Perfect foresight dynamics in binary supermodular games*

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Abstract

This paper studies equilibrium selection in binary supermodular games based on perfect foresight dynamics. We provide complete characterizations of absorbing and globally accessible equilibria and apply them to two subclasses of games. First, for *unanimity games*, it is shown that our selection criterion is not in agreement with that in terms of Nash products, and an example is presented in which two strict Nash equilibria are simultaneously globally accessible when the friction is sufficiently small. Second, a class of *games with invariant diagonal* are proposed and shown to generically admit an absorbing and globally accessible equilibrium for small frictions. *Journal of Economic Literature* Classification Numbers: C72, C73.

KEYWORDS: equilibrium selection; perfect foresight dynamics; supermodular game; strategic complementarity; unanimity game; invariant diagonal game.

1 Introduction

In this paper we study N-player *bipolar games* (Selten 1995), where each player has *binary* actions 0 and 1, and the profiles $\mathbf{0}$ ("all 0") and $\mathbf{1}$ ("all 1") are strict Nash equilibria. We also assume a certain monotonicity condition of the incentive functions which is equivalent to the game being *supermodular*. Such games form a simple and natural class of coordination games for which the problem of *equilibrium selection* arises.

We employ the approach of *perfect foresight dynamics* due to Matsui and Matsuyama (1995); see also Hofbauer and Sorger (1999, 2002), Oyama (2002), Oyama et al. (2008, OTH henceforth), and Takahashi (2008).[1] An N-player normal form game is played repeatedly in a random-matching fashion in a large society of N continua of agents (one for each player role of the game). Opportunities at which agents can revise their actions arrive according to independent Poisson processes, which along with time discounting constitutes the friction of the model. Each agent, given a revision opportunity, takes a best response to the future course of play in the society. This forward-looking behavior of agents has an effect to destabilize some strict Nash equilibria when the degree of friction is small. A Nash equilibrium a^* is *globally accessible* if for any initial action distribution, there exists an equilibrium path that converges to a^*; a^* is *linearly absorbing* if the path linearly converges to a^* is the unique equilibrium path from each initial action distribution in a neighborhood of a^*. If an equilibrium that is globally accessible (linearly absorbing, resp.) is also linearly absorbing (globally accessible, resp.), then it is a unique such equilibrium.

In Section 3, for arbitrary binary supermodular games we obtain complete characterizations for linear absorption and for global accessibility of a strict Nash equilibrium. These characterizations are applied to two subclasses in Sections 4 and 5. First, for *unanimity games*, we show that our selection criterion is not in agreement with that in terms of Nash products. In fact, the perfect foresight dynamics fails to select a single Nash equilibrium for some unanimity games. A nondegenerate example (Example 4.1) demonstrates that both the two strict Nash equilibria can be globally accessible for a small friction. Second, for *games with invariant diagonal*, we obtain the generic existence of a linearly absorbing and globally accessible equilibrium for a small friction. This equilibrium maximizes the potential along the diagonal. In general the potential does not extend to the whole state space; in fact we provide a simple example for this (Example 5.1). Nevertheless, the maximizer of the potential along the diagonal turns out to have the stability properties, as if the potential extended to the whole state

[1] Other studies include Burdzy et al. (2001), Kojima (2006), Kojima and Takahashi (2007, 2008), Matsui and Oyama (2006), Oyama and Tercieux (2009), Rapp (2008), Tercieux (2006). Economic applications of this class of dynamics include Matsuyama (1991, 1992), Kaneda (1995, 2003), Oyama (2009), and Amaya (2010).

space.[2]

In OTH (2008) we analyzed many-action supermodular games with monotone potentials and showed that a monotone potential maximizer is globally accessible and linearly absorbing for small frictions. However, it need not always exist, and in fact, the results obtained in OTH (2008) invoking monotone potentials do not cover the two subclasses studied in the present paper. This will be illustrated in Examples 4.1 and 5.1 mentioned above.

2 Preliminaries

2.1 Binary Supermodular Games

We consider an $N \geq 2$ player binary game $G = (I, A, (u_i)_{i \in I})$, where $I = \{1, \ldots, N\}$ is the set of players, $A = \{0, 1\}$ the action set for each player, and $u_i \colon A^N \to \mathbb{R}$ the payoff function for player $i \in I$. Payoff functions are extended to mixed strategy profiles in the usual way. We denote by $p_i \in [0, 1]$ the probability assigned by player i on action 1, and hence that assigned on action 0 is $1 - p_i$. The *incentive function* $d_i \colon [0, 1]^N \to \mathbb{R}$ for player i is defined by

$$d_i(p_1, \ldots, p_N) = u_i\big(1, (p_j, 1 - p_j)_{j \neq i}\big) - u_i\big(0, (p_j, 1 - p_j)_{j \neq i}\big).$$

We identify $a = (a_i)_{i \in I} \in A^N$ with the vector $p = (p_1, \ldots, p_N) \in [0, 1]^N$ such that $p_i = 0$ if $a_i = 0$ and $p_i = 1$ if $a_i = 1$. We assume that action profiles $\mathbf{0}$, where all players play 0, and $\mathbf{1}$, where all players play 1, are strict Nash equilibria, i.e., $d_i(\mathbf{0}) < 0 < d_i(\mathbf{1})$ for all $i \in I$. We further assume that d_i is nondecreasing in each p_j ($j \neq i$) so that the game is supermodular. Thus, the game is a *bipolar supermodular game* (Selten 1995).

2.2 Perfect Foresight Dynamics

Given an N-player binary game as described above, which will be called the stage game, we consider the following dynamic societal game. Society consists of N large populations of infinitesimal agents, one for each role in the stage game. In each population, agents are identical and anonymous. At each point in time, one agent is selected randomly from each population and matched to form an N-tuple and play the stage game. Agents cannot switch actions at every point in time. Instead, every agent must make a commitment to a particular action for a random time interval. Time instants at which each agent can switch actions follow a Poisson process with the arrival rate $\lambda > 0$. The processes are independent across agents.

[2] Kim (1996) compares several equilibrium selection approaches for the class of *symmetric* binary supermodular games. These games are known to have a potential on the whole state space. For general potential games, Hofbauer and Sorger (2002) show that the potential maximizer has the stability properties under the perfect foresight dynamics.

We choose without loss of generality the unit of time in such a way that $\lambda = 1$.

The action distribution in population $i \in I$ at time $t \in \mathbb{R}_+$ is denoted by

$$\phi(t) = (\phi_1(t), \ldots, \phi_N(t)) \in [0,1]^N,$$

where $\phi_i(t)$ $(1-\phi_i(t)$, resp.) is the fraction of agents in population i who are committing to action 1 (action 0, resp.) at time t. The initial condition $\phi(0)$ is exogenously given. Due to the assumption that the switching times follow independent Poisson processes with arrival rate $\lambda = 1$, $\phi_i(\cdot)$ is Lipschitz continuous with Lipschitz constant 1, which implies in particular that it is differentiable at almost all $t \geq 0$. We call such a path $\phi(\cdot)$ a feasible path.

Definition 2.1. A path $\phi\colon \mathbb{R}_+ \to [0,1]^N$ is said to be *feasible* if it is Lipschitz continuous, and for almost all $t \geq 0$ there exists $\alpha(t) = (\alpha_i(t))_{i \in I} \in [0,1]^N$ such that

$$\dot{\phi}(t) = \alpha(t) - \phi(t). \tag{2.1}$$

Denote by Φ the set of feasible paths, which is convex and compact in the topology of uniform convergence on compact intervals. In Equation (2.1), $\alpha_i(t) \in [0,1]$ denotes the fraction of the agents in population i who have a revision opportunity and choose action 1 during the short time interval $[t, t+dt)$. In particular, if for some action profile $a = (a_i)_{i \in I} \in A^N$, $\alpha_i(t) = a_i$ for all $i \in I$ and all $t \geq 0$, then the resulting feasible path, which converges linearly to a, is called a *linear path* to a.

An agent in population i anticipates the future evolution of the action distribution, and, if given the opportunity to switch actions, commits to an action that maximizes his expected discounted payoff. Since the duration of the commitment has an exponential distribution with mean 1, the expected discounted payoff of committing to action $h \in A$ at time t with a given anticipated path $\phi \in \Phi$ is represented by

$$\begin{aligned} V^\theta_{ih}(\phi)(t) &= (1+\theta)\int_0^\infty \int_t^{t+s} e^{-\theta(z-t)} u_i\big(h, (\phi_j(z), 1-\phi_j(z))_{j\neq i}\big)\, dz\, e^{-s} ds \\ &= (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} u_i\big(h, (\phi_j(s), 1-\phi_j(s))_{j\neq i}\big)\, ds, \end{aligned}$$

where $\theta > 0$ is the common rate of time preference relative to $\lambda = 1$. We view the discounted average duration of a commitment, $\theta/\lambda = \theta$, as the *degree of friction*. Note that V is well-defined whenever $\theta > -1$, and particularly for $\theta = 0$. Denote

$$\mathit{\Delta} V^\theta_i(\phi)(t) = V^\theta_{i1}(\phi)(t) - V^\theta_{i0}(\phi)(t) = (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} d_i(\phi(s))\, ds.$$

Given a feasible path $\phi \in \Phi$, let $\mathcal{B}_i(\phi)(t) \subset [0,1]$ be the set of best responses in mixed strategies to $\phi_{-i} = (\phi_j)_{j \neq i}$ at time t, i.e.,

$$\mathcal{B}_i(\phi)(t) = \begin{cases} \{1\} & \text{if } \Delta V_i^\theta(\phi)(t) > 0, \\ [0,1] & \text{if } \Delta V_i^\theta(\phi)(t) = 0, \\ \{0\} & \text{if } \Delta V_i^\theta(\phi)(t) < 0. \end{cases}$$

Denote $\mathcal{B}(\phi)(t) = \prod_{i \in I} \mathcal{B}_i(\phi)(t) \subset [0,1]^N$. A perfect foresight path is an equilibrium path of the dynamic model, that is, a feasible path along which each agent optimizes against the correctly anticipated future path.

Definition 2.2. A feasible path ϕ is said to be a *perfect foresight path* from $p \in [0,1]^N$ if for almost all $t \geq 0$,

$$\dot{\phi}(t) \in \mathcal{B}(\phi)(t) - \phi(t), \quad \phi(0) = p. \tag{2.2}$$

If $\dot{\phi}_i(t) > -\phi_i(t)$ ($\dot{\phi}_i(t) < 1 - \phi_i(t)$, resp.), then action 1 (action 0, resp.) is taken by some positive fraction of the agents in population i having a revision opportunity during the short time interval $[t, t+dt)$. The definition says that such an action must be a best response to the path ϕ itself. It is known that a perfect foresight path exists for each initial condition (see OTH 2008, Subsection 2.3).

Since $\mathbf{1}$ and $\mathbf{0}$ are Nash equilibria of the stage game, the constant paths $\bar{\phi}$ and $\bar{\psi}$ such that $\bar{\phi}(t) = \mathbf{1}$ and $\bar{\psi}(t) = \mathbf{0}$ for all $t \geq 0$ are perfect foresight paths. Nevertheless, there may exist another perfect foresight path from a Nash equilibrium which converges to a different Nash equilibrium. In fact, in 2×2 coordination games, there exists a perfect foresight path from the risk-dominated equilibrium to the risk-dominant equilibrium for small $\theta > 0$, but not vice versa (Matsui and Matsuyama 1995). Following Matsui and Matsuyama (1995) and OTH (2008), we employ the following stability concepts ($B_\varepsilon(a^*)$ denotes the ε-neighborhood of $a^* \in A^N$ in $[0,1]^N$).

Definition 2.3. (a) $a^* \in A^N$ is *absorbing* if there exists $\varepsilon > 0$ such that any perfect foresight path from any $p \in B_\varepsilon(a^*)$ converges to a^*.

(b) $a^* \in A^N$ is *linearly absorbing* if there exists $\varepsilon > 0$ such that for any $p \in B_\varepsilon(a^*)$, the linear path to a^* is a unique perfect foresight path from p.

(c) $a^* \in A^N$ is *accessible* from $p \in [0,1]^N$ if there exists a perfect foresight path from p that converges to a^*. a^* is *globally accessible* if it is accessible from any p.

Any absorbing or globally accessible state must be a Nash equilibrium of the stage game (OTH 2008, Proposition 2.1). In supermodular games, absorption and linear absorption are equivalent (OTH 2008, Proposition 3.3). We are interested in a (unique, by definition) Nash equilibrium that is linearly absorbing and globally accessible when the degree of friction $\theta > 0$ is sufficiently small.

3 General Results

In this section, we give complete characterizations for the strict Nash equilibrium **1** to be globally accessible and to be absorbing (or, equivalently, linearly absorbing), respectively. By reversing the orders of actions, the results can be applied to the other Nash equilibrium **0**. The subsequent sections then consider two subclasses of binary supermodular games.

For $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$, let $\phi_{\mathbf{T}}^{\mathrm{u}}$ be the feasible path given by

$$(\phi_{\mathbf{T}}^{\mathrm{u}})_i(t) = \begin{cases} 0 & \text{if } t < T_i \\ 1 - e^{-(t-T_i)} & \text{if } t \geq T_i, \end{cases} \tag{3.1}$$

which goes upwards (hence denoted with superscript "u") starting at **0** and converging to **1**. Along $\phi_{\mathbf{T}}^{\mathrm{u}}$, agents in population $i \in I$ start choosing action 1 at time T_i.

Denote $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. For $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$, let $\psi_{\mathbf{T}}^{\mathrm{d}}$ be the feasible path given by

$$(\psi_{\mathbf{T}}^{\mathrm{d}})_i(t) = \begin{cases} 1 & \text{if } t < T_i \\ e^{-(t-T_i)} & \text{if } t \geq T_i \end{cases} \qquad \text{for } i \in S, \tag{3.2}$$

and

$$(\psi_{\mathbf{T}}^{\mathrm{d}})_i(t) = 1 \qquad \text{for } i \notin S, \tag{3.3}$$

where $S = \{i \in I \mid T_i \neq \infty\}$. Let $\mathbf{0}_S$ be the action profile such that i chooses 0 if $i \in S$ and 1 if $i \notin S$. Along $\psi_{\mathbf{T}}^{\mathrm{d}}$, which goes downwards (hence denoted with "d") from **1** to **0**, agents in population $i \in S$ start choosing action 0 at time T_i, while those in population $i \notin S$ always play action 1.

First, we provide a necessary and sufficient condition for the state **1** to be globally accessible for any small degree of friction. The condition is stated in terms of existence of a subpath of the form (3.1).

Proposition 3.1. *There exists $\bar{\theta} > 0$ such that the strict Nash equilibrium* **1** *is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if there exists $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$ such that for all $i \in I$,*

$$\Delta V_i^0(\phi_{\mathbf{T}}^{\mathrm{u}})(T_i) > 0.$$

Proof. See Appendix. □

Next, we provide a necessary and sufficient condition for the state **1** to be absorbing for any degree of friction. The condition is stated in terms of nonexistence of a superpath of the form (3.2)–(3.3) with $\mathbf{0}_S$ being a Nash equilibrium of the stage game such that the players in S have strict incentives.

Proposition 3.2. *The strict Nash equilibrium* $\mathbf{1}$ *is absorbing for all* $\theta > 0$ *if and only if for any* $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ *such that* $S = \{i \in I \mid T_i \neq \infty\}$ *is nonempty and* $\mathbf{0}_S$ *is a Nash equilibrium with* $d_i(\mathbf{0}_S) < 0$ *for all* $i \in S$*, there exists* $i \in S$ *such that*

$$\Delta V_i^0(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \geq 0.$$

Proof. See Appendix. □

Importantly, the characterizations of global accessibility and absorption in Propositions 3.1 and 3.2 are stated in terms of finite dimensional objects $\mathbf{T}$, whereas their definitions are given in terms of perfect foresight paths which belong to the infinite dimensional set Φ of feasible paths. It is the supermodularity that allows us to obtain these simple characterizations.

Consider games such that for any Nash equilibrium $\mathbf{0}_S$ with $S \neq \emptyset, I$, at least one player in S and at least one player in $I \setminus S$ are indifferent between the actions 0 and 1 (i.e., $d_i(\mathbf{0}_S) = 0$ holds for some $i \in S$ and for some $i \notin S$). For such games, it follows from Propositions 3.1 and 3.2 that if one of the strict Nash equilibria $\mathbf{0}$ and $\mathbf{1}$ is not globally accessible for any sufficiently small $\theta > 0$ (not absorbing for any $\theta > 0$, resp.), then the other one is absorbing for any $\theta > 0$ (globally accessible for any sufficiently small $\theta > 0$, resp.). Among such games are unanimity games as considered in the next section.

4 Unanimity Games

This section considers N-player unanimity games. The stage game is given by

$$u_i(a) = \begin{cases} y_i & \text{if } a = \mathbf{0} \\ z_i & \text{if } a = \mathbf{1} \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $y_i, z_i > 0$. The incentive function for player i is then given by

$$d_i(p_1, \cdots, p_N) = z_i \prod_{j \neq i} p_j - y_i \prod_{j \neq i} (1 - p_j).$$

Note that this game is supermodular.

For $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$, let

$$\begin{aligned} \pi_i(\mathbf{T}) &= \int_{T_i}^{\infty} e^{-(t - T_i)} \prod_{j \neq i} \left[0 \vee \left\{ 1 - e^{-(t - T_j)} \right\} \right] dt \\ &= \int_{\max_j T_j}^{\infty} e^{-(t - T_i)} \prod_{j \neq i} \left\{ 1 - e^{-(t - T_j)} \right\} dt, \end{aligned} \tag{4.2}$$

$$\rho_i(\mathbf{T}) = \int_{T_i}^{\infty} e^{-(t - T_i)} \prod_{j \neq i} \left\{ 1 \wedge e^{-(t - T_j)} \right\} dt. \tag{4.3}$$

4.1 Global Accessibility

For a feasible path $\phi^{\mathrm{u}}_{\mathbf{T}}$ defined by (3.1) with a given $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}^N_+$, the discounted payoff difference is given by

$$\Delta V_i^0(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) = z_i\pi_i(\mathbf{T}) - y_i\rho_i(\mathbf{T}),$$

so that $\Delta V_i^0(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$ if and only if $z_i/y_i > \rho_i(\mathbf{T})/\pi_i(\mathbf{T})$. Thus we immediately have the following from Proposition 3.1.

Proposition 4.1. *Suppose that the stage game is a unanimity game given by (4.1). Then there exists $\bar{\theta} > 0$ such that $\mathbf{1}$ is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if there exists $\mathbf{T} \in \mathbb{R}^N_+$ such that for all $i \in I$,*

$$\frac{z_i}{y_i} > \frac{\rho_i(\mathbf{T})}{\pi_i(\mathbf{T})}.$$

Symmetrically, there exists $\bar{\theta} > 0$ such that $\mathbf{0}$ is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if there exists $\mathbf{T} \in \mathbb{R}^N_+$ such that for all $i \in I$,

$$\frac{y_i}{z_i} > \frac{\rho_i(\mathbf{T})}{\pi_i(\mathbf{T})}.$$

4.2 Absorption

For a feasible path $\psi^{\mathrm{d}}_{\mathbf{T}}$ defined by (3.2) with a given $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}^N_+$, the discounted payoff difference is given by

$$\Delta V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) = z_i\rho_i(\mathbf{T}) - y_i\pi_i(\mathbf{T}),$$

so that $\Delta V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \geq 0$ if and only if $z_i/y_i \geq \pi_i(\mathbf{T})/\rho_i(\mathbf{T})$. Thus we have the following from Proposition 3.2. Observe that in this case, S satisfies the condition in Proposition 3.2 only if $S = I$.

Proposition 4.2. *Suppose that the stage game is a unanimity game given by (4.1). Then $\mathbf{1}$ is absorbing for all $\theta > 0$ if and only if for any $\mathbf{T} \in \mathbb{R}^N_+$, there exists $i \in I$ such that*

$$\frac{z_i}{y_i} \geq \frac{\pi_i(\mathbf{T})}{\rho_i(\mathbf{T})}.$$

Symmetrically, $\mathbf{0}$ is absorbing for all $\theta > 0$ if and only if for any $\mathbf{T} \in \mathbb{R}^N_+$, there exists $i \in I$ such that

$$\frac{y_i}{z_i} \geq \frac{\pi_i(\mathbf{T})}{\rho_i(\mathbf{T})}.$$

4.3 Two-Player Case

In the case where $N = 2$, there exists $\mathbf{T} \in \mathbb{R}^2_+$ such that

$$\frac{z_1}{y_1} > \frac{\rho_1(\mathbf{T})}{\pi_1(\mathbf{T})}, \qquad \frac{z_2}{y_2} > \frac{\rho_2(\mathbf{T})}{\pi_2(\mathbf{T})}$$

if and only if $z_1 z_2 > y_1 y_2$. Therefore, by Propositions 4.1 and 4.2, $\mathbf{1}$ is absorbing and globally accessible for any small degree of friction if and only if $\mathbf{1}$ has the higher Nash product over $\mathbf{0}$. In the two-player case, this is equivalent to that $\mathbf{1}$ risk-dominates $\mathbf{0}$.

4.4 Three-Player Case

When $N \geq 3$, the complete characterizations given in Propositions 4.1 and 4.2 turn out to be rather complex. Here we consider three-player binary games with a symmetry between players 2 and 3. We demonstrate that even for this simple class of games, both Nash equilibria $\mathbf{1}$ and $\mathbf{0}$ may be simultaneously globally accessible states when the friction is small.

Specifically, we consider the case where

$$(z_1/y_1, z_2/y_2, z_3/y_3) = (r, s, s). \tag{4.4}$$

We can exploit the symmetry due to the following fact. Note here that if $T_i = T_j$, then $\pi_i(\mathbf{T}) = \pi_j(\mathbf{T})$ and $\rho_i(\mathbf{T}) = \rho_j(\mathbf{T})$.

Lemma 4.3. *Suppose that the stage game is given by* (4.1)*. Then* $\mathbf{1}$ *is globally accessible for any small degree of friction if and only if there exists* $\mathbf{T}$ *such that for all* $i \in I$*,*

$$\frac{z_i}{y_i} > \frac{\rho_i(\mathbf{T})}{\pi_i(\mathbf{T})}, \tag{4.5}$$

and

$$\frac{z_i}{y_i} \geq \frac{z_j}{y_j} \Rightarrow T_i \leq T_j. \tag{4.6}$$

Proof. It suffices to show that if there exists $\mathbf{T}$ that satisfies (4.5), then there exists $\mathbf{T}'$ that satisfies both (4.5) and (4.6).

Take $\mathbf{T}$ that satisfies (4.5) and define $\mathbf{T}'$ by

$$T'_i = \min_{j\colon z_j/y_j \leq z_i/y_i} T_j$$

for each i. Note that $T'_i \leq T_i$ for any i.

Here we fix any i. By definition, there exists j such that $T'_i = T_j$ and $z_j/y_j \leq z_i/y_i$. Take such a j. Note that $\mathbf{T}_{-j} \geq \mathbf{T}'_{-j}$ and $T_j = T'_j$. Since $\mathbf{T}$ satisfies (4.5), π_j is decreasing in $\mathbf{T}_{-j}$, and ρ_j is increasing in $\mathbf{T}_{-j}$, we have

$$\frac{z_j}{y_j} > \frac{\rho_j(\mathbf{T})}{\pi_j(\mathbf{T})} \geq \frac{\rho_j(\mathbf{T}')}{\pi_j(\mathbf{T}')}.$$

On the other hand, $\pi_i(\mathbf{T}') = \pi_j(\mathbf{T}')$ and $\rho_i(\mathbf{T}') = \rho_j(\mathbf{T}')$ since $T'_i = T'_j$. Therefore, it follows from $z_j/y_j \leq z_i/y_i$ that

$$\frac{z_i}{y_i} \geq \frac{z_j}{y_j} > \frac{\rho_j(\mathbf{T}')}{\pi_j(\mathbf{T}')} = \frac{\rho_i(\mathbf{T}')}{\pi_i(\mathbf{T}')},$$

which completes the proof. □

A direct computation utilizing Lemma 4.3 shows that **1** is globally accessible for a small friction if and only if

$$\begin{aligned} & r < s \quad \text{and} \quad r > \frac{2}{(s-1)\sqrt{9s^2 - 12s + 12} + 3s^2 - 5s + 4} \\ & \text{or} \\ & r \geq s \quad \text{and} \quad r > \frac{2}{s} - 1. \end{aligned} \tag{4.7}$$

The condition for the global accessibility of **0** is given by replacing r and s with $1/r$ and $1/s$.

In the game given by (4.4), **1** has the higher Nash product over **0** if $rs^2 > 1$. A direct comparison between $r > 1/s^2$ and the above expressions gives the following sufficient condition in terms of Nash product.

Proposition 4.4. *In the three-player unanimity game given by* (4.4)*, the Nash equilibrium with a higher Nash product is globally accessible for any small degree of friction.*

Equivalently, if an equilibrium is absorbing for any degree of friction, then its Nash product is no lower than that of the other equilibrium (recall that, for unanimity games, one equilibrium is absorbing if and only if the other is not globally accessible).

The converse of Proposition 4.4 is not true, as the following example shows (the result of which has been reported without proof in OTH 2008).

Example 4.1. Let $y_1 = 6 + c > 0$, $y_2 = y_3 = 1$, and $z_1 = z_2 = z_3 = 2$, where $c > -6$ (see Figure 1). This game is a modified version of an example in Morris and Ui (2005, Example 1).[3] Substituting $r = 2/(6 + c)$ and $s = 2$ into (4.7) shows that **1** is globally accessible for a small friction if and only if $c < 2\sqrt{6}$, while substituting $r = (6 + c)/2$ and $s = 1/2$ into (4.7) shows that **0** is globally accessible for a small friction if and only if $c > 0$. Therefore, if $0 < c < 2\sqrt{6}$, the game has two globally accessible states simultaneously when the friction is small. Note that **1** (**0**, resp.) has a higher Nash product if $c < 2$ ($c > 2$, resp.).

[3] One can verify that **0** is not an MP-maximizer for any c, while **1** is an MP-maximizer if and only if $c < -2$.

	0	1
0	$6+c$, 1, 1	0, 0, 0
1	0, 0, 0	0, 0, 0

0

	0	1
0	0, 0, 0	0, 0, 0
1	0, 0, 0	2, 2, 2

1

Figure 1: Unanimity game with multiple globally accessible states

5 Binary Games with Invariant Diagonal

This section considers N-player binary supermodular games with invariant diagonal. A binary game is said to have an *invariant diagonal* if the incentive functions satisfy

$$d_1(\xi, \dots, \xi) = \cdots = d_N(\xi, \dots, \xi) \tag{5.1}$$

for all $\xi \in [0, 1]$. This class of games includes games with "equistable biforms" introduced in Selten (1995). We maintain the assumption that d_i is nondecreasing in each p_j $(j \neq i)$ so that the game is supermodular.

Denote by $D(\xi)$ the restriction of any d_i to the diagonal $\xi = p_1 = \cdots = p_N$. Observe that $D(\xi)$ is nondecreasing in ξ. This game has a potential function along the diagonal, which is defined by

$$v(\xi) = \int_0^{\xi} D(\xi')\, d\xi'. \tag{5.2}$$

Games with an invariant diagonal may not have a potential function defined on the whole state space, so that the result of Hofbauer and Sorger (2002) does not apply. Furthermore, they may not have a monotone potential function as considered in OTH (2008), one of the most general concepts of potential; see Example 5.1 below. The following result demonstrates that the supermodularity allows us to obtain equilibrium selection in the absence of a *global* potential function; a potential function defined on the diagonal suffices in supermodular games.

Proposition 5.1. *Suppose that the stage game is a binary supermodular game with an invariant diagonal. Let v be the potential function along the diagonal given by* (5.2)*. If $v(1) > v(0)$, then*

(a) *there exists $\bar{\theta} > 0$ such that $\mathbf{1}$ is globally accessible for all $\theta \in (0, \bar{\theta})$;*

(b) *$\mathbf{1}$ is absorbing for all $\theta > 0$.*

Proof. (a) Along the linear path ϕ from $\mathbf{0}$ to $\mathbf{1}$, which is given by $\phi_i(t) = 1 - e^{-t}$ for all $i \in I$,

$$\Delta V_i^0(\phi)(0) = \int_0^\infty e^{-s} D(1 - e^{-s})\, ds = \int_0^1 D(\xi)\, d\xi = v(1).$$

Hence, if $v(1) > v(0) = 0$, then $\Delta V_i^0(\phi)(0) > 0$, implying that $\mathbf{1}$ is globally accessible for any small $\theta > 0$ by Proposition 3.1.

(b) If $v(1) > v(0) = 0$, then there exists $\xi^0 < 1$ such that $v(\xi^0) > 0$. Take such a ξ^0 and any perfect foresight path ϕ with $\phi_i(0) \geq \xi^0$ for all $i \in I$. Note that $\phi_i(t) \geq \xi^0 e^{-t}$. Then,

$$\begin{aligned}
\Delta V_i^\theta(\phi)(0) &= (1+\theta) \int_0^\infty e^{-(1+\theta)s} d_i((\phi_i(s))_{i \in I})\, ds \\
&\geq (1+\theta) \int_0^\infty e^{-(1+\theta)s} D(\xi^0 e^{-s})\, ds \\
&\geq \int_0^\infty e^{-s} D(\xi^0 e^{-s})\, ds \\
&= \frac{1}{\xi^0} \int_0^{\xi^0} D(\xi)\, d\xi = \frac{v(\xi^0)}{\xi^0} > 0,
\end{aligned}$$

where the first inequality follows from the monotonicity of d_i, and the second inequality follows from the stochastic dominance relation between the distributions on $[0, \infty)$ with the density functions $(1+\theta)e^{-(1+\theta)s}$ and e^{-s}. Hence, we have $\phi_i(t) = 1 - (1 - \phi_i(0))e^{-t}$ for all $t \geq 0$, and therefore, ϕ converges to $\mathbf{1}$, implying that $\mathbf{1}$ is absorbing (independently of $\theta > 0$). $\square$

Similarly, if $v(0) > v(1)$, then $\mathbf{0}$ is globally accessible for any small $\theta > 0$ and absorbing for any $\theta > 0$. Therefore, for generic binary supermodular games with invariant diagonal, either $\mathbf{0}$ or $\mathbf{1}$ is a unique absorbing and globally accessible state for any small degree of friction (even though there may be other strict equilibria).

Remark 5.1. A state $x^* \in \prod_i \Delta(A_i)$ is *linearly stable* if for any $x \in \prod_i \Delta(A_i)$, the linear path from x to x^* is a perfect foresight path. One can verify that for binary supermodular games with invariant diagonal, if $v(1) > v(0)$, then $\mathbf{1}$ is linearly stable for any small degree of friction $\theta > 0$.

Remark 5.2. The result here does not follow from Remark 1A in Matsui and Matsuyama (1995), since we work with a multi-population setting with the state space $[0,1]^N$ and allow for asymmetric paths out of the diagonal of $[0,1]^N$, which in principle must be taken into account for stability considerations. It is thanks to the diagonal invariance (5.1) that it suffices to consider only symmetric paths in (5.2).

Remark 5.3. The above result extends to the class of games with "monotone diagonal". Let $D_i(\xi) = d_i(\xi, \ldots, \xi)$ and $v_i(\xi) = \int_0^\xi D_i(\xi')\, d\xi'$. It can be shown precisely in the same way as in Proposition 5.1 that if $v_i(1) > v_i(0)$ for all $i \in I$, then $\mathbf{1}$ is globally accessible for any small $\theta > 0$ and absorbing for any $\theta > 0$.

Example 5.1. Consider the following three player game (see Figure 2). If all three players match their actions, then their payoffs are given by $u_i(\mathbf{0}) = a > 0$ and $u_i(\mathbf{1}) = d > 0$. For other action profiles, if i matches $i + 1$ with action 0, then i's payoff is $b > 0$; if i matches $i + 1$ with action 1, then i's payoff is $c > 0$; otherwise, all players receive payoff 0. Suppose here that $a > b$ and $d > c$. Note that this game is supermodular and has an invariant diagonal.[4] Proposition 5.1 implies that if $2a+b > c+2d$, then $\mathbf{0}$ is absorbing and globally accessible for a small friction, while if $2a + b < c + 2d$, then $\mathbf{1}$ is absorbing and globally accessible for a small friction.

The selection criterion based on MP-maximization, on the other hand, yields a limited prediction: One can verify that $\mathbf{0}$ is an MP-maximizer if and only if $a > c + d$, while $\mathbf{1}$ is an MP-maximizer if and only if $a + b < d$. For this game, the notion of u-dominance introduced by Kojima (2006) gives the same condition: $\mathbf{0}$ is u-dominant if and only if $a > c + d$, while $\mathbf{1}$ is u-dominant if and only if $a + b < d$.[5]

Spatial dominance due to Hofbauer (1999) selects a different equilibrium for this game, namely, the equilibrium with the larger best response region on the diagonal, i.e., $\mathbf{0}$ is spatially dominant if and only if $a + b > c + d$, while $\mathbf{1}$ is spatially dominant if and only if $a + b < c + d$.

	0	1
0	a, a, a	$0, 0, b$
1	$0, b, 0$	$c, 0, 0$

0

	0	1
0	$b, 0, 0$	$0, c, 0$
1	$0, 0, c$	d, d, d

1

Figure 2: Game with invariant diagonal

[4]This game is not a (weighted) potential game, since it has a better reply cycle.

[5]In general, MP-maximization and u-dominance give different conditions.

Appendix

A.1 Supermodularity and Monotonicity

For $p, q \in [0,1]^N$, write $p \leq q$ if $p_i \leq q_i$ for all $i = 1, \ldots, N$. Due to the supermodularity assumption on the stage game, $d_i(p) \leq d_i(q)$ if $p \leq q$, the perfect foresight dynamics has several nice monotonicity properties as demonstrated in OTH (2008), some of which will be used in the subsequent proofs.

For $\phi, \psi \in \Phi$, define $\phi \precsim \psi$ by $\phi(t) \leq \psi(t)$ for all $t \geq 0$. The monotonicity of d_i is immediately inherited by ΔV_i^θ. That is, it holds that if $\phi \precsim \psi$, then $\Delta V_i^\theta(\phi)(t) \leq \Delta V_i^\theta(\psi)(t)$ for all $i \in I$ and all $t \geq 0$.

We say that a feasible path ϕ is a *superpath* if

$$\Delta V_i^\theta(\phi)(t) > 0 \Rightarrow \dot{\phi}_i(t) = 1 - \phi_i(t)$$

for all $i \in I$ and almost all $t \geq 0$; a feasible path ψ is a *subpath* if

$$\Delta V_i^\theta(\phi)(t) < 0 \Rightarrow \dot{\phi}_i(t) = -\phi_i(t)$$

for all $i \in I$ and almost all $t \geq 0$ Note that ϕ is a perfect foresight path if and only if it is both a superpath and a subpath. We have the following.

Lemma A.1 (OTH 2008, Lemma 3.3)**.** *Let $p, q \in [0,1]^N$ be such that $q \leq p$.*

(a) *If there exists a superpath ϕ with $\phi(0) = p$, then there exists a perfect foresight path ψ^* with $\psi^*(0) = q$ such that $\psi^* \precsim \phi$.*

(b) *If there exists a subpath ψ with $\psi(0) = q$, then there exists a perfect foresight path ϕ^* with $\phi^*(0) = p$ such that $\psi \precsim \phi^*$.*

We will need the following lemma.

Lemma A.2. *For all $i \in I$ and all $t \geq 0$,*

(a) *for any $\mathbf{T} \in \mathbb{R}_+^N$, $\Delta V_i^\theta(\phi_{\mathbf{T}}^{\mathrm{u}})(t)$ is decreasing in $\theta \geq 0$,*

(b) *for any $\mathbf{T} \in \bar{\mathbb{R}}_+^N$ with $S = \{i \in I \,|\, T_i \neq \infty\}$, $\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(t)$ is nondecreasing in $\theta \geq 0$, and is increasing in $\theta \geq 0$ if $d_i(\mathbf{1}) > d_i(\mathbf{0}_S)$.*

This lemma is a consequence of the stochastic dominance relation among distributions on $[t, \infty)$ induced by discount rates: the distribution on $[t, \infty)$ with density function $(1+\theta)e^{-(1+\theta)(s-t)}$ strictly stochastically dominates the one with density function $(1+\theta')e^{-(1+\theta')(s-t)}$ for $0 \leq \theta < \theta'$. The statements follow from the facts that $d_i((\phi_{\mathbf{T}}^{\mathrm{u}})(s))$ is nondecreasing in $s \geq 0$ and increasing in $s \geq \max_{j \in I} T_j$, and that $d_i((\psi_{\mathbf{T}}^{\mathrm{d}})(s))$ is nonincreasing in $s \geq 0$, and decreasing in $s \geq \max_{j \in S} T_j$ if $d_i(\mathbf{1}) > d_i(\mathbf{0}_S)$.

A.2 Proofs of Propositions 3.1 and 3.2

We first prove the global accessibility results. Lemma A.3 gives a necessary and sufficient condition for the state **1** to be globally accessible for a given degree of friction, from which Proposition 3.1 follows.

Lemma A.3. *Let $\theta > 0$ be given. The strict Nash equilibrium* **1** *is globally accessible for θ if and only if there exists $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}^N_+$ such that for all $i \in I$,*

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0.$$

Proof. "If" part: Suppose that there exists $\mathbf{T} = (T_i)_{i\in I}$ such that for all i,

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0.$$

Since $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(t)$ is increasing in t, $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(t) \geq 0$ holds for all $i \in I$ and all $t \geq T_i$. By the definition of $\phi^{\mathrm{u}}_{\mathbf{T}}$, this implies that $\phi^{\mathrm{u}}_{\mathbf{T}}$ is a subpath. It follows from Lemma A.1 that for any $x \in \prod_i \Delta(A_i)$, there exists a perfect foresight path ϕ^* from x such that $\phi^{\mathrm{u}}_{\mathbf{T}} \precsim \phi^*$. Since $\phi^{\mathrm{u}}_{\mathbf{T}}$ converges to **1**, ϕ^* also converges to **1**. Therefore, **1** is globally accessible.

"Only if" part: Suppose that **1** is globally accessible, so that there exists a perfect foresight path ϕ such that $\phi(0) = \mathbf{0}$ and $\lim_{t\to\infty} \phi(t) = \mathbf{1}$. Take such a perfect foresight path ϕ and let

$$T_i = \inf\{t \geq 0 \mid \dot{\phi}_i(t) > -\phi_i(t)\}$$

for each $i \in I$. Note that $T_i < \infty$ for all $i \in I$.

For $\mathbf{T} = (T_i)_{i\in I}$ defined above, define $\phi^{\mathrm{u}}_{\mathbf{T}}$ as in (3.1). Since $\phi \precsim \phi^{\mathrm{u}}_{\mathbf{T}}$, we have

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq \Delta V_i^\theta(\phi)(T_i) \geq 0$$

due to the supermodularity. □

Proof of Proposition 3.1. "If" part: Take a $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}^N_+$ such that

$$\Delta V_i^0(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$$

for all $i \in I$. Since $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i)$ is continuous in θ, there exists $\bar{\theta} > 0$ such that for all $\theta \in (0, \bar{\theta})$,

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$$

for all $i \in I$, implying that **1** is globally accessible for all $\theta \in (0, \bar{\theta})$ by Lemma A.3.

"Only if" part: Suppose that **1** is globally accessible for a small $\theta > 0$. Then, by Lemma A.3 there exists $\mathbf{T}$ such that

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0$$

for all $i \in I$. Since $\Delta V_i^\theta(\phi_{\mathbf{T}}^{\mathrm{u}})(T_i)$ is decreasing in θ by Lemma A.2, it follows that

$$\Delta V_i^0(\phi_{\mathbf{T}}^{\mathrm{u}})(T_i) > \Delta V_i^\theta(\phi_{\mathbf{T}}^{\mathrm{u}})(T_i) \geq 0$$

for all $i \in I$. □

Next we prove the absorption results. The following lemma is a special case of Proposition 3 in Takahashi (2008).

Lemma A.4. *Let $\theta > 0$ be given. The state $\mathbf{1}$ is absorbing for θ if and only if for any $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ such that $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty, there exists $i \in S$ such that*

$$\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) > 0.$$

Lemma A.5 gives a necessary and sufficient condition for the state $\mathbf{1}$ to be absorbing for a given degree of friction, from which Proposition 3.2 follows.

Lemma A.5. *Let $\theta > 0$ be given. The strict Nash equilibrium $\mathbf{1}$ is absorbing for θ if and only if for any $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ such that $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium with $d_i(\mathbf{0}_S) < 0$ for all $i \in S$, there exists $i \in S$ such that*

$$\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) > 0.$$

Proof. In light of Lemma A.4, it suffices to show that for any $\mathbf{T}$ such that $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is not a Nash equilibrium with $d_i(\mathbf{0}_S) < 0$ for all $i \in S$, there exists $i \in S$ such that $\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) > 0$. Suppose not, and choose $\mathbf{T}$ and S such that S is maximal among all the subsets that violate the condition. Thus $\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \leq 0$ for all $i \in S$. We first claim that $d_i(\mathbf{0}_S) < 0$ for all $i \in S$. Indeed, for any $i \in S$, by supermodularity we have $d_i(\mathbf{0}_S) \leq \Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \leq 0$, and since

$$d_i(\psi_{\mathbf{T}}^{\mathrm{d}}(t)) = \sum_{S' \subset S \setminus \{i\}} \prod_{j \notin S' \cup \{i\}} (\psi_{\mathbf{T}}^{\mathrm{d}})_j(t) \prod_{j \in S'} \bigl(1 - (\psi_{\mathbf{T}}^{\mathrm{d}})_j(t)\bigr) d_i(\mathbf{0}_{S'})$$

and $\mathbf{0}_\emptyset$ $(= \mathbf{1})$ is a strict Nash equilibrium so that $d_i(\mathbf{0}_\emptyset) > 0$, it must be that $d_i(\mathbf{0}_{S'}) < 0$ for some $S' \subset S$, implying that $d_i(\mathbf{0}_S) < 0$ by supermodularity. Therefore, there must exist $j \notin S$ such that $d_j(\mathbf{0}_S) < 0$ (so that $\mathbf{0}_S$ is not a Nash equilibrium). Choose such a j.

Define $\mathbf{T}' = (T_1', \ldots, T_N')$ by $T_i' = T_i$ for $i \neq j$ and T_j' as a sufficiently large but finite number. Then $\psi_{\mathbf{T}'}^{\mathrm{d}} \precsim \psi_{\mathbf{T}}^{\mathrm{d}}$, so that

$$\Delta V_i^\theta(\psi_{\mathbf{T}'}^{\mathrm{d}})(T_i') \leq \Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \leq 0$$

for $i \in S$ by the supermodularity. Moreover, since $\Delta V_j^\theta(\psi^{\mathrm{d}}_{\mathbf{T}'})(T'_j)$ converges to $d_j(\mathbf{0}_S) < 0$ as $T'_j \to \infty$, we have

$$\Delta V_j^\theta(\psi^{\mathrm{d}}_{\mathbf{T}'})(T'_j) < 0.$$

This contradicts the maximality of S. □

Proposition 3.2 follows immediately from the following.

Lemma A.6. *The following conditions are equivalent:*

(a) $\mathbf{1}$ *is absorbing for all* $\theta > 0$*;*

(b) *there exists* $\bar{\theta}$ *such that* $\mathbf{1}$ *is absorbing for all* $\theta \in (0, \bar{\theta})$*;*

(c) *for any* $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}^N_+$ *such that* $S = \{i \in I \,|\, T_i \neq \infty\}$ *is nonempty and* $\mathbf{0}_S$ *is a Nash equilibrium with* $d_i(\mathbf{0}_S) < 0$ *for all* $i \in S$*, there exists* $i \in S$ *such that*

$$\Delta V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \geq 0.$$

Proof. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Suppose that there exists $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty, $\mathbf{0}_S$ is a Nash equilibrium with $d_i(\mathbf{0}_S) < 0$ for all $i \in S$, and $\Delta V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) < 0$ for all $i \in S$. Fix such a $\mathbf{T}$. Since $\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i)$ is continuous in θ, there exists $\bar{\theta} > 0$ such that for all $\theta \in (0, \bar{\theta})$,

$$\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) < 0$$

for all $i \in S$, implying that $\mathbf{1}$ is not absorbing for any $\theta \in (0, \bar{\theta})$ by Lemma A.5.

(c) $\Rightarrow$ (a): Suppose (c). For each $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium, take $i \in S$ as in (c).

By the monotonicity of d_i, we have $d_i(\mathbf{1}) \geq d_i(\mathbf{0}_S)$. If $d_i(\mathbf{1}) = d_i(\mathbf{0}_S)$, then for any $\theta > 0$,

$$\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) = d_i(\mathbf{1}) > 0$$

by the monotonicity of d_i. If $d_i(\mathbf{1}) > d_i(\mathbf{0}_S)$, then $\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i)$ is increasing in θ by Lemma A.2, so that for any $\theta > 0$,

$$\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) > \Delta V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \geq 0.$$

It follows that $\mathbf{1}$ is absorbing for all $\theta > 0$ by Lemma A.5. □

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