Online Appendix to:
Contagion and uninvadability in local interaction games: The bilingual game and general supermodular games

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Abstract

In this supplementary appendix, we provide proofs of Lemma 3 and Theorem 3 in the main paper (Sections B.2 and B.4), as well as examine alternative definitions of contagion (Section B.1), study multidimensional lattice networks (Section B.3), provide some more examples of interest (Section B.5), consider the case where Pareto dominance and risk dominance coincide (Section B.6), and discuss the implications of our results in the context of incomplete information games (Section B.7).

Online Appendix

B.1. Equivalent Definitions of Contagion in Supermodular Games

In this appendix, we discuss three other definitions of contagion, and show that all of them are equivalent to the original one for any (generic) symmetric supermodular game (S, u), where the smallest and the largest actions are denoted by $\underline{s}$ and $\overline{s}$, respectively. (None of the results here relies on the particular payoff structure of the bilingual game.) We use the partial order $\sigma \leq \sigma'$ whenever $\sigma(x) \leq \sigma'(x)$ for any $x \in X$.

Recall that in the main text, we consider the sequential best response dynamics, where at most one player revises his action in each period (property (i) in Definition 1). Instead, we can define the simultaneous (generalized, resp.) best response dynamics, where all (some, resp.) players revise their actions at a time.

Definition B.1. Given a local interaction game (X, P, S, u), a sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *simultaneous best response sequence* if $\sigma^t(x) \in BR(\sigma^{t-1}|x)$ for all $x \in X$ and $t \geq 1$. A sequence $(\sigma^t)_{t=0}^{\infty}$ is a *generalized best response sequence* if it satisfies the following properties: (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR(\sigma^{t-1}|x)$; and (iii) if $\lim_{t\to\infty} \sigma^t(x) = s$, then for all $T \geq 0$, $s \in BR(\sigma^t|x)$ for some $t \geq T$.

For clarity, we add adjective "sequential" to the original notion of best response sequences. Generalized best response sequences subsume both sequential and simultaneous best response sequences as special cases.

Using simultaneous or generalized best response sequences, we obtain two new definitions of contagion.[1]

Definition B.2. Given a pairwise game (S, u), action s^* is *contagious by simultaneous* (*generalized*, resp.) *best responses in network* (X, P) if there exists a finite subset Y of X such that every simultaneous (generalized, resp.) best response sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0(x) = s^*$ for all $x \in Y$ satisfies $\lim_{t\to\infty} \sigma^t(x) = s^*$ for each $x \in X$.

We refer to the notion of contagion in Definition 2 as "contagion by sequential best responses". By definition, contagion by generalized best responses implies both contagion by sequential best responses and by simultaneous best responses. Here we show the converse.

In the next lemma, we show that if s^* is contagious by sequential best responses, then there exist two sequential best response sequences that converge to s^* monotonically (one increasingly and the other decreasingly), and that any generalized best response sequence that starts between the two sequences also converges to s^*. This lemma is used to prove both Proposition B.1 below and Theorem 3 in the main text.

Lemma B.1. *Fix a network (X, P) and a supermodular game (S, u). Suppose that s^* is contagious by sequential best responses in (X, P). Then there exist two sequential best response sequences $(\sigma_-^t)_{t=0}^{\infty}$ and $(\sigma_+^t)_{t=0}^{\infty}$ such that*

(1) $\sigma_-^t(x) \leq s^* \leq \sigma_+^t(x)$ *for all* $x \in X$ *and* $t \geq 0$*;*

(2) $\sigma_-^0(x) = \underline{s}$ *and* $\sigma_+^0(x) = \overline{s}$ *for all but finitely many* $x \in X$*;*

(3) $\sigma_-^t(x) \in \{\sigma_-^{t-1}(x), \min BR(\sigma_-^{t-1}|x)\}$ *and* $\sigma_+^t(x) \in \{\sigma_+^{t-1}(x), \max BR(\sigma_+^{t-1}|x)\}$ *for all* $x \in X$ *and* $t \geq 1$*;*

(4) $\lim_{t\to\infty} \sigma_-^t(x) = \lim_{t\to\infty} \sigma_+^t(x) = s^*$ *for all* $x \in X$*; and*

[1] The notion of contagion used in Morris [9] is similar to contagion by simultaneous best responses, but requires only that for each $x \in X$, $\sigma^t(x) = s^*$ for some $t \geq 0$.

(5) $\min BR(\sigma^0_-|x) \geq \sigma^0_-(x)$ *and* $\max BR(\sigma^0_+|x) \leq \sigma^0_+(x)$ *for all* $x \in X$.

Moreover,

(6) *for any generalized best response sequence* $(\tilde{\sigma}^t)_{t=0}^\infty$ *with* $\sigma^0_- \leq \tilde{\sigma}^0 \leq \sigma^0_+$, *we have* $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$ *for all* $x \in X$.

Proof. Suppose that s^* is contagious by sequential best responses in (X, P) (and hence a strict Nash equilibrium of (S, u)). Let $Y \subset X$ be a finite set as in Definition 2, and let $(\phi^t_-)_{t=0}^\infty$ be the sequential best response sequence such that $\phi^0_-(x) = s^*$ for all $x \in Y$, $\phi^0_-(x) = \underline{s}$ for all $x \in X \setminus Y$, and $\phi^t_-(x) \in \{\phi^{t-1}_-(x), \min BR(\phi^{t-1}_-|x)\}$ for all $x \in X$ and $t \geq 1$. By definition, $\lim_{t\to\infty} \phi^t_-(x) = s^*$ for all $x \in X$.

The sequence $(\phi^t_-)_{t=0}^\infty$ satisfies properties (1)–(4), but not necessarily property (5). From $(\phi^t_-)_{t=0}^\infty$, we construct another sequence that satisfies property (5) as well. Let $\psi^0_- = \phi^0_-$ and

$$\psi^t_-(x) = \begin{cases} \psi^{t-1}_-(x) & \text{if } \phi^t_-(x) \leq \psi^{t-1}_-(x), \\ \min BR(\psi^{t-1}_-|x) & \text{if } \phi^t_-(x) > \psi^{t-1}_-(x). \end{cases}$$

Clearly, $(\psi^t_-)_{t=0}^\infty$ is a sequential best response sequence. By the construction of $(\phi^t_-)_{t=0}^\infty$ and $(\psi^t_-)_{t=0}^\infty$ along with the supermodularity of u and s^* being a Nash equilibrium of (S, u), one can show by induction on t that $\phi^t_-(x) \leq \psi^t_-(x) \leq s^*$ for all $x \in X$ and $t \geq 0$. Thus for each $x \in X$, $(\psi^t_-(x))_{t=0}^\infty$ is weakly increasing and converges to s^*.

Since s^* is a strict Nash equilibrium of (S, u), we can take a finite but sufficiently large subset Z of $\bigcup_{x\in Y} \Gamma(x)$ such that for any $x \in Y$, the best response of player x is s^* if all players in Z play s^* (recall that $\Gamma(x)$ is the set of the neighbors of player x). Let T be sufficiently large so that $\psi^T_-(x) = s^*$ for all $x \in Z$.

We claim that $\min BR(\psi^T_-|x) \geq \psi^T_-(x)$ for all $x \in X$. For $x \in Y$, since all players in Z play s^* in period T, we have $\min BR(\psi^T_-|x) = s^* \geq \psi^T_-(x)$. For $x \in X \setminus Y$, we first have $\min BR(\psi^0_-|x) \geq \underline{s} = \psi^0_-(x)$. Next, assume that $\min BR(\psi^{t-1}_-|x) \geq \psi^{t-1}_-(x)$. By the construction of $(\psi^t_-(x))_{t=0}^\infty$, $\psi^t_-(x)$ is equal to either $\psi^{t-1}_-(x)$ or $\min BR(\psi^{t-1}_-|x)$. In both cases, we have $\min BR(\psi^{t-1}_-|x) \geq \psi^t_-(x)$. Since $(\psi^t_-)_{t=0}^\infty$ is weakly increasing, we have $\min BR(\psi^t_-|x) \geq \min BR(\psi^{t-1}_-|x)$ by the supermodularity of u. Hence, $\min BR(\psi^t_-|x) \geq \psi^t_-(x)$.

Now let $\sigma^t_- = \psi^{t+T}_-$ for $t \geq 0$. Then $(\sigma^t_-)_{t=0}^\infty$ satisfies properties (1)–(5). In particular, along the sequential best response sequence $(\psi^t_-)_{t=0}^\infty$, at most

T players change actions by period T, so that $\sigma^0_-(x) = \psi^T_-(x) = \underline{s}$ except for finitely many x. The construction of $(\sigma^t_+)_{t=0}^\infty$ is analogous.

For property (6), pick any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^\infty$ with $\sigma^0_- \leq \tilde{\sigma}^0 \leq \sigma^0_+$. For each $x \in X$, let $\underline{\tilde{\sigma}}^t(x) = \inf_{\tau \geq t} \tilde{\sigma}^\tau(x)$, and $\tilde{\sigma}_-(x) = \liminf_{t\to\infty} \tilde{\sigma}^t(x)$ $(= \lim_{t\to\infty} \underline{\tilde{\sigma}}^t(x))$.

Claim 1. $\liminf_{t\to\infty} \min BR(\tilde{\sigma}^t|x) \geq \min BR(\tilde{\sigma}_-|x)$ *for all* $x \in X$.

Proof. Fix any $x \in X$. By the supermodularity of u, we have $\min BR(\tilde{\sigma}^t|x) \geq \min BR(\underline{\tilde{\sigma}}^t|x)$ for all $t \geq 0$. Therefore, we have

$$\begin{aligned}\liminf_{t\to\infty} \min BR(\tilde{\sigma}^t|x) &\geq \liminf_{t\to\infty} \min BR(\underline{\tilde{\sigma}}^t|x) \\ &\geq \min BR\left(\lim_{t\to\infty} \underline{\tilde{\sigma}}^t \;\middle|\; x\right) = \min BR(\tilde{\sigma}_-|x),\end{aligned}$$

where the second inequality follows from the lower semicontinuity of $\min BR(\cdot|x)$ in the product topology on S^X. □

Claim 2. $\tilde{\sigma}_-(x) \geq \min BR(\tilde{\sigma}_-|x)$ *for all* $x \in X$.

Proof. Fix any $x \in X$. By Claim 1, there exists $T_1 \geq 0$ such that $\min BR(\tilde{\sigma}^t|x) \geq \min BR(\tilde{\sigma}_-|x)$ for all $t \geq T_1$. By (ii) and (iii) in Definition B.1, there exists $T_2 \geq T_1$ such that $\tilde{\sigma}^{T_2}(x) \geq \min BR(\tilde{\sigma}_-)$. By (ii) in Definition B.1, we also have $\tilde{\sigma}^t(x) \geq \tilde{\sigma}^{T_2}(x) \wedge \min_{T_2 \leq \tau < t} \min BR(\tilde{\sigma}^\tau|x)$ for all $t \geq T_2$. Therefore, by Claim 1 it follows that $\tilde{\sigma}^t(x) \geq \min BR(\tilde{\sigma}_-)$ for all $t \geq T_2$, and hence $\tilde{\sigma}_-(x) \geq \min BR(\tilde{\sigma}_-|x)$, as desired. □

Claim 3. $\sigma^t_- \leq \tilde{\sigma}_-$ *for all* $t \geq 0$.

Proof. We proceed by induction. First, we want to show $\sigma^0_- \leq \tilde{\sigma}_-$. By assumption, $\sigma^0_- \leq \tilde{\sigma}^0$. Assume that $\sigma^0_- \leq \tilde{\sigma}^{t-1}$, and consider any $x \in X$ such that $\tilde{\sigma}^t(x) \neq \tilde{\sigma}^{t-1}(x)$. Then by the property (5) and the supermodularity of u, $\sigma^0_-(x) \leq \min BR(\sigma^0_-|x) \leq \min BR(\tilde{\sigma}^{t-1}|x) \leq \tilde{\sigma}^t(x)$. Therefore, we have $\sigma^0_- \leq \tilde{\sigma}^t$ for all $t \geq 0$, and hence $\sigma^0_- \leq \tilde{\sigma}_-$.

Next, assume that $\sigma^{t-1}_- \leq \tilde{\sigma}_-$, and let $x \in X$ be such that $\sigma^t_-(x) \neq \sigma^{t-1}_-(x)$. Then by the property (3), the induction hypothesis, the supermodularity of u, and Claim 2, we have $\sigma^t_-(x) = \min BR(\sigma^{t-1}_-|x) \leq \min BR(\tilde{\sigma}_-|x) \leq \tilde{\sigma}_-(x)$. Thus we have $\sigma^t_- \leq \tilde{\sigma}_-$. □

Symmetrically, defining $\tilde{\sigma}_+(x) = \limsup_{t\to\infty} \tilde{\sigma}^t(x)$, we can show that $\tilde{\sigma}_+ \leq \sigma^t_+$ for all $t \geq 0$. For each $x \in X$, since $\lim_{t\to\infty} \sigma^t_-(x) = \lim_{t\to\infty} \sigma^t_+(x) = s^*$, we have $\tilde{\sigma}_-(x) = \tilde{\sigma}_+(x) = s^*$, and hence $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$.

This completes the proof of Lemma B.1. □

Proposition B.1. *Fix a network (X, P) and a supermodular game (S, u). Then s^* is contagious by sequential best responses in (X, P) if and only if it is contagious by generalized best responses in (X, P).*

Proof. The "if" part holds by definition. To show the "only if" part, suppose that s^* is contagious by sequential best responses in (X, P). Let $(\sigma_-^t)_{t=0}^{\infty}$ and $(\sigma_+^t)_{t=0}^{\infty}$ be sequential best response sequences as in Lemma B.1. Let Y be a finite subset of X such that $\sigma_-^0(x) = \underline{s}$ and $\sigma_+^0(x) = \overline{s}$ for all $x \in X \setminus Y$. Then for any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^{\infty}$ with $\tilde{\sigma}^0(x) = s^*$ for all $x \in Y$, we have $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$, and hence by Lemma B.1, $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$ for all $x \in X$. Thus s^* is contagious by generalized best responses in (X, P). $\square$

Similarly, we can prove the equivalence between contagion by simultaneous best responses and contagion by generalized best responses. Here we assume that the set of neighbors $\Gamma(x)$ is finite for each player $x \in X$, which is satisfied in all of our examples.

Proposition B.2. *Fix a network (X, P) such that $\Gamma(x)$ is finite for each $x \in X$ and a supermodular game (S, u). Then s^* is contagious by simultaneous best responses in (X, P) if and only if it is contagious by generalized best responses in (X, P).*

Proof. The "if" part holds by definition. The proof of the "only if" part is to mimic the proofs of Lemma B.1 and the "only if" part of Proposition B.1. Indeed, as in the proof of Lemma B.1, we take a simultaneous best response sequence $(\phi_-^t)_{t=0}^{\infty}$, modify it to obtain a generalized (not necessarily simultaneous) best response sequence $(\psi_-^t)_{t=0}^{\infty}$, and then define $(\sigma_-^t)_{t=0}^{\infty}$ by $\sigma_-^t = \psi_-^{t+T}$ for sufficiently large T. The only difference lies here, where it is not the case in general that "at most T players change actions by period T". Instead, we assume without loss of generality that action $\underline{s}$ (as well as action $\overline{s}$) is a Nash equilibrium of (S, u), and resort to the finiteness of $\Gamma(x)$ to show that in each step of $(\psi_-^t)_{t=0}^{T}$, only finitely many players have minimum best responses other than action $\underline{s}$. $\square$

Another definition is to only require *some* sequential best response sequence to converge.

Definition B.3. Given a pairwise game (S, u), action s^* is *weakly contagious in network* (X, P) if there exists a finite subset Y of X such that for any initial action configuration σ^0 such that $\sigma^0(x) = s^*$ for any $x \in Y$, there exists a sequential best response sequence (σ^t) such that $\lim_{t\to\infty} \sigma^t(x) = s^*$ for any $x \in X$.

By definition, contagion implies weak contagion. The converse does not always hold. A counterexample is given by the trivial payoff function $u \equiv 0$, where all actions are weakly contagious but none of them is contagious. Nevertheless, we can show that weak contagion is equivalent to contagion for generic supermodular games.

We say that a game (S, u) is *generic for* (X, P) if no player has multiple best responses to any action configuration on (X, P). If each player has finitely many neighbors, then genericity excludes at most countably many hyperplanes in the payoff parameter space.

Proposition B.3. *Fix a network* (X, P) *and a generic supermodular game* (S, u) *for* (X, P)*. Then* s^* *is weakly contagious in* (X, P) *if and only if it is contagious by generalized best responses in* (X, P)*.*

Proof. Once again, the proof is almost the same as the proofs of Lemma B.1 and Proposition B.1. We only need to make the following two changes.

First, in the first paragraph of the proof of Lemma B.1, given a finite set $Y \subset X$ as in Definition B.3, let $(\phi_-^t)_{t=0}^\infty$ be a sequential best response sequence such that $\phi_-^0(x) = s^*$ for all $x \in Y$, $\phi_-^0(x) = \underline{s}$ for all $x \in X \setminus Y$, and $\lim_{t\to\infty} \phi_-^t(x) = s^*$ for all $x \in X$. Here it follows from the genericity of (S, u) that we have $\phi_-^t(x) \in \{\phi_-^{t-1}(x), BR(\phi_-^{t-1}|x)\}$ for any $x \in X$ and $t \geq 1$, where with an abuse of notation, $BR(\phi_-^{t-1}|x)$ denotes the unique best response.

Second, a weakly contagious action is always a Nash equilibrium of (S, u), but may not be a *strict* Nash equilibrium. Here again, the genericity assumption guarantees that the weakly contagious action s^* is a strict Nash equilibrium. □

B.2. Proof of Lemma 3

Given a payoff function $f\colon S \times S \to \mathbb{R}$, we write BR_f for the best correspondence for the local interaction game (X, P, S, f):

$$\begin{aligned} BR_f(\sigma|x) = \bigl\{h \in S \bigm| {}& \textstyle\sum_{y\in\Gamma(x)} P(y|x) f(h, \sigma(y)) \\ &\geq \textstyle\sum_{y\in\Gamma(x)} P(y|x) f(h', \sigma(y)) \text{ for all } h' \in S\bigr\}. \end{aligned}$$

(Thus the best response correspondence for the local interaction game (X, P, S, u) as defined in (2.3) is now denoted BR_u.) Recall that $BR_f(\sigma|x) = br_f(\pi(\sigma|x))$. We show a result stronger than Lemma 3, that a strict MP-maximizer is uninvadable by sequences that satisfy properties (i) and (ii) in Definition 1. Such sequences do not necessarily satisfy property (iii) in Definition 1, so that some players may have no opportunity to revise their suboptimal actions. We call those sequences partial best response sequences.

Definition B.4. Given a network (X, P) and for a payoff function $f\colon S \times S \to \mathbb{R}$, a sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *partial best response sequence* in local interaction game (X, P, S, f) if it satisfies the following properties: (i) for all $t \geq 1$, there is at most one $x \in X$ such that $\sigma^t(x) \neq \sigma^{t-1}(x)$; and (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR_f(\sigma^{t-1}|x)$.

The following result is due to Morris [8, Proposition 6.1].

Lemma B.2. *Suppose that s^* is a potential maximizer of (S, u) with a potential function v. For any unbounded network (X, P) and any partial best response sequence $(\sigma^t)_{t=0}^{\infty}$ in local interaction game (X, P, S, u) with $\sigma_P^0(S \setminus \{s^*\}) < \infty$, there exists $M < \infty$ such that $\sigma_P^t(S \setminus \{s^*\}) \leq M$ for all $t \geq 0$.*

Lemma 3 is a direct corollary of the following.

Lemma B.3. *Suppose that $s^* \in \{\underline{s}, \overline{s}\}$ is a strict MP-maximizer of (S, u) with a strict MP-function v. If u or v is supermodular, then for any unbounded network (X, P) and any partial best response sequence $(\sigma^t)_{t=0}^{\infty}$ in local interaction game (X, P, S, u) with $\sigma_P^0(S \setminus \{s^*\}) < \infty$, there exists $M < \infty$ such that $\sigma_P^t(S \setminus \{s^*\}) \leq M$ for all $t \geq 0$.*

Proof. Let $s^* \in \{\underline{s}, \overline{s}\}$ be a strict MP-maximizer of (S, u) with a strict MP-function v. We only consider the case where $s^* = \underline{s}$. Fix any network (X, P). Let $(\sigma^t)_{t=0}^{\infty}$ be any partial best response sequence in (X, P, S, u). such that $\sigma_P^0(S \setminus \{\underline{s}\}) < \infty$.

Let $(\hat{\sigma}^t)_{t=0}^{\infty}$ be defined by $\hat{\sigma}^0 = \sigma^0$ and for $t \geq 1$,

$$\hat{\sigma}^t(x) = \begin{cases} \max BR_v(\hat{\sigma}^{t-1}|x) & \text{if } \sigma^t(x) \neq \sigma^{t-1}(x), \\ \hat{\sigma}^{t-1}(x) & \text{otherwise.} \end{cases}$$

Clearly, $(\hat{\sigma}^t)_{t=0}^{\infty}$ is a partial best response sequence in (X, P, S, v). Therefore, by Lemma B.2, there exists M such that $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show that if u or v is supermodular, then

$$\sigma^t \leq \hat{\sigma}^t \tag{$\star_t$}$$

for all $t \geq 0$. Then, $\sigma_P^t(S \setminus \{\underline{s}\}) \leq \hat{\sigma}_P^t(S \setminus \{\underline{s}\})$ for all t, and since $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t, it follows that $\sigma_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show by induction that $(\star_t)$ holds for all $t \geq 0$. First, $(\star_0)$ trivially holds by the definition of $\hat{\sigma}^0$. Next, assume $(\star_{t-1})$. It implies that for all $x \in X$, $\pi(\sigma^{t-1}|x) \precsim \pi(\hat{\sigma}^{t-1}|x)$. Let $x \in X$ be such that $\sigma^t(x) \neq \sigma^{t-1}(x)$, and hence $\hat{\sigma}^t(x^t) = \max br_v(\pi(\hat{\sigma}^{t-1}|x^t))$ by construction. If u is supermodular, then

$$\begin{aligned}\sigma^t(x^t) &\leq \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_u(\pi(\hat{\sigma}^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t),\end{aligned}$$

where the second inequality follows from the supermodularity of u, and the third inequality follows from (3.5). If v is supermodular, then

$$\begin{aligned}\sigma^t(x^t) &\leq \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t),\end{aligned}$$

where the second inequality follows from (3.5), and the third inequality follows from the supermodularity of v. Therefore, in each case, $(\star_t)$ holds. $\square$

We show in passing that Lemma 3 extends to generalized best response sequences (Definition B.1) in any network where each player has finitely many neighbors.

Definition B.5. Given a pairwise game (S, u), action s^* is *uninvadable by generalized best response sequences in network* (X, P) if there exists no generalized best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$ and $\lim_{t\to\infty} \sigma_P^t(S \setminus \{s^*\}) = \infty$.

Proposition B.4. *Let (S, u) be any game with totally ordered action set S. If $s^* \in \{\underline{s}, \overline{s}\}$ is a strict MP-maximizer of (S, u) with a strict MP-function v and if u or v is supermodular, then s^* is uninvadable by generalized best responses in (X, P) such that $\Gamma(x)$ is finite for all $x \in X$.*

Proof. Let $s^* \in \{\underline{s}, \overline{s}\}$ be a strict MP-maximizer of u with a strict MP-function v. We only consider the case where $s^* = \underline{s}$. Fix any network (X, P) such that $\Gamma(x)$ is finite for all $x \in X$. Let $(\sigma^t)_{t=0}^{\infty}$ be any generalized best response sequence in (X, P, S, u) such that $\sigma_P^0(S \setminus \{\underline{s}\}) < \infty$. We will construct a nondecreasing partial best response sequence $(\hat{\sigma}^\tau)_{\tau=0}^{\infty}$ in (X, P, S, v) such that

$$\sigma^t \leq \bar{\sigma} \tag{$\star\star_t$}$$

for all $t \geq 0$, where $\bar{\sigma}$ is defined by $\bar{\sigma}(x) = \lim_{\tau\to\infty} \hat{\sigma}^\tau(x)$ for all $x \in X$. Then, we have, for all $t \geq 0$,

$$\sigma_P^t(S \setminus \{\underline{s}\}) \leq \bar{\sigma}_P(S \setminus \{\underline{s}\}) = \lim_{\tau\to\infty} \hat{\sigma}_P^\tau(S \setminus \{\underline{s}\}) < \infty$$

as desired, where the last inequality (the finiteness of $\lim_{\tau\to\infty}\hat{\sigma}^{\tau}_P(S\setminus\{\underline{s}\})$) follows from Lemma B.2.

We construct such a sequence $(\hat{\sigma}^{\tau})_{\tau=0}^{\infty}$ as follows. Pick a sequence $(x^{\tau})_{\tau=1}^{\infty}$ in X such that $\{\tau\geq 1 \mid x^{\tau}=x\}$ is infinite for each $x\in X$.[2] Then, let $\hat{\sigma}^0=\sigma^0$, and for each $\tau\geq 1$, let $\hat{\sigma}^{\tau}(x^{\tau})=\max\{\max BR_v(\hat{\sigma}^{\tau-1}|x^{\tau}),\hat{\sigma}^{\tau-1}(x^{\tau})\}$ and $\hat{\sigma}^{\tau}(x)=\hat{\sigma}^{\tau-1}(x)$ for $x\neq x^{\tau}$. By construction, $(\hat{\sigma}^{\tau})_{\tau=0}^{\infty}$ is a partial best response sequence in (X,P,S,v), and for each $x\in X$, $(\hat{\sigma}^{\tau}(x))_{t=0}^{\infty}$ is nondecreasing. Denote $\bar{\sigma}(x)=\lim_{\tau\to\infty}\hat{\sigma}^{\tau}(x)$. Note that $\bar{\sigma}\geq\hat{\sigma}^{\tau}$ for all $\tau\geq 0$.

Claim 1. $\max BR_v(\bar{\sigma}|x)\leq\bar{\sigma}(x)$ *for all* $x\in X$.

Proof. Fix any $x\in X$. By the finiteness of $\Gamma(x)$, there exists T such that $\hat{\sigma}^{\tau}(y)=\bar{\sigma}(y)$ for all $y\in\Gamma(x)$ and all $\tau\geq T$. By the construction of $(\hat{\sigma}^{\tau})_{\tau=0}^{\infty}$, there exists $\tau'>T$ such that $x^{\tau'}=x$, and with such a τ' we have $\max BR_v(\bar{\sigma}|x)=\max BR_v(\hat{\sigma}^{\tau'-1}|x)\leq\hat{\sigma}^{\tau'}(x)\leq\bar{\sigma}(x)$. □

Now we show by induction that $(\star\star_t)$ holds for all $t\geq 0$. First, $(\star\star_0)$ holds by the construction of $(\hat{\sigma}^{\tau})_{\tau=0}^{\infty}$. Next, assume $(\star\star_{t-1})$. It implies that for all $x\in X$, $\pi(\sigma^{t-1}|x)\precsim\pi(\bar{\sigma}|x)$. Let $x\in X$ be such that $\sigma^t(x)\neq\sigma^{t-1}(x)$, and hence $\sigma^t(x)\in br_u(\pi(\sigma^{t-1}|x))$. If u is supermodular, then $\sigma^t(x)\leq\max br_u(\pi(\sigma^{t-1}|x))\leq\max br_u(\pi(\bar{\sigma}|x))\leq\max br_v(\pi(\bar{\sigma}|x))\leq\bar{\sigma}(x)$, where the second inequality follows from the supermodularity of u, the third from (3.5), and the fourth from Claim 1. If v is supermodular, then $\sigma^t(x)\leq\max br_u(\pi(\sigma^{t-1}|x))\leq\max br_v(\pi(\sigma^{t-1}|x))\leq\max br_v(\pi(\bar{\sigma}|x))\leq\bar{\sigma}(x)$, where the second inequality follows from (3.5), the third from the supermodularity of v, and the fourth from Claim 1. Therefore, in each case, $(\star\star_t)$ holds. □

B.3. Multidimensional Lattice Networks

We fix the dimension m. A sequence $(P_n)_{n=0}^{\infty}$ of interaction weights on the m-dimensional lattice $\mathbb{Z}^m$ is *well-behaved* if the following conditions are satisfied.

- For each n, P_n is invariant up to translation, i.e., $P_n(x,y)=P_n(x+z,y+z)$ for $x,y,z\in\mathbb{Z}^m$.

[2]For example, enumerate $X=\{x_1,x_2,x_3,\ldots\}$, and for each $\tau\geq 1$, let $\ell(\tau)$ be the largest integer ℓ such that $\ell(\ell+1)/2<\tau$, and let $k(\tau)=\tau-\ell(\tau)(\ell(\tau)+1)/2$ and $x^{\tau}=x_{k(\tau)}$.

- There exist a pair of nonnegative integrable functions $g, \bar{g} \colon \mathbb{R}^m \to \mathbb{R}_+$ such that for almost every $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m$, we have $n^m P_n([n\nu]|0) \to g(\nu)$ as $n \to \infty$ (pointwise convergence), and $n^m P_n([n\nu]|0) \leq \bar{g}(\nu)$ for every n.[3]

- The support of g is connected.

For example, consider n-max distance interactions P_n, where $P_n(x,y) = 1$ if $1 \leq \max_i |x_i - y_i| \leq n$ and $P_n(x,y) = 0$ otherwise. Then $(P_n)_{n=0}^{\infty}$ is well-behaved since $n^m P_n([n\nu]|0)$ converges to 2^{-m} times the indicator function of $\{\nu \in \mathbb{R}^m \mid \max_i |\nu_i| \leq 1\}$.

The next theorem characterizes contagion and uninvadability in the limit of any well-behaved sequence of multidimensional lattice networks. The core of the proof is similar to that of Lemma 1, but we take $n \to \infty$ in order to mitigate the "lumpiness" of interaction weights.

Theorem B.1. *Let (S, u) be the bilingual game given by* (2.2)*. Fix the dimension m and a well-behaved sequence $(P_n)_{n=0}^{\infty}$ of interaction weights on $\mathbb{Z}^m$.* (i) *If $e < e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$, A is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.* (ii) *If $e > e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$, B is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.*

Proof. We will show (i) only. The proof for (ii) is analogous.

By Lemma 2(i-1) and the upper semi-continuity of br, there exist $p \in (0, 1/2)$ and $\varepsilon \in (0, 1/2 - p)$ such that $\max br(\hat{\pi}^a) = A$ and $\max br(\hat{\pi}^b) \leq AB$, where

$$\hat{\pi}^a = \left(\tfrac{1}{2} - \varepsilon, p, \tfrac{1}{2} - p + \varepsilon\right), \quad \hat{\pi}^b = \left(\tfrac{1}{2} - p - \varepsilon, p, \tfrac{1}{2} + \varepsilon\right).$$

Let $g(\nu)$ be the pointwise limit of $n^m P_n([n\nu]|0)$ as $n \to \infty$. Since P_n is symmetric and translation invariant, g is symmetric, i.e., $g(\nu) = g(-\nu)$ for almost all ν. We also have $\int_{\mathbb{R}^m} g(\nu) d\nu = 1$.

Since g is symmetric and has a connected support, for each $\lambda \in \mathbb{R}^m$ whose Euclidean norm $\|\lambda\|$ is 1, there exists a unique $\delta = \delta(\lambda) > 0$ that satisfies

$$\int_{0 \leq \lambda \cdot x \leq \delta} g(x) dx = p$$

and $\delta(\lambda)$ is continuous in λ, since the left hand side is continuous in λ and δ and strictly increasing in δ (whenever the left hand side is less than 1/2).

[3]For $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$, $[\eta] = ([\eta_1], \ldots, [\eta_m])$ denotes the profiles of the largest integers that do not exceed η_i.

For each $r > 0$, let D_r be a disk $\{\nu \in \mathbb{R}^m \mid \|\nu\| \leq r\}$ and R_r be a ring-shaped object $\{\nu \in \mathbb{R}^m \mid r < \|\nu\| \leq r + \delta(\nu/\|\nu\|)\}$. Note that for large r and any boundary point ν of D_r, we have $\lambda \cdot \xi \approx r$ for any boundary point ξ of D_r near ν. By the continuity of $\delta(\cdot)$, the same is true for the boundary of R_r; i.e., for large r and any boundary point ν of D_r, we have $\lambda \cdot \xi \approx r + \delta(\nu/\|\nu\|)$ for any boundary point ξ of R_r near ν. Thus, there exists r_1 such that for any $r \geq r_1$,

$$\nu \in D_r \Longrightarrow \int_{D_r} g(\xi - \nu) d\xi \geq \frac{1}{2} - \frac{\varepsilon}{3}, \; \int_{D_r \cup R_r} g(\xi - \nu) d\xi \geq \frac{1}{2} + p - \frac{\varepsilon}{3},$$
$$\nu \in R_r \Longrightarrow \int_{D_r} g(\xi - \nu) d\xi \geq \frac{1}{2} - p - \frac{\varepsilon}{3}, \; \int_{D_r \cup R_r} g(\xi - \nu) d\xi \geq \frac{1}{2} - \frac{\varepsilon}{3}.$$

For each $k \in \mathbb{N}$, let $\hat{D}_k = \{x \in \mathbb{Z}^m \mid \|x\| \leq k\}$ and $\hat{R}_{k,n} = \{x \in \mathbb{Z}^m \mid k < \|x\| \leq k + n\delta(x/\|x\|)\}$. Since $(P_n)_{n=0}^{\infty}$ is well-behaved, one can apply the dominated convergence theorem to show that there exists n_1 such that for any $n \geq n_1$, any $x \in \mathbb{Z}^m$, and any $k \in \mathbb{N}$,

$$\left| \sum_{y \in \hat{D}_k} P_n(y - x|0) - \int_{D_{k/n}} g(\xi - x/n) d\xi \right| \leq \frac{\varepsilon}{3},$$
$$\left| \sum_{y \in \hat{D}_k \cup \hat{R}_{k,n}} P_n(y - x|0) - \int_{D_{k/n} \cup R_{k/n}} g(\xi - x/n) d\xi \right| \leq \frac{\varepsilon}{3}.$$

Therefore, there exists $n_2 \geq n_1$ such that for any $n \geq n_2$ and any $k \geq r_1 n$,

$$x \in \hat{D}_{k+1} \Longrightarrow \sum_{y \in \hat{D}_k} P_n(y|x) \geq \frac{1}{2} - \varepsilon, \quad \sum_{y \in \hat{D}_k \cup \hat{R}_{k,n}} P_n(y|x) \geq \frac{1}{2} + p - \varepsilon,$$
$$x \in \hat{R}_{k+1,n} \Longrightarrow \sum_{y \in \hat{D}_k} P_n(y|x) \geq \frac{1}{2} - p - \varepsilon, \quad \sum_{y \in \hat{D}_k \cup \hat{R}_{k,n}} P_n(y|x) \geq \frac{1}{2} - \varepsilon.$$

Now let $n \geq n_2$. We show that A is contagious in $(\mathbb{Z}^m, P_n)$. The proof is similar to that of Lemma 1(1). Pick a natural number $K \geq r_1 n$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = A$ for all $x \in \hat{D}_K \cup \hat{R}_{K,n}$. Then one can show by induction on k that for any $k \geq K$, there exists T_k such that for any $T \geq T_k$, we have $\sigma^t(x) = A$ for all $x \in \hat{D}_k$ and $\sigma^0(x) \leq AB$ for all $x \in \hat{R}_{k,n}$.

This argument also shows that A is uninvadable in $(\mathbb{Z}^m, P_n)$ because for any initial configuration that satisfies $\sigma^0_{P_n}(\{AB, B\}) < \infty$, there exists a translation Y of $\hat{D}_K \cup \hat{R}_{K,n}$ such that $\sigma^0(x) = A$ for all $x \in Y$. $\square$

B.4. Proof of Theorem 3

We denote by $\underline{s}$ and $\overline{s}$ the smallest and the largest actions, respectively. We use the partial order $\sigma \leq \sigma'$ whenever $\sigma(x) \leq \sigma'(x)$ for any $x \in X$.

Let φ be a weight-preserving node identification from (X, P) to $(\hat{X}, \hat{P})$ with a finite set E of exceptional nodes. Fix a supermodular game (S, u), and assume that s^* is contagious in (X, P). We show that s^* is contagious in $(\hat{X}, \hat{P})$.

Since s^* is a strict Nash equilibrium of (S, u), there exists a finite subset $F \subset X$ such that $F \supset E$ and s^* is the unique best response for any $\hat{x} \in \varphi(E)$ if all players in $\varphi(F)$ play s^*.

Let $(\sigma_-^t)_{t=0}^\infty$ and $(\sigma_+^t)_{t=0}^\infty$ be sequential best response sequences in (X, P) that satisfy properties (1)–(5) in Lemma B.1. Pick a $T \geq 0$ such that $\sigma_-^T(x) = \sigma_+^T(x) = s^*$ for all $x \in F$, and let $Y = \{x \in X \mid \sigma_-^T(x) \neq \underline{s} \text{ or } \sigma_+^T(x) \neq \overline{s}\}$. Note that $Y \supset F$ and Y is finite.

Define action configurations $\hat{\sigma}_-$ and $\hat{\sigma}_+$ in $(\hat{X}, \hat{P})$ by

$$\hat{\sigma}_-(\hat{x}) = \max_{x \in \varphi^{-1}(\hat{x})} \sigma_-^T(x) \text{ and } \hat{\sigma}_+(\hat{x}) = \min_{x \in \varphi^{-1}(\hat{x})} \sigma_+^T(x)$$

for all $\hat{x} \in \hat{X}$. Note that $\hat{\sigma}_-(\hat{x}) = \hat{\sigma}_+(\hat{x}) = s^*$ for all $\hat{x} \in \varphi(F)$, and $\hat{\sigma}_-(\hat{x}) = \underline{s}$ and $\hat{\sigma}_+(\hat{x}) = \overline{s}$ for all $\hat{x} \in \hat{X} \setminus \varphi(Y)$. Denote by $\widehat{BR}$ the set of best responses defined in $(\hat{X}, \hat{P})$.

Claim 1. *$\min \widehat{BR}(\hat{\sigma}_-|\hat{x}) \geq \hat{\sigma}_-(\hat{x})$ and $\hat{\sigma}_+(\hat{x}) \leq \max \widehat{BR}(\hat{\sigma}_+|\hat{x})$ for all $\hat{x} \in \hat{X}$.*

Proof. We only show the first inequality; the proof of the second is analogous. For any $\hat{x} \in \varphi(E)$, since $\hat{\sigma}_-(\hat{y}) = s^*$ for all $\hat{y} \in \varphi(F)$, we have $\widehat{BR}(\hat{\sigma}_-|\hat{x}) = \{s^*\}$ by the construction of F. For any $\hat{x} \in X \setminus \varphi(E)$, let $\bar{\sigma}_-^T = \hat{\sigma}_- \circ \varphi$, and let $\bar{x} \in \arg\max_{x \in \varphi^{-1}(\hat{x})} \sigma_-^T(x)$. Then we have $\min \widehat{BR}(\hat{\sigma}_-|\hat{x}) = \min BR(\bar{\sigma}_-^T|\bar{x}) \geq BR(\sigma_-^T|\bar{x}) \geq \sigma_-^T(\bar{x}) = \hat{\sigma}_-(\hat{x})$, where the first equality follows from the weight-preserving property of φ, the first inequality from the supermodularity of u, and the second inequality from property (5) in Lemma B.1. $\square$

Let $\hat{Y} = \varphi(Y)$, which is finite. Pick any sequential best response sequence $(\hat{\sigma}^t)$ in $(\hat{X}, \hat{P})$ such that $\hat{\sigma}^0(\hat{x}) = s^*$ for all $\hat{x} \in \hat{Y}$. We want to show that $\lim_{t \to \infty} \hat{\sigma}^t(\hat{x}) = s^*$ for all $\hat{x} \in \hat{X}$.

Claim 2. *$\hat{\sigma}_- \leq \hat{\sigma}^t \leq \hat{\sigma}_+$ for all $t \geq 0$.*

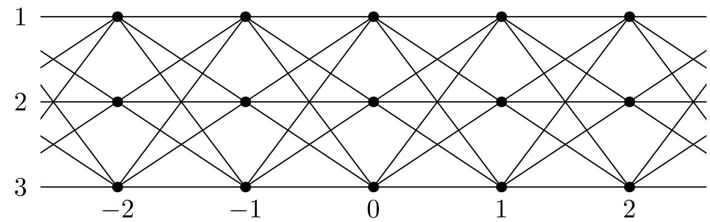

Figure B.1: Replicated linear network

Proof. We only show the first inequality; the proof of the second is analogous. First, we have $\hat{\sigma}^0 \geq \hat{\sigma}_-$ by construction. Next, assume $\hat{\sigma}^{t-1} \geq \hat{\sigma}_-$. If $\hat{\sigma}^t(\hat{x}) \neq \hat{\sigma}^{t-1}(\hat{x})$, then we have $\hat{\sigma}^t(\hat{x}) \geq \min \widehat{BR}(\hat{\sigma}^{t-1}|\hat{x}) \geq \min \widehat{BR}(\hat{\sigma}_-|\hat{x}) \geq \hat{\sigma}_-(\hat{x})$, where the first inequality follows from the definition of sequential best response sequence, the second follows from the supermodularity of u, and the third from Claim 1. □

Claim 2 implies in particular that $\hat{\sigma}^t(\hat{x}) = s^*$ for all $\hat{x} \in \varphi(F)$ and all $t \geq 0$.

Given the sequence $(\hat{\sigma}^t)_{t=0}^{\infty}$ in $(\hat{X}, \hat{P})$, let $(\tilde{\sigma}^t)_{t=0}^{\infty}$ be the corresponding sequence in (X, P) defined by $\tilde{\sigma}^t = \hat{\sigma}^t \circ \varphi$ for all $t \geq 0$. First, by Claim 2, we have $\sigma_-^0 \leq \sigma_-^T \leq \hat{\sigma}_- \circ \varphi \leq \tilde{\sigma}^0 \leq \hat{\sigma}_+ \circ \varphi \leq \sigma_+^T \leq \sigma_+^0$. Second, $(\tilde{\sigma}^t)_{t=0}^{\infty}$ is a generalized best response sequence in (X, P) as defined in Definition B.1. (Notice that players in $\varphi^{-1}(\hat{x})$ change actions simultaneously.) Indeed, for $x \in X \setminus E$, we have $BR(\tilde{\sigma}^t|x) = \widehat{BR}(\hat{\sigma}^t|\varphi(x))$ for all $t \geq 0$ by the weight-preserving property of φ, while for $x \in E$, we have $\tilde{\sigma}^t(x) = s^*$ and $BR(\tilde{\sigma}^t|x) = \{s^*\}$ for all $t \geq 0$ by construction. Thus, by Lemma B.1(6), $(\sigma^t(x))_{t=0}^{\infty}$ converges to s^* for all $x \in X$, and hence $(\hat{\sigma}^t(\hat{x}))_{t=0}^{\infty}$ also converges to s^* for all $\hat{x} \in \hat{X}$. □

B.5. Examples

Example B.1 (Line versus replicated lines)**.** Let $(\{1, \ldots, m\} \times \mathbb{Z}, P)$ be a *replicated linear network*, where for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \{1, \ldots, m\} \times \mathbb{Z}$, we have $P(x, y) = P(x+z, y+z)$ (sums in the first coordinate are defined modulo m) and $P(x, y) = 0$ whenever $x_2 = y_2$.[4] An example of replicated linear network with $m = 3$ is depicted in Figure B.1. The mapping $\varphi \colon \{1, \ldots, m\} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(k, i) = i$ is a weight-preserving node

[4]The "thick line graph" in Immorlica et al. [5, Figure 2] is a special case of replicated linear network.

identification (with no exceptional node) from this network to the linear network $(\mathbb{Z}, \hat{P})$ with $\hat{P}(i,j) = \sum_{k=1}^{m} P((1,i),(k,j))$. In fact, one can show that the two networks are equally contagion-inducing in the class of all supermodular games. In particular, Theorem 2 extends to replicated linear networks.

Example B.2 (Line versus max distance)**.** Consider the m-dimensional lattice with n-max distance interactions, i.e., the network $(\mathbb{Z}^m, P)$ where $P(x,y) = 1$ if $1 \leq \max_i |x_i - y_i| \leq n$ and $P(x,y) = 0$ otherwise. Define the mapping $\varphi\colon \mathbb{Z}^m \to \mathbb{Z}$ by

$$\varphi(x_1, \ldots, x_m) = x_1 + (n+1)x_2 + \cdots + (n+1)^{m-1}x_m$$

for any $(x_1, \ldots, x_m) \in \mathbb{Z}^m$. Then φ is a weight-preserving node identification (with no exceptional node) from this network to the linear network $(\mathbb{Z}, \hat{P})$ with $\hat{P}(x,y) = \#(\varphi^{-1}(y-x) \cap [-n,n]^m)$ for any $x, y \in \mathbb{Z}$ with $x \neq y$.[5] Thus, by Theorem 3, the n-max distance interaction network is less contagion-inducing than some linear network. Combined with Theorem 2, this implies that for the bilingual game, action A is not contagious in the n-max distance interaction network if $e > e^*$.

Example B.3 (Regions versus lattice)**.** Consider the network depicted in Figure B.2, where the players are divided into infinitely many "regions", and each region consists of three players: $X = \{1,2,3\} \times \mathbb{Z}$, and with equal weights, player (k,i) interacts with players (ℓ, j) such that $\ell \neq k$ and $j = i$, or $\ell = k$ and $j = i \pm 1$. Then the mapping $\varphi\colon \mathbb{Z}^2 \to \{1,2,3\} \times \mathbb{Z}$ defined by $\varphi(x_1, x_2) = (k, x_2)$ such that $k \equiv x_1 \pmod 3$ is a weight-preserving node identification from the two-dimensional lattice to the regions network (with no exceptional node). Thus, by Theorem 3 and in a similar manner as in Example 2, one can show that the regions network is strictly more contagion-inducing than the two-dimensional lattice in the class of all supermodular games. This is in contrast to the class of 2×2 coordination games, where the two networks have the same contagion threshold $1/4$ (Examples 2 and 4 in Morris [9]).

Example B.4 (Line versus Figure 4)**.** Theorems 2 and 3 imply that there exists no weight-preserving node identification from the network in Figure 4 to any linear network.

[5] $\#X$ denotes the cardinality of X.

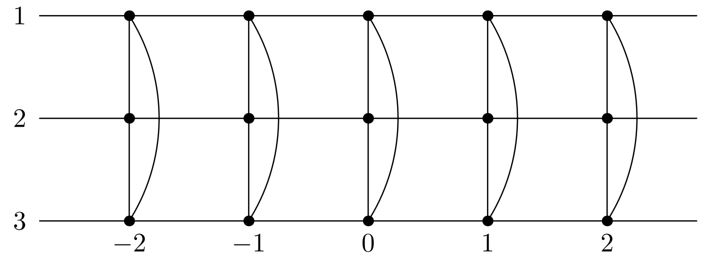

Figure B.2: Regions

Example B.5 (Line versus regions)**.** Consider the regions network as depicted in Figure B.2. In 2×2 coordination games, the regions network has contagion threshold 1/4, whereas the linear network in Figure 1 has contagion threshold 1/2. Also, in a similar manner as in Example 2, one can show that there is a set of parameter values of the bilingual game such that B is contagious in the regions network, but not in the linear network. Therefore, the regions network is incomparable to the linear network in the class of all supermodular games.

B.6. The Case Where Pareto Dominance and Risk Dominance Coincide

For completeness, we report the contagion and uninvadability result also for the case where action A is both Pareto-dominant and pairwise risk-dominant. The bilingual game (S, u) now satisfies

$$c \leq d < a, \;\; d - b < a - c, \;\; \text{and } e > 0. \tag{B.1}$$

Theorem B.2. *Let* (S, u) *be the bilingual game given by* (2.2a) *and* (B.1)*. A is always contagious and uninvadable.*

Proof. In light of Lemma 1(1-i) and Lemma 3, it suffices to show that condition (3.1) holds for some p and that A is a strict MP-maximizer. If $e \leq (d-b)/2$, we have $(c-b)e < (a-d)(d-b)/2$. Therefore, these follow from the argument in case (α) in the proof of Lemma 2(1) and Claims 1–3 in the proof of Lemma A.2. If $e > (d-b)/2$, they follow from the symmetric arguments for A in place of B as in case (β) in the proof of Lemma 2(1) and Lemma A.4. □

Goyal and Janssen [4, Theorem 3] show the contagion part of this theorem in their circular network.

Immorlica et al. [5] consider the current case with a payoff parameter restriction $a = 1 - q$, $b = c = 0$, and $d = q$, so the game is given by

$$\begin{array}{c} \\ A \\ AB \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \quad A \quad & \quad AB \quad & \quad B \quad \end{array} \\ \begin{pmatrix} 1-q & 1-q & 0 \\ 1-q-e & 1-q-e & q-e \\ 0 & q & q \end{pmatrix} \end{array}, \qquad 0 < q < \frac{1}{2}.$$

This game is a potential game with action A being the potential maximizer (more generally, the bilingual game is a potential game whenever $b = c$). Immorlica et al. [5] focus on the class $\mathcal{N}_\Delta$ of Δ-regular networks; for each $\Delta \in \mathbb{N}$, a Δ-regular network is a network where each player has Δ neighbors with constant weights. They consider the "epidemic region" $\Omega(X, P) \subset (0, 1/2) \times \mathbb{R}_{++}$, the set of parameter values (q, e) for which action A spreads contagiously in network (X, P), and show that for any fixed Δ, there exists a point $(q, e) \notin \Omega_\Delta := \bigcup_{(X,P) \in \mathcal{N}_\Delta} \Omega(X, P)$, and in particular, Ω_Δ is not convex. On the other hand, since contagion in Lemma 1(1-i) can be induced by a Δ-regular network with some Δ (see Footnote 17 in the main text), our Theorem B.2 implies that $\bigcup_{\Delta \in \mathbb{N}} \Omega_\Delta = (0, 1/2) \times \mathbb{R}_{++}$, which is convex.

B.7. Interpretations in Incomplete Information Games

Local interaction games and incomplete information games, though capturing different economic or social situations, share the same formal structures and thus belong to a more general class of "interaction games" (Morris [7, 8], Morris and Shin [11]): in local interaction games, each node interacts with a set of neighbors and payoffs are given by the weighted sum of those from the interactions; in incomplete information games, each type interacts with a subset of types and payoffs are given by the expectation of those from the interactions.[6] Indeed, Morris [7, 8] demonstrates, in spite of some technical differences, that several tools and results in the context of incomplete information games can be utilized also in the context of local interaction games, and vice versa.[7] In this section, we interpret our results in the language of incomplete information games, thereby shedding new light on two existing lines of literature, robustness to incomplete information and global

[6]For example, with the incomplete information interpretation, the linear network in Figure 1 is essentially equivalent to the information structure of the email game of Rubinstein [17].

[7]For example, the contagion threshold of a network due to Morris [9] is essentially equivalent to the belief potential of an information system due to Morris et al. [10].

games. We also discuss our symmetry assumption of interaction weights in relation to the common prior assumption in incomplete information games.

B.7.1. Robustness to Incomplete Information

A Nash equilibrium (s_1^*, s_2^*) of a two-player game (S, u) is said to be *robust to incomplete information* if any ε-incomplete information perturbation of (S, u) with ε sufficiently small has a Bayesian Nash equilibrium that plays (s_1^*, s_2^*) with high probability, where an ε-incomplete information perturbation of (S, u) refers to an incomplete information game in which the set T^u of type profiles whose payoffs are given by u has ex ante probability $1 - \varepsilon$ while types outside T^u ("crazy types") may have very different payoff functions (Kajii and Morris [6]).[8] Robustness to incomplete information corresponds to uninvadability in networks in that both notions require that a small amount of "crazy types" should not affect the aggregate behavior.

Indeed, they have the same characterizations in many classes of games. For example, in parallel with Lemma 3, an MP-maximizer of a game (S, u) with MP-function v is robust to incomplete information if u or v is supermodular (Morris and Ui [12]). Combining this result with Lemma 4, we obtain a sufficient condition for robustness in the bilingual game.

Conversely, a necessary condition for robustness is obtained by constructing ε-incomplete information perturbations in which a given action profile is contagious, where an action s^* is said to be contagious in an ε-incomplete information perturbation if s^* is a dominant action for types outside T^u and playing s^* everywhere is a unique rationalizable strategy. Specifically, in any symmetric 3×3 supermodular game (S, u), adjusting the proof of Lemma 1, for any $\varepsilon > 0$ one can construct ε-incomplete information perturbations in which 0 (2, resp.) is contagious if (3.1) ((3.2), resp.) holds for some $p \in (0, 1/2)$, or (3.3) holds for some $q, r \in (0, 1)$ with $r \leq q$ (Oyama and Takahashi [14]). The necessary condition thus follows by applying this result to the bilingual game combined with Lemma 2.

These arguments characterize, exactly as in Theorem 1, when an equilibrium in the bilingual game is robust to incomplete information.

Proposition B.5. *Let* (S, u) *be the bilingual game given by* (2.2)*.* (i) (A, A) *is a unique robust equilibrium if* $e < e^*$*.* (ii) (B, B) *is a unique robust equilibrium if* $e > \max\{e^*, e^{**}\}$*.* (iii) *No equilibrium is robust if* $e^* < e < \max\{e^*, e^{**}\}$*.*

[8]Kajii and Morris [6] consider games with any finite number of players.

B.7.2. Global Games

Global games constitute a subclass of incomplete information games, where the underlying state θ is drawn from the real line, and each player i receives a noisy signal $x_i = \theta + \nu\varepsilon_i$ with ε_i being a noise error independent across players and from θ. Under supermodularity and state-monotonicity in payoffs, it has been shown by a contagion argument that an essentially unique equilibrium survives iterative deletion of dominated strategies as $\nu \to 0$, while the limit equilibrium may depend on the distribution of noise terms ε_i (Frankel et al. [3]).

Global game perturbations in the class of all incomplete information perturbations can be viewed as linear networks in the class of all networks. In global games, the distribution of the opponent's signal x_j conditional on x_i is (approximately) invariant up to translation (for small $\nu > 0$) due to the assumption of state-independent noise errors, which parallels the translation invariance in linear networks. In fact, in the context of local interactions, by adopting the argument of Frankel et al. [3], one can show that a generic supermodular game has at least one contagious action, and hence if an action is uninvadable, then it is also contagious and no other action is uninvadable.[9]

Basteck and Daniëls [1] prove that in any global game, independently of the noise distribution, action profile $(0,0)$ ($(2,2)$, resp.) is played at θ as $\nu \to 0$ if the game at that state θ is a symmetric 3×3 supermodular game that satisfies (3.1) ((3.2), resp.) for some $p \in (0, 1/2)$. Together with Lemma 2(1), this leads to the following characterization of global-game noise-independent selection in the bilingual game, the same characterization as in Theorem 2.

Proposition B.6. *Let (S, u) be the bilingual game given by* (2.2)*.* (i) (A, A) *is a noise-independent global game selection if $e < e^*$.* (ii) (B, B) *is a noise-independent global game selection if $e > e^*$.*

Since this characterization is different from that in Proposition B.5, global games are not a critical class of incomplete information games that determines whether or not an action profile is robust to incomplete information. See Oyama and Takahashi [14] for further discussions.

Global games have been extended to multidimensional states and signals while maintaining the assumption of state-independent noise errors. (Indeed, multidimensional states and signals are already accommodated in Carlsson and van Damme [2].) Recently, Oury [13] shows that if an action is played in some one-dimensional global game of supermodular games

[9] For the bilingual game, these results also follow from our Theorem 1.

independently of the noise distribution, then it is also played in any multidimensional global game. This result, combined with that of Oyama and Takahashi [14], implies that Proposition B.6 extends to multidimensional global games.

B.7.3. Non-Common Priors and Asymmetric Interaction Weights

All results reported in Sections B.7.1 and B.7.2 rely on the implicit assumption that in incomplete information perturbations the players share a common prior probability distribution, from which each player derives his conditional beliefs based on the information he has. This common prior assumption corresponds in our local interaction context to the assumption that the weight function P on interactions is symmetric, i.e., $P(x, y) = P(y, x)$ for all $x, y \in X$. The symmetry of the weight function naturally arises when the value $P(x, y)$ represents the duration (within a period) or intimacy of the interaction between x and y. Alternatively, if asymmetric weights are allowed, the situation corresponds to one of non-common priors.

Oyama and Tercieux [15, 16] study contagion and robustness under non-common priors, where players may have heterogeneous priors in ε-incomplete information perturbations and the probability of crazy types is no larger than ε with respect to all the players' priors. They show that under non-common priors, any strict Nash equilibrium of a complete information game is contagious in some ε-perturbations, and that generically, a game has a robust equilibrium if and only if it is dominance solvable, in which case the unique surviving action profile is robust.

Their results have a direct translation in our local interactions context: under asymmetric weights, any strict Nash equilibrium of a pairwise game is contagious, and generically, a game has an uninvadable action if and only if it is dominance solvable, in which case the unique surviving action is uninvadable.

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