the relation to global-game noise-independent selection. Global games, first introduced by Carlsson and van Damme (1993) for binary games, represent an important class of incomplete information games in which equilibrium uniqueness arises through contagion effects along higher order beliefs. General supermodular games (with many players and many actions) are studied by FMP in the following setting. A state of the world θ is drawn from the real line and determines the payoffs of the players, and each player observes a noisy signal $\theta+\nu\eta_i$, where η_i is a noise error that is independent of the state θ, and $\nu > 0$ is a scale parameter. It is assumed that the payoff differences are monotone in opponents' actions (supermodularity) and in the state θ (state monotonicity), and the players have a dominant action when θ is sufficiently small or large (dominance regions). In this setting, FMP show that as the signal noise vanishes, the game has a unique equilibrium that survives iterative dominance, while the selected equilibrium may depend on the noise distribution. FMP provide a symmetric 4×4 example in which different equilibria survive depending on the noise distribution. They also provide sufficient conditions for the selection to be noise-independent. In particular, they give a heuristic argument that generic symmetric 3×3 supermodular games have a noise-independent selection, which is formally proved by Basteck and Daniëls (2010).

The crucial difference between the robustness and the global game approaches (besides the technical difference whether the type space is discrete or continuous) is that the latter considers a certain subclass of payoff perturbations, while the former allows for all perturbations. In fact, as proved by Oury and Tercieux (2007) and Basteck et al. (2010), robustness implies global-game noise-independent selection in supermodular games: i.e., if an action profile is robust in the game given by the payoffs at θ, then it must be played at θ in the global game independently of the noise distribution.[14]

Our Proposition 1 shows that the converse of this result does not hold: we demonstrated non-existence of robust equilibrium in symmetric 3×3 supermodular games, a class of games that admit noise independence in global games, thus implying that a global-game noise-independent selection may not be a robust equilibrium. That is, the global game perturbation is in general not the only possible perturbation that yields a unique equilibrium outcome.

Corollary 2. *A global-game noise-independent selection may not be a robust equilibrium.*

More specifically, the result by Basteck and Daniëls (2010) in fact shows that if the game $\mathbf{g}$ satisfies the condition (1) in Lemma 1, then $(2, 2)$ is the global game selection of $\mathbf{g}$. The incomplete information elaboration we constructed in the proof of Lemma 1 can thus be seen as a type space representation of a global game with some noise structure with a one-dimensional state space as in FMP. On the other hand, their noise-independence result implies that the elaboration we constructed in the proof of Lemma 2 for

[14]See Morris and Shin (2003, Section 4.5) for a heuristic argument for this claim.

contagion of $(0,0)$ cannot be generated by a global game perturbation. In global games, due to the assumption that the noise term is independent of the state θ, a player's belief, given a signal observation, over the difference between his signal and that of the opponent is (approximately) invariant in the signal value when the noise is sufficiently small. The "one-dimensional" elaboration in the proof of Lemma 1 has the corresponding property that each player's beliefs (except for the boundary types 0 and 1) over the opponent's are invariant up to translation. This property is shared, to the best of our knowledge, by all the existing contagion constructions in the robustness literature (e.g., KM (Lemma 5.5) and Morris (1999, Section 7)). By contrast, the "two-dimensional" elaboration in the proof of Lemma 2 does not satisfy this property, where the posterior beliefs are invariant only in translation $\tau \mapsto \tau + 1$ in the second coordinate and the beliefs of types (α, τ) and those of types (β, τ) are entirely different. Such an elaboration allowed us to obtain the contagion that would not occur in the perturbations generated by global games with state-independent noise errors. To conclude, it is the state-independence assumption on the noise errors that delineates the boundary of the class of global game perturbations.

References

Basteck, C. and T. R. Daniëls (2010). "Every Symmetric 3×3 Global Game of Strategic Complementarities is Noise Independent," SFB 649 Discussion Paper 2010-061, Humboldt University.

Basteck, C., T. R. Daniëls, and F. Heinemann (2010). "Characterising Equilibrium Selection in Global Sames with Strategic Complementarities," SFB 649 Discussion Paper 2010-008, Humboldt University.

Carlsson, H. and E. van Damme (1993). "Global Games and Equilibrium Selection," *Econometrica* **61**, 989-1018.

Frankel, D. M., S. Morris, and A. Pauzner (2003). "Equilibrium Selection in Global Games with Strategic Complementarities," *Journal of Economic Theory* **108**, 1-44.

Galesloot, B. M. and S. Goyal (1997). "Costs of Flexibility and Equilibrium Selection," *Journal of Mathematical Economics* **28**, 249-264.

Goyal, S. and M. C. W. Janssen (1997). "Non-Exclusive Conventions and Social Coordination," *Journal of Economic Theory* **77**, 34-57.

Honda, J. (2010). "Noise-Independent Selection in Global Games and Monotone Potential Maximizer: A Symmetric 3×3 Example," forthcoming in *Journal of Mathematical Economics*.

Kajii, A. and S. Morris (1997). "The Robustness of Equilibria to Incomplete Information," *Econometrica* **65**, 1283-1309.

Morris, S. (1999). "Potential Methods in Interaction Games," mimeo.

Morris, S. and H. S. Shin (2003). "Global Games: Theory and Applications," in M. Dewatripont, L. P. Hansen, and S. J. Turnovsky, eds., *Advances in Economics and Econometrics: Theory and Applications: Eighth World Congress, Volume 1*, Cambridge University Press, Cambridge.

Morris, S. and T. Ui (2005). "Generalized Potentials and Robust Sets of Equilibria," *Journal of Economic Theory* **124**, 45-78.

Oury, M. (2009). "Multidimensional Global Games," mimeo.

Oury, M. and O. Tercieux (2007). "Contagion in Games with Strategic Complementarities," mimeo.

Oyama, D. and S. Takahashi (2009). "Monotone and Local Potential Maximizers in Symmetric 3×3 Supermodular Games," *Economics Bulletin* **29**, 2132-2144.

Oyama, D. and S. Takahashi (2010). "Contagion and Uninvadability in Social Networks with Bilingual Option," ssrn.com/abstract=1846531.

Oyama, D. and O. Tercieux (2009). "Iterated Potential and Robustness of Equilibria," *Journal of Economic Theory* **144**, 1726-1769.

Rubinstein, A. (1989). "The Electronic Mail Game: Strategic Behavior under 'Almost Common Knowledge'," *American Economic Review* **79**, 385-391.

Ui, T. (2001). "Robust Equilibria of Potential Games," *Econometrica* **69**, 1373-1380.

Uno, H. (2011). "Nested Potentials and Robust Equilibria," CORE Discussion Paper 2011/9.

$t_1 \backslash t_2$	0	1	2	3	4	5	$\cdots$
0			$\left(\frac{1}{2}-p\right)\varepsilon$				
1			$p\varepsilon$	$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)$			
2	$\left(\frac{1}{2}-p\right)\varepsilon$	$p\varepsilon$		$p\varepsilon(1-\varepsilon)$	$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)^2$		
3		$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)$	$p\varepsilon(1-\varepsilon)$		$p\varepsilon(1-\varepsilon)^2$	$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)^3$	
$\vdots$			$\ddots$	$\ddots$		$\ddots$	$\ddots$

Table 1: Contagion of action 2

$t_1 \backslash t_2$	$(\alpha,0)$	$(\alpha,1)$	$(\alpha,2)$	$(\alpha,3)$	$(\alpha,4)$	$\cdots$
$(\alpha,0)$		$Q\varepsilon$				
$(\alpha,1)$	$Q\varepsilon$		$Q\varepsilon(1-\varepsilon)$			
$(\alpha,2)$		$Q\varepsilon(1-\varepsilon)$		$Q\varepsilon(1-\varepsilon)^2$		
$(\alpha,3)$			$Q\varepsilon(1-\varepsilon)^2$		$Q\varepsilon(1-\varepsilon)^3$	
$\vdots$				$\ddots$		$\ddots$

$t_1 \backslash t_2$	$(\beta,0)$	$(\beta,1)$	$(\beta,2)$	$(\beta,3)$	$\cdots$
$(\alpha,0)$	$R_0\varepsilon$				
$(\alpha,1)$	$R_1\varepsilon$	$R_0\varepsilon(1-\varepsilon)$			
$(\alpha,2)$		$R_1\varepsilon(1-\varepsilon)$	$R_0\varepsilon(1-\varepsilon)^2$		
$(\alpha,3)$			$R_1\varepsilon(1-\varepsilon)^2$	$R_0\varepsilon(1-\varepsilon)^3$	
$\vdots$				$\ddots$	$\ddots$

$t_1 \backslash t_2$	$(\alpha,0)$	$(\alpha,1)$	$(\alpha,2)$	$(\alpha,3)$	$(\alpha,4)$	$\cdots$
$(\beta,0)$	$R_0\varepsilon$	$R_1\varepsilon$				
$(\beta,1)$		$R_0\varepsilon(1-\varepsilon)$	$R_1\varepsilon(1-\varepsilon)$			
$(\beta,2)$			$R_0\varepsilon(1-\varepsilon)^2$	$R_1\varepsilon(1-\varepsilon)^2$		
$(\beta,3)$				$R_0\varepsilon(1-\varepsilon)^3$	$R_1\varepsilon(1-\varepsilon)^3$	
$\vdots$					$\ddots$	$\ddots$

Table 2: Contagion of action 0, where $Q = \frac{1-q}{2(1+q)}$, $R_0 = \frac{q+r}{2(1+q)}$, $R_1 = \frac{q-r}{2(1+q)}$