Contagion and Uninvadability in Social Networks with Bilingual Option*

DAISUKE OYAMA

Faculty of Economics, University of Tokyo
Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
oyama@e.u-tokyo.ac.jp

AND

SATORU TAKAHASHI

Department of Economics, National University of Singapore
AS2 Level 6, 1 Arts Link, Singapore 117570, Singapore
ecsst@nus.edu.sg

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Abstract

In a setting where an infinite population of players interact locally and repeatedly, we study the impacts of payoff structures and network structures on contagion of a convention. For the "bilingual game", where each player chooses one of two conventions or adopts both (i.e., chooses the "bilingual option") at an additional cost, we completely characterize when a convention spreads contagiously from a finite subset of players to the entire population in some network, and conversely, when a convention is never invaded by the other convention in any network. In particular, depending on the cost of bilingual option, the risk-dominated but Pareto-dominant convention can be contagious, and furthermore, for a fixed profile of payoff parameter values, the risk-dominant and the Pareto-dominant conventions can each be contagious in respective networks. Non-linear networks can be critical in determining the contagious conventions in some range of parameter values. We also introduce the concept of weight-preserving node identification and show that if there is a weight-preserving node identification from one network to another, then the latter is more contagion-inducing than the former in all supermodular games. We demonstrate that with our three-action game, our analysis reveals strictly finer structures of networks than that with two-action coordination games. *Journal of Economic Literature* Classification Numbers: C72, C73, D83.

KEYWORDS: equilibrium selection; strategic complementarity; bilingual game; local interaction; network; contagion; uninvadability.

1 Introduction

Behavior initiated by a small group of individuals, such as adoption of languages or technology standards, can in the long run spread over a large population through local interactions. Such a phenomenon is called contagion (also known as diffusion or epidemics) and has attracted much attention (Blume (1993, 1995), Ellison (1993, 2000), Morris (2000)[1]). In general, whether contagion occurs depends on the payoffs of the game played as well as the topology of the underlying network. Focusing on the class of 2×2 coordination games, Morris (2000) shows that an action can spread contagiously in some network if and only if it is a risk-dominant action, and offers an approach to exploring the network topology by assigning to each network the "contagion threshold", the threshold of payoff parameter values for which an action is contagious in that network. He computes the contagion thresholds of various (infinite) networks and in particular shows that a simple linear network has a maximum contagion threshold and hence is most contagion-inducing among all networks, i.e., whenever contagion occurs in some network, it occurs in that network.

In this paper, we aim at understanding the impacts of payoff structures and network structures on contagion for general supermodular games. To study the impact of payoff structures, we consider a particular 3×3 game, which we call the bilingual game, obtained by adding a third, "bilingual" option to a 2×2 coordination game. We completely characterize when an action of the bilingual game spreads contagiously in some network (in the universal domain of networks or a restricted domain of linear or lattice networks); in particular, depending on the cost of bilingual option, the risk-dominated (in the original 2×2 game) but Pareto-dominant action can be contagious, and furthermore, for a fixed profile of payoff parameter values, the risk-dominant and the Pareto-dominant actions can each be contagious in respective networks. To study the impact of network structures on contagion, we take an approach of relating a network to the set of parameter values of general $n \times n$ supermodular games for which an action becomes contagious in the network, where a network is viewed as more contagion-inducing than another if the former has a larger set (in terms of set inclusion) than the latter. We develop the concept of weight-preserving node identification to provide a sufficient condition for two networks to be comparable in the above sense; we show that if there is a weight-preserving node identification from one network to another, then the latter is more contagion-inducing than the former. We also apply this approach to the bilingual game and demonstrate that even with a minimal departure from the 2×2 case, it offers a strictly more detailed analysis of network topologies than that by Morris' (2000)

[1] See also Easley and Kleinberg (2010), Goyal (2007), Jackson (2008), Vega-Redondo (2007), Young (1998), among others.

contagion threshold.

To be specific, consider an infinite population of players who are connected with each other through a graph ("social network"). Suppose that each player uses one of two computer programming languages, or two types of technologies in general, A and B. The payoffs from each interaction with his neighbors are given by the following 2×2 coordination game:[2]

	A	B
A	a, a	b, c
B	c, b	d, d

where $a > c$ and $d > b$, so that (A, A) and (B, B) are strict Nash equilibria. We assume that $a > d$, i.e., (A, A) Pareto-dominates (B, B), while $a - c < d - b$, i.e., (B, B) risk-dominates (A, A). We further assume that $d \geq c$, so that coordination on some action is always better than miscoordination. Morris (2000) shows that the risk-dominant action B spreads contagiously from a finite subset of players to the entire population in some network, and that it is never invaded by the other action A in any network. Thus, in 2×2 coordination games, the risk-dominant action is always both contagious and uninvadable. Observe that B is a best response if at least a proportion $q = (a-c)/\{(a-c)+(d-b)\}$ of neighbors play B. Morris (2000) defines the contagion threshold of a network to be the number such that contagion occurs in that network if and only if the payoff parameter q is smaller than this number. In particular, the linear network with nearest neighbor interactions, as depicted in Figure 1, has a maximum contagion threshold 1/2.

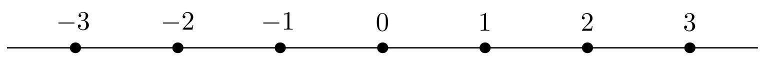


Figure 1: Nearest neighbor linear interaction

Now suppose that players can adopt a combination of the two actions, a "bilingual option" AB, with an additional cost $e > 0$. A player who plays AB receives a (gross) payoff a (d, resp.) from an interaction with an A-player (B-player, resp.). When two AB-players interact, they adopt the superior action A and receive a. This situation is described by the following payoff matrix:[3]

[2]With only two actions, the model is a special case of the "threshold model" (Granovetter (1973)). For related studies in computer science, see, e.g., Easley and Kleinberg (2010) or Wortman (2008).

[3]This game has been studied by Galesloot and Goyal (1997), Goyal and Janssen (1997), Immorlica et al. (2007), and Easley and Kleinberg (2010).

	A	AB	B
A	a, a	$a, a - e$	b, c
AB	$a - e, a$	$a - e, a - e$	$d - e, d$
B	c, b	$d, d - e$	d, d

where (A, A) and (B, B) are the only pure-strategy Nash equilibria. One may expect that, when the value of the cost parameter e is large, the action AB is not much relevant so that the situation is close to the previous 2×2 case where the risk-dominant action B survives, while as e becomes smaller, AB becomes closer to dominating B so that eventually B will be abandoned and only the Pareto-dominant action A will survive.

For this class of 3×3 games, we first completely characterize when an action is contagious and when it is uninvadable. Conforming to the conjecture in the previous paragraph, we show that if e is large, then the risk-dominant B is contagious and uninvadable, while if e is small, then the Pareto-dominant A is contagious and uninvadable. In the latter case, the region of A-players, together with the "bilingual region" of AB-players, invades that of B-players, where, first, players at the boundary of the B-region, interacting with AB-players, switch to action AB as a "stepping stone", and then, the AB-players interacting with A-players switch to action A.[4] Generically, A or B is contagious, but, in contrast to the 2×2 case, both actions are each contagious if e is in a medium range (which is nonempty and open under an additional condition on parameter values), i.e., A spreads contagiously in some network while B does in another. Indeed, for this range, our construction involves a "non-linear" network to induce the contagion of A.

The above construction motivates the question of how domain restriction on networks affects contagion. A class of networks is called *critical for contagion* if these networks induce all possible contagion, i.e., whenever an action can spread contagiously in some network, it does so within this class of networks. We show that the class of linear or lattice networks that have translation invariant interactions (a generalization of the network in Figure 1) is not critical in determining contagion in the bilingual game, that is, a critical class must contain a "non-linear" network. This is in contrast with the case of 2×2 coordination games, where the linear network in Figure 1 constitutes a critical class.

[4]This process is reminiscent of, but different from, the selection of Pareto-efficient outcomes in evolutionary dynamics with pre-play communication (e.g., Matsui (1991)). In the latter, it incurs no cost to adopt a strategy switching between the Pareto-dominant and the Pareto-dominated actions contingent on the pre-play communication and the Pareto efficiency can prevail under global interactions, while in ours, adopting the bilingual option incurs a strictly positive cost and the Pareto efficiency can prevail only under local interactions.

The analysis described above is to fix a game and vary networks to find one (from the universal domain or a restricted class of networks) in which an action becomes contagious, thereby highlighting the impact of payoff structures on contagion. We next consider a converse exercise to study the impact of network topologies on contagion: fix a network and vary games to find one in which an action becomes contagious. Specifically, we ask the following question: for a pair of networks, which network has a larger set of games for which an action is contagion? We say that a network is *more contagion-inducing* in a class $\mathcal{U}$ of games than another network if for any game in $\mathcal{U}$, any action that is contagious in the latter network is also contagious in the former network. This notion defines a preorder over networks for general $\mathcal{U}$, which is incomplete when $\mathcal{U}$ is taken to be our bilingual game, while it is complete, as represented by the contagion threshold of Morris (2000), when $\mathcal{U}$ is the class of 2×2 coordination games. We then introduce the concept of weight-preserving node identification between two networks, and prove that this concept provides a sufficient condition for a network to induce more contagion than another when $\mathcal{U}$ is the class of all supermodular games. For example, we can construct a weight-preserving node identification from a two-dimensional lattice network to the linear network in Figure 1, which implies that the latter is more contagion-inducing than the former in all supermoludar games. We also show that our exercise based on our 3×3 game provides a strictly finer analysis of network topologies than that by Morris (2000) based on 2×2 games; we find a pair of networks that are not differentiated in terms of contagion in 2×2 coordination games (i.e., have the same contagion threshold), but are strictly ordered in terms of the power of inducing contagion in the bilingual game.

Local interaction games and incomplete information games have formal connections, and both belong to a more general class of "interaction games" (Morris (1997, 1999)). Accordingly, we interpret our results on local interaction games in the context of incomplete information games, whereby we provide interesting implications on global games (Carlsson and van Damme (1993), Frankel et al. (2003)), and robustness to incomplete information (Kajii and Morris (1997)). We also discuss the common prior assumption translated in our context of local interactions.

In his series of papers, Morris (1997, 1999, 2000) defines general notions of contagion and uninvadability, develops a method using potential functions to provide a sufficient condition on payoffs, independent of the underlying network structure, for uninvadability (and hence a necessary condition for contagion), and gives an example of a symmetric 4×4 game to demonstrate the multiplicity of contagious actions. In particular, in his 4×4 example, contagion of these actions occurs in two different "linear" networks. For our class of games, we utilize the potential method to show uninvadability, while we construct a "non-linear" network to obtain contagion of one of the two equilibrium actions in our bilingual game. We also develop the method of weight-preserving node identification to compare network topologies for general supermodular games (including non-potential games).

Contagious behavior in the bilingual game is analyzed by Goyal and Janssen (1997) and Immorlica et al. (2007), both of whom focus on restricted classes of networks. Goyal and Janssen (1997) consider circle networks (with a continuum of players) and characterize contagion for this class of networks. Their characterization is the same as the one we obtain for the class of linear networks, which implies that circle networks are not critical for contagion. Immorlica et al. (2007) focus on the case where there is no conflict between risk-dominance and Pareto-dominance in the original 2×2 game, and find that for a given natural number Δ and for Δ-regular networks (i.e., networks where each player interacts with exactly Δ neighbors with equal weights), the contagion property in the bilingual game is not necessarily monotonic with respect to the cost parameter e (with a, b, c, and d held fixed); the risk- and Pareto-dominant action is contagious in a given Δ-regular network (for some values of a, b, c, and d) if e is sufficiently small or sufficiently large but not if e takes values in some intermediate interval. This is in contrast with our finding, as well as Goyal and Janssen's (1997), that the contagion property is monotonic in e for the universal domain or the class of all linear or lattice networks. Moreover, we show (in the Appendix) that if an action is both risk- and Pareto-dominant, then regardless of the value of e, it is always contagious for the class of linear networks[5] and uninvadable for the universal domain.

This paper also makes a contribution to the literature on learning in games (e.g., Fudenberg and Levine (1998)). In general, long-run outcomes may depend on fine details of the underlying dynamics, such as simultaneity or sequentiality and order of action revisions. In our formulation, in contrast, contagion and uninvadability phenomena do not depend on such details. In particular, we prove (in the Appendix) that in supermodular games, an action is contagious under sequential best responses if and only if it is contagious under simultaneous best responses.

The remainder of the paper is organized as follows. Section 2 formulates local interaction games for general payoff functions, while Section 3 introduces our bilingual game. Section 4 provides the complete characterization of contagion and uninvadability in the universal domain of networks, and in the proofs, discusses particular networks in which each action is contagious. Section 5 characterizes contagion and uninvadability in restricted domains of linear and lattice networks and establishes their non-criticality for contagion. Section 6 compares networks in terms of strategic behavior on the networks and demonstrates that the analysis based on our three-action game reveals strictly finer structures of networks than that based on two-action games. Section 7 discusses the implications of our results in the context

[5] Goyal and Janssen (1997) obtain a similar result in their setting.

of incomplete information games. Section 8 concludes with a discussion on incorporating randomness into the model.

2 Local Interaction Games

Let $\mathcal{X}$ be a countably infinite set of players, and $P\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$ a function such that

1. $P(x,x)=0$ for all $x\in\mathcal{X}$,
2. $P(x,y)=P(y,x)$ for all $x,y\in\mathcal{X}$, and
3. $0<\sum_{y\in\mathcal{X}}P(x,y)<\infty$ for all $x\in\mathcal{X}$.

A *local interaction system*, or *network*, $(\mathcal{X},P)$ defines an undirected graph with vertices $\mathcal{X}$ and edges weighted by P.[6] (We will use the terms "local interaction system" and "network" interchangeably.) We will restrict our attention to *unbounded* local interaction systems; i.e., $\sum_{(x,y)\in\mathcal{X}\times\mathcal{X}}P(x,y)=\infty$. Write $\Gamma(x)=\{y\in\mathcal{X}\mid P(x,y)>0\}$ for the set of neighbors of player $x\in\mathcal{X}$. Denote

$$P(y|x)=\frac{P(x,y)}{\sum_{y'\in\Gamma(x)}P(x,y')},$$

which is well defined due to property 3 above.

Players have a (common) finite set of actions S and a (common) payoff function $u\colon S\times S\to\mathbb{R}$. With the action set S fixed, a *local interaction game* is represented by the tuple $(\mathcal{X},P,u)$. Let $\Delta(S)$ denote the set of probability distributions over S. Given payoff function u, write $br(\pi)$ for the set of pure best responses to $\pi\in\Delta(S)$:

$$br(\pi)=\{h\in S\mid u(h,\pi)\geq u(h',\pi)\text{ for all }h'\in S\},\tag{2.1}$$

where $u(h,\pi)=\sum_{k\in S}\pi_k u(h,k)$.

An *action configuration* is a function $\sigma\colon\mathcal{X}\to S$. For an action configuration σ, we denote by $\pi(\sigma|x)\in\Delta(S)$ the action distribution, weighted by $P(\cdot|x)$, over the actions of player x's neighbors: i.e.,

$$\pi_h(\sigma|x)=\sum_{y\in\Gamma(x)\colon\sigma(y)=h}P(y|x).$$

The payoff for player $x\in\mathcal{X}$ playing action $s\in S$ is given by the weighted sum (with respect to $P(\cdot|x)$) of payoffs from the interactions with his neighbors:

$$U(s,\sigma|x)=\sum_{y\in\Gamma(x)}P(y|x)u(s,\sigma(y)),$$

[6] One could instead focus on local interaction systems with *constant weights*, where $P(x,y)\in\{0,1\}$ for all $x,y\in\mathcal{X}$. All the results in this paper would remain unchanged since any local interaction system with rational weights can be replicated by a local interaction system with constant weights.

which equals $u(s, \pi(\sigma|x))$. Write $BR(\sigma|x)$ for the set of pure best responses for player x to action configuration σ:

$$BR(\sigma|x) = \{s \in S \mid U(s, \sigma|x) \geq U(s', \sigma|x) \text{ for all } s' \in S\}, \tag{2.2}$$

which equals $br(\pi(\sigma|x))$.

We consider the sequential best response dynamics on network $(\mathcal{X}, P)$ as defined below. (There being finitely many actions, for a sequence of actions $(s^t)_{t=0}^{\infty}$, $\lim_{t\to\infty} s^t = s$ if and only if there exists T such that $s^t = s$ for all $t \geq T$.)

Definition 1. A sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *best response sequence* if it satisfies the following properties: (i) for all $t \geq 1$, there is at most one $x \in \mathcal{X}$ such that $\sigma^t(x) \neq \sigma^{t-1}(x)$; (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR(\sigma^{t-1}|x)$; and (iii) if $\lim_{t\to\infty} \sigma^t(x) = s$, then for all $T \geq 0$, $s \in BR(\sigma^t|x)$ for some $t \geq T$.

Property (i) requires that in each period at most one player revise his action,[7,8] while property (ii) requires that the revising player switch to a myopic best response to the current distribution of his neighbors' actions. Property (iii) requires that actions that are never a best response be abandoned eventually. Properties (i)–(iii) are satisfied with probability one, for example, in dynamics where in each period at most one player is randomly chosen (according to an i.i.d. full support distribution on $\mathcal{X}$) to revise his action and the revising player switches to a myopic best response to the current distribution of his neighbors' actions. In particular, (ii) and (iii) imply that if there exists T such that $s \notin BR(\sigma^t|x)$ for all $t \geq T$, then there exists T' such that $\sigma^t(x) \neq s$ for all $t \geq T'$. Note that for a given initial action configuration, there are in general multiple best response sequences, as properties (i) and (iii) do not specify which player revises actions in which period.

We are concerned with the following questions. Is it possible in some network that if some finite group of players initially play action s^*, then the whole population will eventually play s^*? In this case, s^* is said to be contagious. Conversely, is it always the case in any network that if s^* is played by almost all players, it continues to be played by almost all players? If so, s^* is said to be uninvadable. Below we formally define the relevant concepts following Morris (1997, 1999).

[7] One may interpret our dynamic process as one of overlapping generations, where in each period at most one player exits the society and is replaced by a newborn, and the newborn chooses an action only once upon entry and commits to it through the rest of his life. This interpretation especially fits in the context of language adoption.

[8] All the results in this paper hold even if we allow for simultaneous best responses, under an additional assumption that $\Gamma(x)$ is finite for every player x; see Appendices A.1 and A.3.

Definition 2. Given an unbounded local interaction system $(\mathcal{X}, P)$, action s^* is *contagious in* $(\mathcal{X}, P)$ if there exists a finite subset Y of $\mathcal{X}$ such that every best response sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0(x) = s^*$ for all $x \in Y$ satisfies $\lim_{t\to\infty} \sigma^t(x) = s^*$ for each $x \in \mathcal{X}$. Action s^* is *contagious* if it is contagious in some unbounded local interaction system.

Note that contagion of s^* in $(\mathcal{X}, P)$ requires that, once the finite set Y of initial s^*-players is chosen, s^* be eventually played by all the players along *any* best response sequence.[9]

A game may have multiple contagious actions, and we will show in Section 4.1 that this is indeed the case for some (nonempty and open) set of payoff parameter values in our bilingual game, where we construct two different local interaction systems in which different actions respectively spread contagiously.[10]

For uninvadability, the notion "almost all" is formalized by "except for a set of players whose weight with respect to P is finite". For an action configuration σ and a subset of actions $S' \subset S$, we write

$$\sigma_P(S') = \frac{1}{2} \sum_{(x,y)\colon \sigma(x)\in S' \text{ or } \sigma(y)\in S'} P(x, y).$$

In particular, for an action $s^* \in S$, $\sigma_P(S \setminus \{s^*\}) = (1/2)\sum_{(\sigma(x),\sigma(y))\neq(s^*,s^*)} P(x, y)$, which is the total weight of pairs of players who play action profiles other than (s^*, s^*).

Definition 3. Given an unbounded local interaction system $(\mathcal{X}, P)$, action s^* is *uninvadable in* $(\mathcal{X}, P)$ if there exists no best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$ and $\lim_{t\to\infty} \sigma_P^t(S \setminus \{s^*\}) = \infty$. Action s^* is *uninvadable* if it is uninvadable in all unbounded local interaction systems.[11]

In particular, if s^* is uninvadable in $(\mathcal{X}, P)$, then there exists no best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = s^*$ for all but finitely many $x \in \mathcal{X}$ and $\lim_{t\to\infty} \sigma^t(x) \neq s^*$ for all $x \in \mathcal{X}$. Therefore, if s^* is uninvadable in $(\mathcal{X}, P)$, then actions other than s^* are not contagious in $(\mathcal{X}, P)$; if s^* is contagious in $(\mathcal{X}, P)$, then actions other than s^* are not uninvadable in $(\mathcal{X}, P)$.

Notice that we call an action uninvadable if it is uninvadable for *all* unbounded networks. This strong notion of uninvadability is most relevant

[9] In (generic) supermodular games, this definition of contagion is equivalent to the one that requires only *some* best response sequence to converge or the one that allows for simultaneous best responses; see Appendix A.1.

[10] In principle, two different actions may spread contagiously from different initial groups in one local interaction system, but we are not aware of any such case.

[11] In local interaction systems with constant weights, where $P(x, y) \in \{0, 1\}$ for all $x, y \in \mathcal{X}$, s^* is uninvadable if and only if there exists no best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = s^*$ for all but finitely many $x \in \mathcal{X}$ and $\lim_{t\to\infty} \sigma^t(x) \neq s^*$ for infinitely many $x \in \mathcal{X}$.

when the analyst has no information about the underlying network structure but nonetheless wishes to predict the long-run behavior of the action distribution in the society. If an action s^* is uninvadable, then, once s^* is played by almost all players, the analyst should be confident that s^* will continue to be played by almost all players, whatever the actual network structure is. Recall that we define a contagious action as an action that is contagious in *some* unbounded network (Definition 2). Thus, the notions of uninvadability and contagion are exclusive not only for a given network but also in the universal domain. That is, if s^* is uninvadable, then actions other than s^* are not contagious; if s^* is contagious, then actions other than s^* are not uninvadable.

Our first result (Theorem 1) characterizes uninvadability and contagion for the universal domain of networks. To study how the underlying network structure affects uninvadability/contagion, in Section 5 we will consider several restricted domains of networks and examine whether an action that is invaded (contagious, resp.) in the universal domain becomes uninvadable (remains contagious, resp.) in restricted domains.

3 The Bilingual Game

Hereafter, we consider the class of 3×3 games described in the Introduction. We denote the actions A, AB, and B by 0, 1, and 2, respectively, so that $S = \{0, 1, 2\}$, and let the payoff function $u\colon S \times S \to \mathbb{R}$ be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} a & a & b \\ a-e & a-e & d-e \\ c & d & d \end{pmatrix} \end{array}, \tag{3.1a}$$

where we assume

$$b < c \leq d < a, \quad a - c < d - b, \quad e > 0. \tag{3.1b}$$

Action profiles $(0, 0)$ and $(2, 2)$ are the only pure-strategy Nash equilibria. By the assumption that $d < a$, $(0, 0)$ Pareto-dominates $(2, 2)$, while by $a - c < d - b$, $(2, 2)$ pairwise risk-dominates $(0, 0)$.[12] By the additional assumption that $c \leq d$, this game is *supermodular* with respect to the order on actions $0 < 1 < 2$, i.e., $u(h', k) - u(h, k) \leq u(h', k') - u(h, k')$ if $h < h'$ and $k < k'$.

We will exploit the property of supermodular games, that the best response correspondence is nondecreasing in the stochastic dominance order.

[12] In Appendix A.8, we analyze the case where $(0, 0)$ is both Pareto-dominant and pairwise risk-dominant.

For $\pi, \pi' \in \Delta(S)$, we write $\pi \precsim \pi'$ (and $\pi' \succsim \pi$) if π' stochastically dominates π, i.e., if

$$\sum_{k \geq h} \pi_k \leq \sum_{k \geq h} \pi'_k$$

for all $h \in S$. If u is supermodular, then

$$\max br(\pi) \leq \max br(\pi')$$
$$\min br(\pi) \leq \min br(\pi')$$

whenever $\pi \precsim \pi'$.

4 Characterization for the Universal Domain

In this section, we give a full characterization for contagion and uninvadability in our bilingual game for the universal domain of all networks. In particular, we show that the Pareto-dominant action 0 prevails if the bilingual cost e is small, while the pairwise risk-dominant action 2 survives if e is large. The thresholds will be constructed based on two parameters:

$$e^* = \frac{(a-d)(d-b)}{2(c-b)},$$
$$e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)}.$$

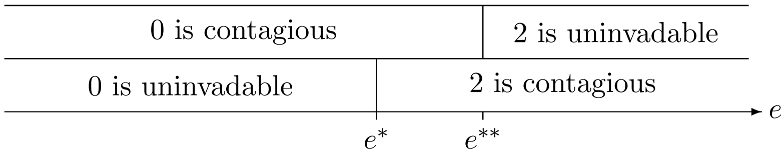
It can be readily verified that $e^* \lesseqqgtr e^{**}$ if $c - b \lesseqqgtr a - c$. The following result characterizes contagious and uninvadable actions in the bilingual game, quantifying our argument in the Introduction.

Theorem 1. *Let u be the bilingual game given by* (3.1).
(i) 0 *is contagious if* $e < \max\{e^*, e^{**}\}$ *and uninvadable if* $e < e^*$. (ii) 2 *is contagious if* $e > e^*$ *and uninvadable if* $e > \max\{e^*, e^{**}\}$.

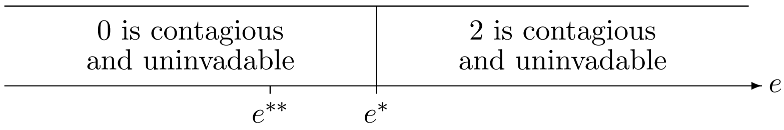
Note that for any (generic) value of e, at least one action is contagious[13] and therefore if an action is uninvadable, then it is also contagious; when $e \in (e^*, e^{**})$ (which is nonempty if $c - b < a - c$), the two actions 0 and 2 are each contagious (in respective networks) and therefore neither action is uninvadable.

One can verify that e^* and e^{**} increase, and thus the contagion and uninvadability regions (in the space of e) of action 0 expand, as action 0 becomes less risky (i.e., b increases or c decreases), or more efficient (i.e., a increases with $a - c$ held fixed or d decreases with $d - b$ held fixed). This

[13] In fact, one can show that any generic supermodular game has at least one contagious action, by appropriately translating the contagion argument of Frankel et al. (2003) into our local interactions context. See Section 7.2.



(1) $c - b < a - c$



(2) $c - b \geq a - c$

comparative statics is in contrast with that in the 2×2 case, where the risk-dominance and hence the characterizations for contagion and uninvadability are not affected by any payoff change with the ratio of $a - c$ and $d - b$ held fixed.

In Subsections 4.1 and 4.2, we prove the contagion and the uninvadability parts of Theorem 1, respectively.

Example 1. Let $a = 11$, $b = 0$, $c = 3$, and $d = 10$: the game is represented by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 11 & 11 & 0 \\ 11 - e & 11 - e & 10 - e \\ 3 & 10 & 10 \end{pmatrix} \end{array}.$$

Thus, $c - b = 3 < a - c = 8$, and $e^* = 5/3$ and $e^{**} = 40/19$. By Theorem 1, if $e > 40/19$, 2 is contagious and uninvadable; if $5/3 < e < 40/19$, both 0 and 2 are contagious; and if $e < 5/3$, 0 is contagious and uninvadable.

4.1 Contagion

In this subsection, we prove the contagion part of Theorem 1: (i) 0 is contagious if $e < \max\{e^*, e^{**}\}$; and (ii) 2 is contagious if $e > e^*$.

We decompose the proof into two lemmas. Lemma 1 provides sufficient conditions for contagion of actions 0 and 2 in general 3×3 supermodular games. Lemma 2 then checks by direct computation when those conditions are satisfied in the bilingual game. Our main theoretical contribution is in the proof of Lemma 1, where we explicitly construct networks in which contagion occurs as desired.

To better understand how contagion occurs in the bilingual game, consider a population of players indexed by integers $x \in \mathcal{X} = \mathbb{Z}$, where player

x interacts with players $x \pm 1$ with equal weights; see Figure 1 in the Introduction.

Suppose that at time $t = 0$, all players play B except for players -1, 0, and 1 who play A, and assume that the bilingual cost e is small so that $e < (a - d)/2$ (where $(a - d)/2 \leq e^*$). We demonstrate that A spreads contagiously. (For concreteness, we here consider a particular best response sequence, while one can verify that contagion occurs for all best response sequences as the definition requires.) Note that, since A is pairwise risk-dominated by B, no player is willing to switch from B to A. Suppose that player 2 adjusts his action at $t = 1$. With his two neighbors playing A and B, respectively, he abandons B and switches to AB since $e < (a - d)/2 \leq (a - c)/2$. Suppose next that player 3 revises his action at $t = 2$. Since he has one AB-neighbor and one B-neighbor, by $e < (a - d)/2$ he abandons B and switches to A or AB (depending on the payoff parameter values); let us assume that he chooses AB. Now let player 2 revise back again at $t = 3$. This time his neighbors are playing A and AB (instead of B), and hence he now switches to A. In this way, the region of A-players spreads, together with the "bilingual" region of AB-players between the A- and the B-regions; see Table 1.

	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$t = 0$	$\cdots$	B	A	A	A	B	B	B	$\cdots$
$t = 1$	$\cdots$	B	A	A	A	AB	B	B	$\cdots$
$t = 2$	$\cdots$	B	A	A	A	AB	AB	B	$\cdots$
$t = 3$	$\cdots$	B	A	A	A	A	AB	B	$\cdots$

Table 1: Contagion of action A

The above construction, which works only for $e < (a - d)/2$, is extended to obtain contagion of A for $e < e^*$ (and symmetrically that of B for $e > e^*$) in Lemma 1(1) where we construct a "linear" network with four neighbors (two for each side) with appropriately chosen weights (Figure 2). In order to obtain contagion further for the range $[e^*, e^{**})$ (which is nonempty when $c - b < a - c$), however, such a construction does not work[14] and we need to construct a "non-linear" network in Lemma 1(2), in which different players may have different types of interacting neighborhoods (Figure 3).

For $p \in (0, 1/2)$ and $q, r \in (0, 1)$, $r \leq q$, let

$$\pi^a = \left(\tfrac{1}{2}, p, \tfrac{1}{2} - p\right), \quad \pi^b = \left(\tfrac{1}{2} - p, p, \tfrac{1}{2}\right),$$

and

$$\pi^c = \left(\tfrac{1+q}{2}, 0, \tfrac{1-q}{2}\right), \quad \pi^d = \left(\tfrac{1-r}{2}, 0, \tfrac{1+r}{2}\right), \quad \pi^e = \left(0, \tfrac{q+r}{2q}, \tfrac{q-r}{2q}\right),$$

[14]We formalize and prove this statement in Section 5.

$$\rho^c = \left(\tfrac{1-q}{2}, 0, \tfrac{1+q}{2}\right), \quad \rho^d = \left(\tfrac{1+r}{2}, 0, \tfrac{1-r}{2}\right), \quad \rho^e = \left(\tfrac{q-r}{2q}, \tfrac{q+r}{2q}, 0\right).$$

The conditions for contagion of actions 0 and 2 are stated in terms of best responses to the above mixed actions.

Lemma 1. *Let u be any 3×3 supermodular game.*

(1) (i) *If for some $p \in (0, 1/2)$,*

$$\max br(\pi^a) = 0, \quad \max br(\pi^b) \leq 1, \tag{4.1}$$

then 0 is contagious. (ii) *If for some $p \in (0, 1/2)$,*

$$\min br(\pi^a) \geq 1, \quad \min br(\pi^b) = 2, \tag{4.2}$$

then 2 is contagious.

(2) (i) *If for some $q, r \in (0, 1)$ with $r \leq q$,*

$$\max br(\pi^c) = 0, \quad \max br(\pi^d) \leq 1, \quad \max br(\pi^e) = 0, \tag{4.3}$$

then 0 is contagious. (ii) *If for some $q, r \in (0, 1)$ with $r \leq q$,*

$$\min br(\rho^c) = 2, \quad \min br(\rho^d) \geq 1, \quad \min br(\rho^e) = 2, \tag{4.4}$$

then 2 is contagious.

Proof. (1) Since cases (i) and (ii) are symmetric, we only show case (i). Let $p \in (0, 1/2)$ satisfy (4.1). We construct a local interaction system $(\mathcal{X}, P)$ in which action 0 spreads contagiously from a finite set of players $Y \subset \mathcal{X}$.

Let $\mathcal{X} = \mathbb{Z}$, and P be defined by

$$P(x, y) = \begin{cases} p & \text{if } |x - y| = 1 \\ \frac{1}{2} - p & \text{if } |x - y| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The defined local interaction system is depicted in Figure 2.

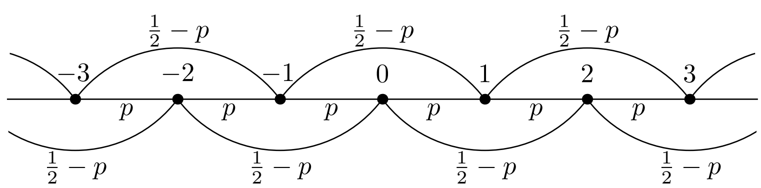


Figure 2: Linear interaction

We will use the following properties of this system.

Observation 1.

(a) If $\sigma(x-2) = \sigma(x-1) = 0$ and $\sigma(x+1) \leq 1$ (or symmetrically if $\sigma(x-1) \leq 1$ and $\sigma(x+1) = \sigma(x+2) = 0$), then $\max BR(\sigma|x) = 0$.

(b) If $\sigma(x-2) = 0$ and $\sigma(x-1) \leq 1$ (or symmetrically if $\sigma(x+1) \leq 1$ and $\sigma(x+2) = 0$), then $\max BR(\sigma|x) \leq 1$.

Proof. (a) Suppose that $\sigma(x-2) = \sigma(x-1) = 0$ and $\sigma(x+1) \leq 1$. Then by construction, the distribution over the actions of player x's neighbors, $\pi(\sigma|x) \in \Delta(S)$, satisfies

$$\pi(\sigma|x)(0) \geq P(x-2, x-1|x) = \frac{1}{2},$$
$$\pi(\sigma|x)(0) + \pi(\sigma|x)(1) \geq P(x-2, x-1, x+1|x) = \frac{1}{2} + p,$$

which implies that $\pi(\sigma|x) \precsim \pi^a = (1/2, p, 1/2 - p)$. By the assumption (4.1) and the supermodularity of u, it follows that $\max BR(\sigma|x) = 0$.

(b) Suppose that $\sigma(x-2) = 0$ and $\sigma(x-1) \leq 1$. Then by construction,

$$\pi(\sigma|x)(0) \geq P(x-2|x) = \frac{1}{2} - p,$$
$$\pi(\sigma|x)(0) + \pi(\sigma|x)(1) \geq P(x-2, x-1|x) = \frac{1}{2},$$

which implies that $\pi(\sigma|x) \precsim \pi^b = (1/2 - p, p, p, 1/2)$. By the assumption (4.1) and the supermodularity of u, it follows that $\max BR(\sigma|x) \leq 1$. ∎

Continuing the proof of Lemma 1(1), let $Y = \{-3, \ldots, 2\}$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. We want to show that

$$\lim_{t \to \infty} \sigma^t(x) = 0 \qquad (\diamondsuit_x)$$

holds for all $x \in \mathcal{X}$. We only consider players $x \geq 0$; the analogous argument applies to $x < 0$.

We first show $(\diamondsuit_0)$ and $(\diamondsuit_1)$, or more strongly, that

$$\sigma^t(x) = 0 \text{ for } x = -2, \ldots, 1$$
$$\sigma^t(x) \leq 1 \text{ for } x = -3, 2$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t-1$, then we have $\sigma^t(x) = 2$ for $x = -2, \ldots, 1$ and $\sigma^t(x) \leq 1$ for $x = -3, 2$ by properties (a) and (b) in Observation 1, respectively.

Assume $(\diamondsuit_{x-2})$ and $(\diamondsuit_{x-1})$. Then, there exists T_0 such that $\sigma^t(x-2) = \sigma^t(x-1) = 0$ for all $t \geq T_0$. By Observation 1(b), this implies that there exists T_1 such that $\sigma^t(x) \leq 1$ for all $t \geq T_1$. By Observation 1(b) applied for $x+1$ in place of x, this implies that there exists T_2 such that $\sigma^t(x+1) \leq 1$ for all $t \geq T_2$. By Observation 1(a), this implies that there exists T_3 such that $\sigma^t(x) = 0$ for all $t \geq T_3$, meaning that $(\diamondsuit_x)$ holds.

(2) Since cases (i) and (ii) are symmetric, we only show case (i). Let $q, r \in (0,1)$, $r \leq q$, satisfy (4.3). We construct a local interaction system $(\mathcal{X}, P)$ in which action 0 spreads contagiously from a finite set of players $Y \subset \mathcal{X}$.

Let $\mathcal{X} = \{\alpha, \beta\} \times \mathbb{Z}$, and P be defined by

$$P((\alpha,i),(\alpha,j)) = \begin{cases} 1-q & \text{if } |i-j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$P((\alpha,i),(\beta,j)) = P((\beta,j),(\alpha,i)) = \begin{cases} q+r & \text{if } i = j \\ q-r & \text{if } i = j+1 \text{ and } j \geq 0 \\ q-r & \text{if } i = j-1 \text{ and } j \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$P((\beta,i),(\beta,j)) = 0$ for all i, j.

The defined local interaction system is depicted in Figure 3.

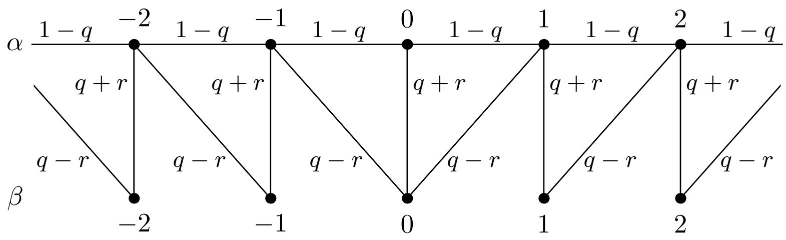


Figure 3: Non-linear interaction

We will use the following properties of this system.

Observation 2.

(c) For $i \geq 1$, if $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = \sigma(\beta, i) = 0$ (or symmetrically for $i \leq -1$, if $\sigma(\alpha, i+1) = \sigma(\beta, i+1) = \sigma(\beta, i) = 0$), then $\max BR(\sigma|(\alpha,i)) = 0$.

(d) For $i \geq 1$, if $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = 0$ (or symmetrically for $i \leq -1$, if $\sigma(\alpha, i+1) = \sigma(\beta, i+1) = 0$), then $\max BR(\sigma|(\alpha,i)) \leq 1$.

(e) If $\sigma(\alpha, i) \leq 1$, then $\max BR(\sigma|(\beta,i)) = 0$.

Proof. (c) Suppose that $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = \sigma(\beta, i) = 0$. Then by construction,

$$\pi(\sigma|(\alpha,i))(0) \geq P((\alpha,i-1),(\beta,i-1),(\beta,i)|(\alpha,i)) = \frac{1+q}{2},$$

which implies $\pi(\sigma|(\alpha,i)) \precsim \pi^c = ((1+q)/2, 0, (1-q)/2)$. By the assumption (4.3) and the supermodularity of u, it follows that $\max BR(\sigma|(\alpha,i)) = 0$.

(d) Suppose that $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = 0$. Then by construction,

$$\pi(\sigma|(\alpha, i))(0) \geq P((\alpha, i-1), (\beta, i-1)|(\alpha, i)) = \frac{1-r}{2},$$

which implies $\pi(\sigma|(\alpha, i)) \precsim \pi^d = ((1-r)/2, 0, (1+r)/2)$. By the assumption (4.3) and the supermodularity of u, it follows that $\max BR(\sigma|(\alpha, i)) \leq 1$.

(e) Suppose that $\sigma(\alpha, i) \leq 1$. Then by construction,

$$\pi(\sigma|(\beta, i))(0) + \pi(\sigma|(\beta, i))(1) \geq P((\alpha, i)|(\beta, i)) = \frac{q+r}{2q},$$

which implies $\pi(\sigma|(\beta, i)) \precsim \pi^e = (0, (q+r)/(2q), (q-r)/(2q))$. By the assumption (4.3) and the supermodularity of u, it follows that $\max BR(\sigma|(\beta, i)) = 0$. ▮

Continuing the proof of Lemma 1(2), let $Y = \{(\alpha, i) \mid i = -1, 0, 1\} \cup \{(\beta, i) \mid i = -1, 0, 1\}$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. We want to show that

$$\lim_{t\to\infty} \sigma^t(\alpha, i) = 0 \text{ and } \lim_{t\to\infty} \sigma^t(\beta, i) = 0 \qquad (\heartsuit_i)$$

holds for all $i \in \mathbb{Z}$. We only consider $i \geq 0$; the analogous argument applies to $i < 0$.

We first show $(\heartsuit_1)$, or more strongly, that

$$\sigma^t(\alpha, i) = \sigma^t(\beta, i) = 0 \text{ for } i = -1, 0, 1$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t-1$, then we have $\sigma^t(\alpha, i) = \sigma^t(\beta, i) = 0$, $i = -1, 0, 1$, by properties (c) and (e) in Observation 2, respectively.

Assume $(\heartsuit_{i-1})$. Then, there exists T_0 such that $\sigma^t(\alpha, i-1) = \sigma^t(\beta, i-1) = 0$ for all $t \geq T_0$, By Observation 2(d), this implies that there exists T_1 such that $\sigma^t(\alpha, i) \leq 1$ for all $t \geq T_1$. By Observation 2(e), this implies that there exists T_2 such that $\sigma^t(\beta, i) = 0$ for all $t \geq T_2$. By Observation 2(c) this implies that there exists T_3 such that $\sigma^t(\alpha, i) = 0$ for all $t \geq T_3$. We thus obtain $(\heartsuit_i)$. ▮

Denote

$$e^\sharp = \frac{(d-b)\{2(a-c)-(d-b)\}}{2(a-c)}.$$

Verify that $e^{**} \lesseqgtr e^\sharp$ if $c - b \lesseqgtr a - c$. The following result characterizes when the hypotheses in Lemma 1 are satisfied in the bilingual game.

Lemma 2. *Let u be the bilingual game given by* (3.1).

(1) (i) *Condition* (4.1) *holds for some $p \in (0, 1/2)$ if $e < e^*$.* (ii) *Condition* (4.2) *holds for some $p \in (0, 1/2)$ if $e > e^*$.*

(2) *Condition* (4.3) *holds for some $0 < r \leq q < 1$ if $e < \min\{e^{**}, e^\sharp\}$.*

Proof. See Appendix A.2. ∎

Proof of the Contagion Part of Theorem 1. (i) Suppose that $e < \max\{e^*, e^{**}\}$. If $\max\{e^*, e^{**}\} = e^*$, then condition (4.1) holds for some $p \in (0, 1/2)$ by Lemma 2(1-i), and hence 0 is contagious by Lemma 1(1-i). If $\max\{e^*, e^{**}\} = e^{**}$, in which case $c - b < a - c$ and thus $\min\{e^{**}, e^\sharp\} = e^{**}$, then condition (4.3) holds for some $0 < r \leq q < 1$ by Lemma 2(2-i), and hence 0 is contagious by Lemma 1(2-i). In both cases, 0 is contagious.

(ii) Suppose that $e > e^*$. Then condition (4.2) holds for some $p \in (0, 1/2)$ by Lemma 2(1-ii), and hence 2 is contagious by Lemma 1(1-ii). ∎

To prove the contagion of action 0 for $e < e^*$ (or of action 2 for $e > e^*$), one can use other networks than the four-neighbor construction as in Figure 2, where interaction weights have appropriately been chosen according to the payoff parameter values. For example, one can instead employ a constant-weight linear network with sufficiently many neighbors, i.e., a network $(\mathcal{X}, P)$ such that $\mathcal{X} = \mathbb{Z}$ and $P(x, y) = 1$ if $1 \leq |x - y| \leq n$, where n is sufficiently large, and $P(x, y) = 0$ otherwise. Indeed, in Section 5.2 we will show that action 0 (action 2) is contagious for $e < e^*$ ($e > e^*$) in lattice networks in which players interact with neighbors within n steps away in each dimension, where n is sufficiently large depending on the payoff parameter values.

On the other hand, for the range $[e^*, e^{**})$ (which is nonempty when $c - b < a - c$), linear networks such as the one in Figure 2 do not induce the contagion of action 0. In Section 5.1, we will show that if $e > e^*$, action 2 is uninvadable (and thus 0 cannot be contagious) in the class of linear networks, those networks $(\mathbb{Z}, P)$ that satisfy the translation invariance: $P(x, y) = P(x + z, y + z)$ for all $x, y, z \in \mathbb{Z}$. This implies that it is necessary to consider non-linear networks, such as the one in Figure 3, for $e \in [e^*, e^{**})$.

4.2 Uninvadability

In this subsection, we prove the uninvadability part of Theorem 1: (i) 0 is uninvadable if $e < e^*$; and (ii) 2 is uninvadable if $e > \max\{e^*, e^{**}\}$.

The condition for uninvadability is stated by using the concept of *monotone potential maximizer* (*MP-maximizer*) due to Morris and Ui (2005). We employ its refinement, strict MP-maximizer, due to Oyama et al. (2008). For our purpose, we define it only for the smallest and the largest actions, which we denote by $\underline{s}$ and $\overline{s}$, respectively.[15] For a function $f\colon S \times S \to \mathbb{R}$ and a probability distribution $\pi \in \Delta(S)$, write $br_f(\pi) = \arg\max_{h \in S} f(h, \pi)$. (Thus the best response correspondence br for the game u as defined in

[15]Here, we define for actions, rather than action profiles, since we only consider symmetric action profiles of symmetric games.

(2.1) is now denoted br_u.) Function f is symmetric if $f(h,k) = f(k,h)$ for all $h, k \in S$ (i.e., it is a symmetric $|S| \times |S|$ matrix).

Definition 4. (i) $\underline{s}$ is a *strict MP-maximizer* of u if there exists a symmetric function $v\colon S \times S \to \mathbb{R}$ with $v(\underline{s},\underline{s}) > v(h,k)$ for all $(h,k) \neq (\underline{s},\underline{s})$ such that for all $\pi \in \Delta(S)$,

$$\max br_u(\pi) \leq \max br_v(\pi). \tag{4.5}$$

Such a function v is called a *strict MP-function* for $\underline{s}$.

(ii) $\overline{s}$ is a *strict MP-maximizer* of u if there exists a symmetric function $v\colon S \times S \to \mathbb{R}$ with $v(\overline{s},\overline{s}) > v(h,k)$ for all $(h,k) \neq (\overline{s},\overline{s})$ such that for all $\pi \in \Delta(S)$,

$$\min br_u(\pi) \geq \min br_v(\pi). \tag{4.6}$$

Such a function v is called a *strict MP-function* for $\overline{s}$.

A strict MP-maximizer is a strict Nash equilibrium and, in supermodular games, is unique if it exists (Oyama et al. (2008)).

Lemma 3. *Let u be any game. If $s^* = \underline{s}, \overline{s}$ is a strict MP-maximizer of u with a strict MP-function v and if u or v is supermodular, then s^* is uninvadable.*

Proof. See Appendix A.3. ▮

In our framework, games that are strategically equivalent to a game defined by a symmetric function are called *potential games* (Monderer and Shapley (1996)); namely, a game u is a potential game if there exists a symmetric function v such that $u(h,k) - u(h',k) = v(h,k) - v(h',k)$ for all $h, h' \in S$ and all $k \in S$. Such a function v is called a potential function, and s^* is called a potential maximizer (P-maximizer) of u if (s^*, s^*) uniquely maximizes v. Clearly, if u has a potential function v, we have the *equality* $br_u(\pi) = br_v(\pi)$ for all $\pi \in \Delta(S)$. The monotone potential condition in Definition 4 weakens this equality with the *inequality* (4.5) for $s^* = \underline{s}$ or (4.6) for $s^* = \overline{s}$. For potential games, best response of each revising player along a best response sequence $(\sigma^t)_{t=0}^{\infty}$ increases the value of the function $V(t) = (1/2)\sum_{(x,y)\in\mathcal{X}\times\mathcal{X}} P(x,y)(v(\sigma^t(x),\sigma^t(y)) - v(s^*,s^*))$, the potential function extended to the local interaction system $(\mathcal{X}, P)$, and thus if the P-maximizer s^* is initially played by almost all players (according to the interaction weights P) so that $V(0)$ is close to 0, then $V(t)$ remains close to 0 implying that s^* continues to be played by almost all players; that is, it is uninvadable (see Morris (1999, Proposition 6.1) or our Lemma A.2). For monotone potential games with MP-maximizer $\underline{s}$ (the symmetric argument applies to $\overline{s}$), if the original game u or the MP-function v is supermodular, then the MP condition (4.5) guarantees that for any best response sequence in u there is a best response sequence in v that bounds the former from above, and therefore, the uninvadability of $\underline{s}$ in the potential game v implies that in the original game u (see Lemma A.3).

Given Lemma 3, the contagion part of Theorem 1 implies that in the bilingual game, action 0 (2, resp.) is never a strict MP-maximizer if $e > e^*$ ($e < \max\{e^*, e^{**}\}$, resp.), and hence, no strict MP-maximizer exists if $e^* < e < \max\{e^*, e^{**}\}$. The following lemma establishes existence of a strict MP-maximizer for the remaining cases (except for knife-edge values of e). The uninvadability part of Theorem 1 follows from this lemma and Lemma 3.

Lemma 4. *Let u be the bilingual game given by* (3.1).

(i) 0 *is a strict MP-maximizer if* $e < e^*$. (ii) 2 *is a strict MP-maximizer if* $e > \max\{e^*, e^{**}\}$.

Proof. See Appendix A.4. ▮

In 2×2 coordination games, a risk-dominant equilibrium is a strict MP-maximizer. Beyond 2×2 games, no general method to find an MP-maximizer has been known (except for some special cases). A strict MP-maximizer is shown, by ad hoc construction, to generically exist in symmetric 3×3 supermodular games such that the three symmetric action profiles are all Nash equilibria (Morris (1999), Oyama and Takahashi (2009)), whereas it fails to exist in some 3×3 games with two strict Nash equilibria, e.g., in our bilingual game with $e^* < e < e^{**}$ (see also Honda (2011)). The proof of Lemma 4 is here again by ad hoc construction of an MP-function involving tedious computations.

5 Linear and Lattice Networks

Thus far, we have allowed for the universal domain of all networks, and in particular, for our contagion result (Lemma 1) we had a maximal degree of freedom to choose a network to obtain contagion. In this section, we study several subclasses of networks and derive conditions for contagion and uninvadability in these subclasses.

In particular, we examine whether a given class of networks is critical for contagion or uninvadability. Formally, for a given game u and for a class $\mathcal{C}$ of unbounded networks, action s^* is *contagious in* $\mathcal{C}$ (*uninvadable in* $\mathcal{C}$, resp.) if it is contagious in some network in $\mathcal{C}$ (uninvadable in all networks in $\mathcal{C}$, resp.). We say that a class $\mathcal{C}$ is *critical for contagion* if any action s^* that is contagious in the universal domain is also contagious in $\mathcal{C}$. In that case, one can restrict attention to that class to characterize contagious actions. Conversely, if $\mathcal{C}$ is non-critical for contagion, some action is contagious in no network in $\mathcal{C}$ but in some network outside $\mathcal{C}$. For example, if the game u is a 2×2 coordination game, a risk-dominant equilibrium is contagious in the network in Figure 1, and hence that network forms a (singleton) critical class for contagion. On the other hand, if u is the bilingual game, it follows from our analysis in the previous section that the network in Figure 1 is not critical for some parameter values, while the union of two classes of networks given by Figures 2 and 3 is critical.

In what follows, we consider two classes of simple networks, which we call linear and multidimensional lattice networks, and show that these classes of networks are not critical for contagion in the bilingual game. In other words, it is necessary to consider non-linear networks, such as the one in Figure 3, for some payoff parameter values (i.e., for $e \in [e^*, e^{**})$).

5.1 Linear Networks

We first introduce linear networks and analyze contagion and uninvadability in those networks. A network $(\mathcal{X}, P)$ is *linear* if $\mathcal{X} = \mathbb{Z}$ and interaction weights P are invariant up to translation: $P(x, y) = P(x + z, y + z)$ for any $x, y, z \in \mathbb{Z}$. (Note that any linear network is unbounded.) Clearly, both of the networks in Figure 1 and in Figure 2 are linear. On the other hand, the network in Figure 3 is not linear.[16]

Due to the translation invariance and symmetry of P, for each $y \in \mathbb{Z}$ we have $P(0, y) = P(-y, 0) = P(0, -y)$, hence $P(y|0) = P(-y|0)$. Conversely, conditional weights $P(y|0)$ of player 0 determine translation invariant weights $P(x, y)$ uniquely (up to positive constant multiplication) if $P(0|0) = 0$, and $P(y|0)$ satisfies reflection symmetry, i.e., $P(y|0) = P(-y|0)$ for all $y > 0$.

It follows from the proof of Lemma 1 that in the class of linear networks given in Figure 2, action 0 (2, resp.) is contagious if $e < e^*$ ($e > e^*$, resp.). The following theorem shows that these conditions are also sufficient for uninvadability in the class of all linear networks.

Theorem 2. *Let u be the bilingual game given by* (3.1).

(i) 0 *is contagious and uninvadable in the class of linear networks if* $e < e^*$. (ii) 2 *is contagious and uninvadable in the class of linear networks if* $e > e^*$.

Proof. See Appendix A.5. ∎

Goyal and Janssen (1997, Theorem 2) obtain the same characterization for contagion on a circle network with translation invariant interactions (in a setting with a continuum of players).

The characterization given in Theorem 2 differs from the one for the universal domain given in Theorem 1 when $c - b < a - c$, implying that, for the range of parameter values (e^*, e^{**}), the class of linear networks is *not* critical for contagion.

[16]Even if we map $\mathcal{X} = \{\alpha, \beta\} \times \mathbb{Z}$ to $\mathbb{Z}$ by relabeling (α, i) with $2i$ and (β, i) with $2i+1$, interaction weights do not satisfy translation invariance.

This characterization generalizes to a slightly larger class of networks where each node on a line is replicated into finitely many nodes. Formally, $(\mathcal{X}, P)$ is a *replicated linear network* if $\mathcal{X} = \{1, \dots, m\} \times \mathbb{Z}$, and for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \{1, \dots, m\} \times \mathbb{Z}$, $P(x, y) = P(x+z, y+z)$, where sums in the first coordinate, $x_1 + z_1$ and $y_1 + z_1$, are defined modulo m, and $P(x, y) = 0$ whenever $x_2 = y_2$.[17] For example, the network depicted in Figure 4 is a replicated linear network with $m = 3$, whereas the network in Figure 3 is not.

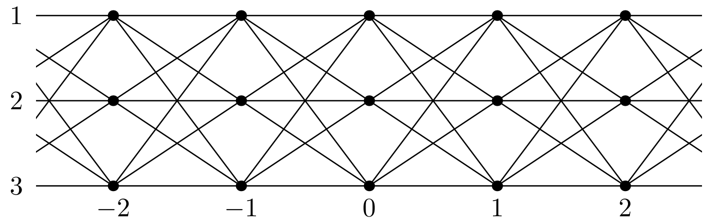


Figure 4: Replicated linear network

Theorem 3. *Let u be the bilingual game given by* (3.1)*.*
(i) 0 *is contagious and uninvadable in the class of replicated linear networks if $e < e^*$.* (ii) 2 *is contagious and uninvadable in the class of replicated linear networks if $e > e^*$.*

The proof is analogous to that of Theorem 2 and thus is omitted. This theorem implies that the class of all replicated linear networks is not critical for contagion.

5.2 Multidimensional Lattice Networks

We next show that the characterization in the previous subsection generalizes to multidimensional lattice networks with translation invariant interaction weights. For the sake of concreteness, we here focus on the m-dimensional lattice with *n-max distance interactions*, where each player interacts with all players within n steps away in each of the m coordinates, i.e., $\mathcal{X} = \mathbb{Z}^m$, and $P(x, y) = 1$ if $1 \leq \max_{i=1,\dots,m} |x_i - y_i| \leq n$ and $P(x, y) = 0$ otherwise. A more general treatment is relegated to Appendix A.6, where we consider a broader class of networks on $\mathbb{Z}^m$ such that interaction weights $P(x, y)$ are translation invariant and conditional weights $P(x|0)$ are approximated (with an appropriate normalization) by a density function on $\mathbb{R}^m$.

For 2×2 coordination games, Morris (2000) demonstrates that the characterization for contagion and uninvadability in the linear lattice still holds

[17] The "thick line graph" in Immorlica et al. (2007, Figure 2) is a special case of replicated linear network.

with higher dimensions as long as the interaction radius n is sufficiently large. We obtain an analogous characterization for our bilingual game.[18]

Theorem 4. *Let u be the bilingual game given by* (3.1)*. Fix the dimension m.*

(i) *If $e < e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$,* 0 *is contagious and uninvadable in the n-max distance interaction network on $\mathbb{Z}^m$.* (ii) *If $e > e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$,* 2 *is contagious and uninvadable in the n-max distance interaction network on $\mathbb{Z}^m$.*

Proof. See Appendix A.6. ∎

The proof is analogous to that of Lemma 1. In the case of $e < e^*$, for example, we show the contagion of action 0 by an induction argument along a sequence of regions of 0-players surrounded by "bilingual" regions. Here, each 0-player region is the set of lattice (i.e., integer-coordinate) points contained in a large m-dimensional ball with an outer m-dimensional ring of 1-players.

To conclude, the class of n-max distance interaction networks (with large n) as well as the class of (replicated) linear networks are not critical for contagion, and hence the network in Figure 3 exhibits fundamentally different properties in strategic behavior from those simple networks.

6 Comparison of Networks

The analysis in the previous sections has been to fix a game and find a network (from the universal domain or a restricted class of networks) in which an action becomes contagious, thereby differentiating, whenever possible, between strict Nash equilibria. In this section, we consider a converse exercise: fix a network and find a game in which an action becomes contagious. More precisely, we ask the following question: For a pair of networks, which network has a larger set (with respect to set inclusion) of payoff parameter values for which a given action is contagious? Such comparison is formalized by the following preorder over the set of networks.

Definition 5. A network $(\hat{\mathcal{X}}, \hat{P})$ is *more contagion-inducing in a class $\mathcal{U}$ of games* than another network $(\mathcal{X}, P)$ if for any game $u \in \mathcal{U}$, an action s^* is contagious in $(\hat{\mathcal{X}}, \hat{P})$ whenever s^* is contagious in $(\mathcal{X}, P)$.

Morris' (2000) approach is in fact in the sprit of this converse exercise for the class of all 2×2 coordination games. He defines the *contagion threshold*

[18] Even if we allow for small n, the class of max distance interaction networks is not critical for contagion. See Example 7 in Section 6.

of a network $(\mathcal{X}, P)$ to be the supremum of $q \in (0,1)$ for which action 1 of the two-action game

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix} \end{array}$$

becomes contagious in $(\mathcal{X}, P)$. The contagion threshold naturally represents the preorder in Definition 5 for the case where $\mathcal{U}$ is the class of 2×2 coordination games, that is, a network is more contagion-inducing in the class of 2×2 coordination games than another network if and only if the contagion threshold of the former is larger than that of the latter. Therefore, the preorder is complete in this special case.

We study this preorder for the case of the bilingual game. We first show that this preorder is incomplete in our case.

Example 2 (Figure 2 versus Figure 3)**.** Consider the bilingual game with $e \in (e^*, e^{**})$. Recall the two networks in the proof of Lemma 1, where we choose p, q, and r that satisfy (4.2) and (4.3). Then action B is contagious in the network of Figure 2, whereas action A is contagious in the network of Figure 3. Thus neither network is more contagion-inducing in the bilingual game than the other.

In principle, comparison of networks based on games with more actions can provide at least a weakly finer analysis of network topologies. The next example shows that our analysis with the bilingual game is strictly finer than that by Morris (2000) with 2×2 coordination games.

Example 3 (Tree versus ladder)**.** Consider the "tree" depicted in Figure 5, where each player is indexed by a finite sequence of 0 and 1, $\mathcal{X} = \bigcup_{n \geq 0} \{0,1\}^n$, and player $x \in \mathcal{X}$ interacts with $x0$, $x1$, and x^- with equal weights, where x^- is the immediate predecessor of x, i.e., the truncation of x that removes the last digit of x. Also consider the "ladder" depicted in Figure 6, where each player is indexed by a pair of α or β and an integer, $\mathcal{X} = \{\alpha, \beta\} \times \mathbb{Z}$, and with equal weights, player (α, i) interacts with $(\alpha, i \pm 1)$ and (β, i), while player (β, i) interacts with (α, i) and $(\beta, i \pm 1)$.

For these networks, we have the following:

- In 2×2 coordination games, the two networks are equally contagion-inducing. The contagion threshold of each network is $1/3$ (Examples 4 and 5 in Morris (2000) with $m = 2$).
- In the bilingual game, the ladder is more contagion-inducing than the tree. This will be proved in Example 4 by invoking Theorem 5 below.
- Moreover, in the bilingual game, the ladder is *strictly* more contagion-inducing than the tree.

To establish the third point, consider a set of parameter values of the bilingual game such that

$$br\left(\tfrac{2}{3}, \tfrac{1}{3}, 0\right) = \{A\}, \tag{6.1}$$

$$br\left(\tfrac{2}{3}, 0, \tfrac{1}{3}\right) = \{AB\}, \tag{6.2}$$

$$br\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = \{B\}, \tag{6.3}$$

that is,

$$2a - c < 2d - b, \qquad \frac{2a - c - d}{3} < e < \min\left\{\frac{d - b}{3}, \frac{2(a - c)}{3}\right\},$$

which are satisfied, for example, by $(a, b, c, d) = (11, 0, 3, 10)$ and $3 < e < 10/3$. For such parameter values, we claim that action B is contagious in the ladder, but not in the tree. First, in the ladder, suppose that initially players $(\alpha, -1)$, $(\alpha, 0)$, $(\beta, -1)$, and $(\beta, 0)$ play B, while all the others play A. Then players $(\alpha, 1)$ and $(\beta, 1)$ will switch from A to AB by condition (6.2) and then switch further from AB to B by (6.3). In this way, all players subsequently switch from A to AB, and to B; thus B is contagious in the ladder. Second, in the tree, for any finite set Y of initial B-players, pick a maximal (longest) element x of Y and assume that all successors of x play A. Then players $x0$ and $x1$ switch from A to AB by (6.2), but by (6.1) all the other successors of x continue to play A for any best response sequence; thus B is not contagious in the tree.

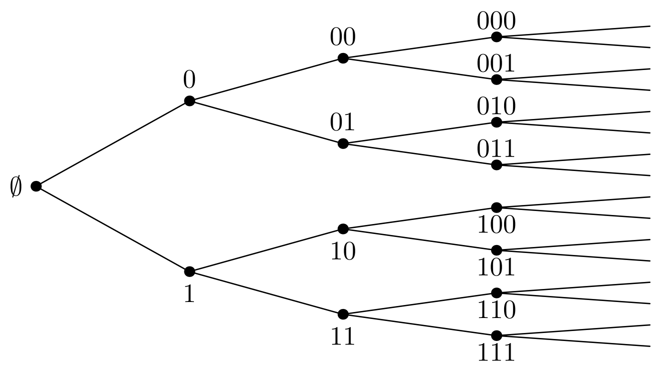


Figure 5: Tree

In Example 3, note that the ladder is obtained by "bundling" (or "identifying") nodes of the tree in the following way: map $\emptyset$ in the tree to $(\alpha, 0)$ in the ladder, 0 to $(\alpha, 1)$, 1 to $(\beta, 0)$, 00 to $(\alpha, 2)$, 11 to $(\beta, -1)$, and "bundle" 01 and 10 and map them to $(\beta, 1)$, and so on. The existence of such a

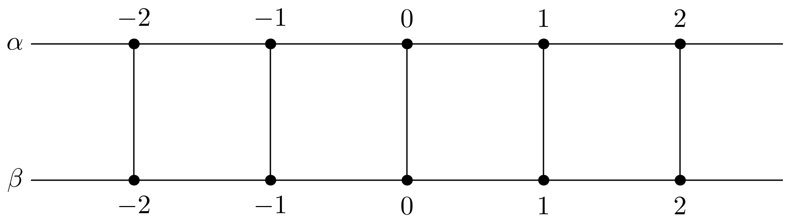


Figure 6: Ladder

map is the key to proving the second point in Example 3. In fact, it implies that any best response sequence in the ladder can be replicated by a best response sequence in the tree, and thus if an action spreads along the latter sequence, so does it along the former, which in turn implies that whenever an action is contagious in the tree, so is it in the ladder. We prove this in Theorem 5 for general networks and general supermodular games. We first formalize the idea of a mapping with "bundling" as described above.

Definition 6. For two networks $(\mathcal{X}, P)$ and $(\hat{\mathcal{X}}, \hat{P})$, a mapping $\varphi\colon \mathcal{X} \to \hat{\mathcal{X}}$ is a *weight-preserving node identification* from $(\mathcal{X}, P)$ to $(\hat{\mathcal{X}}, \hat{P})$ if φ is onto and there exists a finite subset E of $\mathcal{X}$ such that for any $x \in \mathcal{X} \setminus E$ and any $\hat{y} \in \hat{\mathcal{X}}$,

$$\hat{P}(\varphi(x), \hat{y}) = \sum_{y \in \varphi^{-1}(\hat{y})} P(x, y).$$

A node in E is called an *exceptional node*.[19]

The weight-preserving property implies an analogous property on conditional weights as follows:

$$\begin{aligned} \hat{P}(\hat{y}|\varphi(x)) &= \frac{\hat{P}(\varphi(x), \hat{y})}{\sum_{\hat{z} \in \hat{\mathcal{X}}} \hat{P}(\varphi(x), \hat{z})} \\ &= \sum_{y \in \varphi^{-1}(\hat{y})} \frac{P(x, y)}{\sum_{\hat{z} \in \hat{\mathcal{X}}} \sum_{z \in \varphi^{-1}(\hat{z})} P(x, z)} \\ &= \sum_{y \in \varphi^{-1}(\hat{y})} \frac{P(x, y)}{\sum_{z \in \mathcal{X}} P(x, z)} = \sum_{y \in \varphi^{-1}(\hat{y})} P(y|x) \end{aligned}$$

for any $x \in \mathcal{X} \setminus E$ and any $\hat{y} \in \hat{\mathcal{X}}$.

Theorem 5. *Let $\mathcal{U}$ be the class of all supermodular games. If there exists a weight-preserving node identification from $(\mathcal{X}, P)$ to $(\hat{\mathcal{X}}, \hat{P})$, then $(\hat{\mathcal{X}}, \hat{P})$ is more contagion-inducing in $\mathcal{U}$ than $(\mathcal{X}, P)$.*

[19] Allowing for exceptional nodes is essential for constructing a weight-preserving node identification from the tree to the ladder (see Example 4), but the reader may want to assume $E = \emptyset$ upon the first reading.

Proof. See Appendix A.7. ▮

The main idea of the proof is as follows. Suppose that s^* is contagious in $(\mathcal{X}, P)$, and φ is a weight-preserving node identification from $(\mathcal{X}, P)$ to $(\hat{\mathcal{X}}, \hat{P})$. Take any best response sequence $(\hat{\sigma}^t)$ on $(\hat{\mathcal{X}}, \hat{P})$. We construct a sequence (σ^t) on $(\mathcal{X}, P)$ by $\sigma^t(x) = \hat{\sigma}^t(\varphi(x))$ for any $x \in \mathcal{X}$ and $t \geq 0$, which, by the definition of weight-preserving node identification, is almost (but not quite, as we explain below) a best response sequence. Since s^* is contagious in $(\mathcal{X}, P)$, $(\sigma^t(x))$ converges to s^* for any $x \in \mathcal{X}$, and hence $(\hat{\sigma}^t(\hat{x}))$ also converges to s^* for any $\hat{x} \in \hat{\mathcal{X}}$. This argument, however, has two issues: first, along the sequence (σ^t), players in each $\varphi^{-1}(\hat{x})$ change actions simultaneously, which violates property (i) in Definition 1, and second, players at exceptional nodes may not play best responses. We use Lemma A.1 in Appendix A.1 to resolve these issues.

Note that node identification is different from adding or subtracting edges. Even if we construct $(\hat{\mathcal{X}}, \hat{P})$ by adding edges to $(\mathcal{X}, P)$, $(\hat{\mathcal{X}}, \hat{P})$ may be more contagion-inducing than, less contagion-inducing than, or incomparable to $(\mathcal{X}, P)$.[20]

In the next five examples, we construct weight-preserving node identifications between several pairs of networks and illustrate the implications of Theorem 5.

Example 4 (Tree versus ladder, continued)**.** There exists a weight-preserving node identification from the tree to the ladder. In fact, one can construct such a mapping recursively as follows: given that each node $x \in \bigcup_{k=0}^{n} \{0,1\}^k$ of the tree with depth at most n is assigned with a node $\varphi(x)$ of the ladder, for each $x \in \{0,1\}^n$, we assign $x0$ and $x1$ with two of the neighbors of $\varphi(x)$ in the ladder other than $\varphi(x^-)$. We can always do so since each node on the ladder has three neighbors. For example, let

$$\begin{aligned}
&\varphi(\emptyset) = (\alpha, 0), \\
&\varphi(0) = (\alpha, 1),\ \varphi(1) = (\beta, 0), \\
&\varphi(00) = (\alpha, 2),\ \varphi(01) = \varphi(10) = (\beta, 1),\ \varphi(11) = (\beta, -1),\ \ldots .
\end{aligned}$$

Then φ preserves interaction weights except at the root $\emptyset$ (φ does not preserve interaction weights at the root $\emptyset$ because player $\emptyset$ has two neighbors in the tree while player $\varphi(\emptyset)$ has three neighbors in the ladder). Thus, by Theorem 5, the ladder is more contagion-inducing than the tree.

[20] In contrast, Galeotti et al. (2010), for example, obtain the monotonicity of equilibrium actions with respect to adding or subtracting edges in their framework where the action space is a subset of the real line and the payoffs depend on the summation or maximum of neighbors' actions. See also Wolitzky (2012), who shows that the level of cooperation in a repeated game with a monitoring network is monotonic with respect to his notion of network centrality, where a network becomes more central than another network if the former is obtained by adding edges to the latter.

More generally, there exists a weight-preserving node identification from the tree in Figure 5 to any connected network in which each player interacts with exactly 3 neighbors with equal weights. Therefore, by Theorem 5, the latter is (weakly) more contagion-inducing than the former. In fact, this relation extends to the tree with m successors and connected networks with $m + 1$ neighbors.

Example 5 (Line versus lattice)**.** Consider the line depicted in Figure 1, and the two-dimensional lattice depicted in Figure 7 where each player $x = (x_1, x_2) \in \mathbb{Z}^2$ interacts with $(x_1 \pm 1, x_2)$ and $(x_1, x_2 \pm 1)$ with equal weights. Then the mapping $\varphi(x_1, x_2) = x_1 + x_2$ is a weight-preserving node identification from the two-dimensional lattice to the line with no exceptional node. Thus, by Theorem 5, the line is more contagion-inducing (in fact, strictly more contagion-inducing whenever $\mathcal{U}$ contains all 2×2 coordination games) than the two-dimensional lattice.

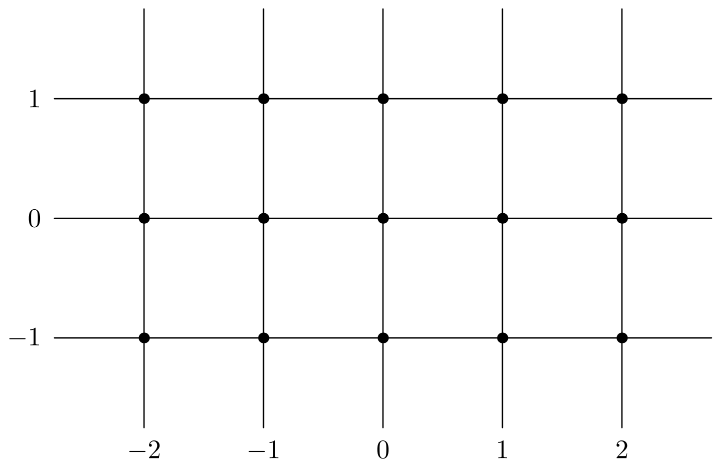


Figure 7: Two-dimensional lattice

Morris (2000) showed that the line is more contagion-inducing than the two-dimensional lattice for the class of 2×2 coordination games by computing the contagion thresholds explicitly. Our Theorem 5 gives an alternative proof to this result, which generalizes to other pairs of networks and to general symmetric supermodular games including the bilingual game.

Example 6 (Line versus replicated lines)**.** Consider a replicated linear network $(\{1, \ldots, m\} \times \mathbb{Z}, P)$ as defined in Section 5.1. Then the mapping $\varphi \colon \{1, \ldots, m\} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(k, i) = i$ is a weight-preserving node identification (with no exceptional node) from this network to the linear network $(\mathbb{Z}, \hat{P})$ with $\hat{P}(i, j) = \sum_{k=1}^{m} P((1, i), (k, j))$. Thus, by Theorem 5, any replicated linear network is less contagion-inducing than some linear network. In light of this, Theorem 3 follows as an immediate corollary of Theorem 2. In fact, one can readily show, in the class of all supermodular games, that $(\{1,\dots,m\}\times\mathbb{Z}, P)$ is also more contagion-inducing than $(\mathbb{Z}, \hat{P})$, and hence the two networks are equally contagion-inducing.

Example 7 (Line versus max distance)**.** Consider the n-max distance interaction network on the m-dimensional lattice $\mathbb{Z}^m$ as defined in Section 5.2. Define the mapping $\varphi\colon \mathbb{Z}^m \to \mathbb{Z}$ by

$$\varphi(x_1,\dots,x_m) = x_1 + (n+1)x_2 + \cdots + (n+1)^{m-1}x_m$$

for any $(x_1,\dots,x_m) \in \mathbb{Z}^m$. Then φ is a weight-preserving node identification (with no exceptional node) from this network to the linear network $(\mathbb{Z}, \hat{P})$ with $\hat{P}(x,y) = \#(\varphi^{-1}(y-x) \cap [-n,n]^m)$ for any $x, y \in \mathbb{Z}$ with $x \neq y$.[21] Thus, by Theorem 5, any max distance interaction network is less contagion-inducing than some linear network. Combined with Theorem 2, this implies that, in the bilingual game, action 0 is not contagious in the class of all max distance interaction networks if $e > e^*$, which in turn implies that this class is not critical for contagion.

Example 8 (Regions versus lattice)**.** Consider the network depicted in Figure 8, where the players are divided into infinitely many "regions", and each region consists of three players: $\mathcal{X} = \{1,2,3\} \times \mathbb{Z}$, and with equal weights, player (k,i) interacts with players (ℓ,j) such that $\ell \neq k$ and $j = i$, or $\ell = k$ and $j = i \pm 1$. Then the mapping $\varphi\colon \mathbb{Z}^2 \to \{1,2,3\} \times \mathbb{Z}$ defined by $\varphi(x_1,x_2) = (k,x_2)$ such that $k \equiv x_1 \pmod 3$ is a weight-preserving node identification from the two-dimensional lattice to the regions network (with no exceptional node). Thus, by Theorem 5, the regions network is more contagion-inducing than the two-dimensional lattice. Moreover, one can show, in a similar manner as in Example 3, that in the bilingual game, the former is strictly more contagion-inducing than the latter. On the other hand, in the class of 2×2 coordination games, the two networks have the same contagion threshold $1/4$ and therefore are equally contagion-inducing (Examples 2 and 4 in Morris (2000)). Thus, this is an example other than Example 3 that shows that our analysis based on the bilingual game is strictly finer than that based on 2×2 coordination games.

The next three examples discuss pairs of networks that do not admit any weight-preserving node identification.

Example 9 (Line versus Figure 3)**.** Theorems 2 and 5 imply that there exists no weight-preserving node identification from the network in Figure 3 to any linear network. For example, the mapping $\varphi(\alpha,i) = \varphi(\beta,i) = i$ from $\{\alpha,\beta\} \times \mathbb{Z}$ to $\mathbb{Z}$ does not preserve interaction weights between the network

[21] $\#X$ denotes the cardinality of X.

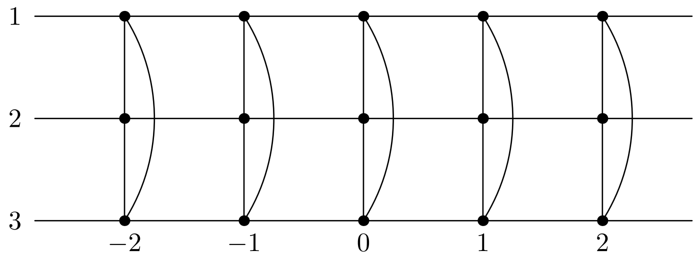


Figure 8: Regions

in Figure 3 and any linear network $(\mathbb{Z}, \hat{P})$. Indeed, it bundles nodes (α, i) and (β, i) with a positive interaction weight, and therefore, if φ preserved interaction weights, we would have $\hat{P}(i,i) > 0$.

Example 10 (Line versus regions)**.** Consider the regions network as depicted in Figure 8. Recall that in 2×2 coordination games, it has contagion threshold $1/4$ and thus is not more contagion-inducing than the linear network in Figure 1. Also, one can show, in a similar manner as in Example 3, that there is a set of parameter values of the bilingual game such that B is contagious in the regions network but not in the linear network. Combining these two facts implies that the regions network is not comparable with the linear network. Therefore, it follows from Theorem 5 that no weight-preserving node identification exists between these two networks. For example, the mapping $\varphi(k,i) = k + i$ from $\{1,2,3\} \times \mathbb{Z}$ to $\mathbb{Z}$ does not preserve interaction weights between the regions network and any linear network on $\mathbb{Z}$.

Example 11 (Line versus line)**.** Consider two linear networks, one depicted in Figure 1 and the other in Figure 2 with any fixed $p \in (0, 1/2)$. Then one can verify that Figure 2 is (strictly) more contagion-inducing than Figure 1 for the class of all supermodular games. However, no node identification from Figure 1 to Figure 2 preserves interaction weights because any weight-preserving node identification is allowed to increase the number of neighbors only for exceptional nodes. Thus the converse of Theorem 5 does not hold, that is, the existence of a weight-preserving node identification is not necessary for two networks to be ordered in terms of the power of inducing contagion.

7 Interpretations in Incomplete Information Games

Local interaction games and incomplete information games, though capturing different economic or social situations, share the same formal structures and thus belong to a more general class of "interaction games" (Morris (1997, 1999), Morris and Shin (2003)): in local interaction games, each node interacts with a set of neighbors and payoffs are given by the weighted sum of those from the interactions; in incomplete information games, each type interacts with a subset of types and payoffs are given by the expectation of those from the interactions.[22,23] Indeed, Morris (1997, 1999) demonstrates, in spite of some technical differences, that several tools and results in the context of incomplete information games can be utilized also in the context of local interaction games, and vice versa.[24] In this section, we interpret our results, in particular the discussions in Section 5, in the language of incomplete information games, thereby shedding new lights on two existing lines of literature, robustness to incomplete information (Kajii and Morris (1997), Morris and Ui (2005)) and global games (Carlsson and van Damme (1993), Frankel et al. (2003)). We also discuss our assumption of symmetry in interaction weights in relation to the common prior assumption in incomplete information games.

7.1 Robustness to Incomplete Information

A Nash equilibrium (s_1^*, s_2^*) of a two-player game u is said to be *robust to incomplete information* in u if any ε-incomplete information perturbation of u with ε sufficiently small has a Bayesian Nash equilibrium that plays (s_1^*, s_2^*) with high probability, where an ε-incomplete information perturbation of u refers to an incomplete information game in which the set T^u of type profiles whose payoffs are given by u has ex ante probability $1 - \varepsilon$ while types outside T^u ("crazy types") may have very different payoff functions (Kajii and Morris (1997)[25]). Robustness to incomplete information corresponds to uninvadability in local interaction systems in that both notions require that a small amount of "crazy types" should not affect the aggregate behavior.

Indeed, they have the same characterizations in many classes of games, including games with an MP-maximizer. In parallel with Lemma 3, an MP-maximizer of a game u with MP-function v is robust to incomplete information if u or v is supermodular (Morris and Ui (2005)). Combining this result with Lemma 4, we obtain a sufficient condition for robustness in

[22]For example, with the incomplete information interpretation, the linear network in Figure 1 is essentially equivalent to the information structure of the email game of Rubinstein (1989).

[23]A class of dynamic games with Poisson action revisions due to Matsui and Matsuyama (1995) (perfect foresight dynamics) also belong to interaction games, where each revising player interacts with a set of past and future players and payoffs are given the discounted sum of flow payoffs from the interactions (Takahashi (2008)).

[24]For example, the contagion threshold of a local interaction system due to Morris (2000) is essentially equivalent to the belief potential of an information system due to Morris et al. (1995).

[25]Kajii and Morris (1997) consider games with any finite number of players.

the bilingual game.

Conversely, a necessary condition for robustness is obtained by constructing ε-incomplete information perturbations in which a given action profile is contagious, where an action s^* is said to be contagious in an ε-incomplete information perturbation if s^* is a dominant action for types outside T^u and playing s^* everywhere is a unique rationalizable strategy. Specifically, in any 3×3 supermodular game u, adjusting the proof of Lemma 1, for any $\varepsilon > 0$ one can construct ε-incomplete information perturbations in which 0 (2, resp.) is contagious if (4.1) ((4.2), resp.) holds for some $p \in (0, 1/2)$, or (4.3) ((4.4), resp.) holds for some $q, r \in (0, 1)$ with $r \leq q$ (Oyama and Takahashi (2011)). The necessary condition thus follows by applying this result to the bilingual game combined with Lemma 2.

These arguments characterize exactly as in Theorem 1 when an equilibrium in the bilingual game is robust to incomplete information.

Proposition 1. *Let u be the bilingual game given by* (3.1).

(i) $(0, 0)$ *is a unique robust equilibrium if* $e < e^*$. (ii) $(2, 2)$ *is a unique robust equilibrium if* $e > \max\{e^*, e^{**}\}$. (iii) *No action profile is robust if* $e^* < e < \max\{e^*, e^{**}\}$.

7.2 Global Games

Global games constitute a subclass of incomplete information games, where the underlying state θ is drawn from the real line, and each player i receives a noisy signal $x_i = \theta + \nu\varepsilon_i$ with ε_i being a noise error independent across players and from θ. Under supermodularity and state-monotonicity in payoffs, it has been shown by a contagion argument that an essentially unique equilibrium survives iterative deletion of dominated strategies as $\nu \to 0$, while the limit equilibrium may depend on the distribution of noise terms ε_i (Frankel et al. (2003)).

Global game perturbations in the class of all incomplete information perturbations can be viewed as linear networks in the class of all networks. In global games, the distribution of the opponent's signal x_j conditional on x_i is (approximately) invariant up to translation (for small $\nu > 0$) due to the assumption of state-independent noise errors, which parallels the translation invariance in linear networks. In fact, in the context of local interactions, by adopting the argument of Frankel et al. (2003) one can show that a generic supermodular game has at least one contagious action, and hence if an action is uninvadable, then it is also contagious and no other action is uninvadable.[26]

Basteck and Daniëls (2011) prove that, in any global game of 3×3 supermodular games independently of the noise distribution, action profile $(0, 0)$ ($(2, 2)$, resp.) is played at θ as $\nu \to 0$ if (4.1) ((4.2), resp.) holds for

[26]In the special case of the bilingual game, these results follow from our Theorem 1.

some $p \in (0, 1/2)$ at that state θ. Together with Lemma 2(1), this leads to the following characterization of global-game noise-independent selection in the bilingual game, the same one as in Theorem 2.

Proposition 2. *Let u be the bilingual game given by* (3.1).

(i) (0, 0) *is a noise-independent global game selection if* $e < e^*$. (ii) (2, 2) *is a noise-independent global game selection if* $e > e^*$.

Since this characterization is different from that in Proposition 1, global games are not a critical class of incomplete information games that determines whether or not an action profile is robust to incomplete information. See Oyama and Takahashi (2011) for further discussions.

Global games have been extended to multidimensional states and signals while maintaining the assumption of state-independent noise errors. (Indeed, multidimensional states and signals are already accommodated in Carlsson and van Damme (1993).) Recently, Oury (2012) shows that if an action is played in some one-dimensional global game of supermodular games independently of the noise distribution, then it is also played in any multidimensional global game. This result, combined with that of Oyama and Takahashi (2011), suggests that even with multidimensional states and signals, global games do not constitute a critical class of incomplete information perturbations. This is consistent with our findings in this paper, especially that multidimensional lattice networks do not constitute a critical class of networks for contagion (Theorem 4 and Example 7).

7.3 Contagion and Robustness under Non-Common Priors

The results on incomplete information games reported in Subsections 7.1 and 7.2 all rely on the implicit assumption that in incomplete information perturbations the players share a common prior probability distribution, from which each player derives his conditional beliefs based on the information he has. This common prior assumption corresponds in our local interaction context to the assumption that the weight function P on interactions is symmetric, i.e., $P(x, y) = P(y, x)$ for all $x, y \in \mathcal{X}$. The symmetry of the weight function naturally arises when the value $P(x, y)$ represents the duration (within a period) or intimacy of the interaction between x and y. Alternatively, if asymmetric weights are allowed, the situation corresponds to one of non-common priors.

Oyama and Tercieux (2010, 2012) study contagion and robustness under non-common priors, where players may have heterogeneous priors in ε-incomplete information perturbations and the probability of crazy types is no larger than ε with respect to all the players' priors. They show that under non-common priors, any strict Nash equilibrium of a complete information game is contagious in some ε-perturbations, and that generically, a game has a robust equilibrium if and only if it is dominance solvable, in which case the unique surviving action profile is robust.

Their results have a direct translation in our local interactions context: under asymmetric weights, any strict Nash equilibrium is contagious, and generically, a game has an uninvadable action if and only if it is dominance solvable, in which case the unique surviving action is uninvadable. To see this, let (s^*, s^*) be any strict Nash equilibrium of u, and $\eta > 0$ be such that $\{s^*\} = br(\pi)$ for all $\pi \in \Delta(S)$ with $\pi_{s^*} \geq 1 - \eta$. Given such an η, define the network $(\mathcal{X}, P^\eta)$ by $\mathcal{X} = \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and $P^\eta(x, x+1) = \eta$, $P^\eta(x+1, x) = 1-\eta$, and $P^\eta(x, y) = 0$ if $|x-y| \neq 1$, where conditional weights $P^\eta(\cdot|\cdot)$ are defined by $P^\eta(y|x) = P^\eta(x, y)/\sum_{y'} P^\eta(x, y')$. Then it is easy to see that along any best response sequence $(\sigma^t)_{t=0}^\infty$ such that $\sigma^0(0) = \sigma^0(1) = s^*$, we have $\lim_{t\to\infty} \sigma^t(x) = s^*$ for all $x \in \mathcal{X}$; that is, s^* is contagious in this network. This immediately implies that if the game has more than two strict Nash equilibria, then no action can be uninvadable. Thus, for generic supermodular games, if the game is not dominance solvable, in which case there are multiple strict equilibria, it has no uninvadable action (for general games, see Oyama and Tercieux (2010)).

One may notice that in the asymmetric network $(\mathcal{X}, P^\eta)$ above, the conditional weights $P^\eta(\cdot|\cdot)$ can be generated by a symmetric weight function $\bar{P}^\eta$ given by $\bar{P}^\eta(x, x+1) = \bar{P}^\eta(x+1, x) = [\eta/(1-\eta)]^x$ and $\bar{P}^\eta(x, y) = 0$ if $|x - y| \neq 1$. This might appear contradicting, for example, the fact that a risk-dominated equilibrium of 2×2 coordination games is not contagious in any network with symmetric weights. However, whenever $\eta < 1/2$ (e.g., s^* is risk-dominated), $\bar{P}^\eta$ does not satisfy the unboundedness condition, as we have $\sum_{x,y} \bar{P}^\eta(x, y) = 2(1-\eta)/(1-2\eta) < \infty$. In this case, the proportion of initial "crazy type" interactions is $(\bar{P}^\eta(0, 1) + \bar{P}^\eta(1, 0) + \bar{P}^\eta(1, 2) + \bar{P}^\eta(2, 1))/\sum_{x,y} \bar{P}^\eta(x, y) = 1 - 2\eta$, which is bounded away from 0 and thus considered to be not small. Indeed, our uninvadability results rely on the joint assumption of symmetry and unboundedness of interaction weights.

8 Conclusion

We have considered contagion and uninvadability of actions for the bilingual game played on a network. By incorporating a third action, our study refines Morris' (2000) approach which analyzes strategic behavior of two-action games on a network to understand the network structures. In principle, an analysis with more actions will reveal, at least weakly, finer structures of networks than that with two actions, whereas conceivably the analysis would become intractable with many actions. We have demonstrated that our three-action bilingual game is simple enough to enable us to obtain a complete characterization (Theorem 1), and yet rich enough to offer a strictly finer analysis than that by Morris (2000) based on two-action coordination games (Example 3). We also developed the concept of weight-preserving node identification to provide a sufficient condition for different networks to be comparable in terms of the power of inducing contagion (Theorem 5). This concept may be of independent interest, and other contexts to which it is applied are yet to be explored in future research.

We close with a brief discussion on incorporating stochastic elements in the model to examine the robustness of our results to various forms of randomness (see also Morris (2000, Section 7)). First, introducing randomness in action revision opportunities, independent across agents, will not change our results, under appropriate restatements of the definitions of contagion and uninvadability by adding "with probability one/zero", as we allow for general (sequential) best response sequences (Definition 1; see also Footnote 9). Second, randomness in initial conditions may foster contagion. In fact, for 2×2 coordination games, the contagion threshold of the two-dimensional lattice is $1/2$ if actions in the initial configuration are given i.i.d. across agents (Lee and Valentinyi (2000)), whereas that in Morris' (2000) sense is $1/4$. Extension to games with more than two actions, including the bilingual game, is left open. Third, persistent randomness in best responses has been considered by Blume (1993) and Ellison (1993) and more recently, Montanari and Saberi (2010), Young (2011), Kreindler and Young (2012), among others, for local interactions. For the bilingual game, if restricted to interactions on circle or lattice networks, stochastic stability under persistent randomness will be characterized by the condition for contagion stated in our Theorems 2–4. A characterization of stochastic stability for global interactions is obtained by Galesloot and Goyal (1997), which is distinct from the one under local interactions. It remains open to characterize stochastic stability for non-linear networks. Fourth, incorporating heterogeneity in players' preferences, for example by allowing for player-specific bilingual cost e_x for each player x, would be an interesting extension. Theorem 1 obviously extends if the corresponding conditions are satisfied uniformly for all players; e.g., 0 is contagious if $e_x < \max\{e^*, e^{**}\}$ for all x. When e_x distributes below and above the cutoffs, coexistence of conventions may arise depending on the correlation between preferences and locations of the players. Following Manski (1993), one may also address the identification problem of local interactions and preference correlation in our model. Fifth, we have applied our general results to various examples of simple networks with some recurrent structures, but have been silent about more complex and less structured networks. As a tractable first step, it would be worthwhile to consider random networks, which generate such networks with high probability (see, e.g., Pastor-Satorras and Vespignani (2001), Watts (2002), López-Pintado (2006, 2008), Berninghaus and Haller (2010), Lelarge (2012)).

Appendix

A.1 Equivalent Definitions of Contagion in Supermodular Games

In this appendix, we discuss three other definitions of contagion, and show that all of them are equivalent to the original one for any (generic) symmetric supermodular game (S, u), where $S = \{0, 1, \dots, n\}$ and $u(h', k) - u(h, k) \le u(h', k') - u(h, k')$ if $h < h'$ and $k < k'$. (None of the results here relies on the particular payoff structure of the bilingual game.) We use the partial order $\sigma \le \sigma'$ whenever $\sigma(x) \le \sigma'(x)$ for any $x \in \mathcal{X}$.

First, recall that in the main text, we consider the sequential best response dynamics, where at most one player revises his action in each period (property (i) in Definition 1). Instead, we can define the simultaneous (resp. generalized) best response dynamics, where all (resp. some) players revise their actions at a time.

Definition A.1. A sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *simultaneous best response sequence* if $\sigma^t(x) \in BR(\sigma^{t-1}|x)$ for all $x \in \mathcal{X}$ and $t \ge 1$. A sequence $(\sigma^t)_{t=0}^{\infty}$ is a *generalized best response sequence* if it satisfies the following properties: (ii) if $\sigma^t(x) \ne \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR(\sigma^{t-1}|x)$; and (iii) if $\lim_{t\to\infty} \sigma^t(x) = s$, then for all $T \ge 0$, $s \in BR(\sigma^t|x)$ for some $t \ge T$.

For clarity, we add adjective "sequential" to the original notion of best response sequences. Generalized best response sequences subsume both sequential and simultaneous best response sequences as special cases.

Using simultaneous or generalized best response sequences, we obtain two new definitions of contagion.[27]

Definition A.2. Action s^* is *contagious by simultaneous* (resp. *generalized*) *best responses in network* $(\mathcal{X}, P)$ if there exists a finite subset Y of $\mathcal{X}$ such that every simultaneous (resp. generalized) best response sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0(x) = s^*$ for all $x \in Y$ satisfies $\lim_{t\to\infty} \sigma^t(x) = s^*$ for each $x \in \mathcal{X}$.

We refer to the notion of contagion in Definition 2 as "contagion by sequential best responses". By definition, contagion by generalized best responses implies both contagion by sequential best responses and by simultaneous best responses. Here we show the converse.

In the next lemma, we show that if s^* is contagious by sequential best responses, then there exist two sequential best response sequences that converge to s^* monotonically (one increasingly and the other decreasingly), and that any generalized best response sequence that starts between the two sequences also converges to s^*. This lemma is used to prove both Proposition A.1 below and Theorem 5 in the main text.

[27]The notion of contagion used in Morris (2000) is similar to contagion by simultaneous best responses, but requires only that for each $x \in \mathcal{X}$, $\sigma^t(x) = s^*$ for some $t \ge 0$.

Lemma A.1. *Fix a network* $(\mathcal{X}, P)$ *and a supermodular game* u*. Suppose that* s^* *is contagious by sequential best responses in* $(\mathcal{X}, P)$*. Then there exist two sequential best response sequences* $(\sigma_-^t)_{t=0}^\infty$ *and* $(\sigma_+^t)_{t=0}^\infty$ *such that*

(1) $\sigma_-^t(x) \leq s^* \leq \sigma_+^t(x)$ *for all* $x \in \mathcal{X}$ *and* $t \geq 0$*;*

(2) $\sigma_-^0(x) = 0$ *and* $\sigma_+^0(x) = n$ *for all but finitely many* $x \in \mathcal{X}$*;*

(3) $\sigma_-^t(x) \in \{\sigma_-^{t-1}(x), \min BR(\sigma_-^{t-1}|x)\}$ *for all* $x \in \mathcal{X}$ *and* $t \geq 1$*;*

(4) $\lim_{t\to\infty} \sigma_-^t(x) = \lim_{t\to\infty} \sigma_+^t(x) = s^*$ *for all* $x \in \mathcal{X}$*; and*

(5) $\min BR(\sigma_-^0|x) \geq \sigma_-^0(x)$ *and* $\max BR(\sigma_+^0|x) \leq \sigma_+^0(x)$ *for all* $x \in \mathcal{X}$*.*

Moreover,

(6) *for any generalized best response sequence* $(\tilde{\sigma}^t)_{t=0}^\infty$ *with* $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$*, we have* $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$ *for all* $x \in \mathcal{X}$*.*

Proof. Suppose that s^* is contagious by sequential best responses in $(\mathcal{X}, P)$ (and hence a strict Nash equilibrium). Let $Y \subset \mathcal{X}$ be a finite set as in Definition 2, and let $(\phi_-^t)_{t=0}^\infty$ be the sequential best response sequence such that $\phi_-^0(x) = s^*$ for all $x \in Y$, $\phi_-^0(x) = 0$ for all $x \in \mathcal{X} \setminus Y$, and $\phi_-^t(x) \in \{\phi_-^{t-1}(x), \min BR(\phi_-^{t-1}|x)\}$ for all $x \in \mathcal{X}$ and $t \geq 1$. By definition, $\lim_{t\to\infty} \phi_-^t(x) = s^*$ for all $x \in \mathcal{X}$.

The sequence $(\phi_-^t)_{t=0}^\infty$ satisfies properties (1)–(4), but not necessarily property (5). From $(\phi_-^t)_{t=0}^\infty$, we construct another sequence that satisfies property (5) as well. Let $\psi_-^0 = \phi_-^0$ and

$$\psi_-^t(x) = \begin{cases} \psi_-^{t-1}(x) & \text{if } \phi_-^t(x) \leq \psi_-^{t-1}(x), \\ \min BR(\psi_-^{t-1}|x) & \text{if } \phi_-^t(x) > \psi_-^{t-1}(x). \end{cases} \tag{A.1}$$

Clearly, $(\psi_-^t)_{t=0}^\infty$ is a sequential best response sequence. By the construction of $(\phi_-^t)_{t=0}^\infty$ and $(\psi_-^t)_{t=0}^\infty$ along with the supermodularity and s^* being a Nash equilibrium, one can show by induction on t that $\phi_-^t(x) \leq \psi_-^t(x) \leq s^*$ for all $x \in \mathcal{X}$ and $t \geq 0$. Thus for each $x \in \mathcal{X}$, $(\psi_-^t(x))_{t=0}^\infty$ is weakly increasing and converges to s^*.

Since s^* is a strict Nash equilibrium, we can take a finite but sufficiently large subset Z of $\bigcup_{x\in Y} \Gamma(x)$ such that for any $x \in Y$, the best response of player x is s^* if all players in Z play s^* (recall that $\Gamma(x)$ is the set of the neighbors of player x). Let T be sufficiently large so that $\psi_-^T(x) = s^*$ for all $x \in Z$.

We claim that $\min BR(\psi_-^T|x) \geq \psi_-^T(x)$ for all $x \in \mathcal{X}$. For $x \in Y$, since all players in Z play s^* in period T, we have $\min BR(\psi_-^T|x) = s^* \geq \psi_-^T(x)$. For $x \in \mathcal{X} \setminus Y$, we first have $\min BR(\psi_-^0|x) \geq 0 = \psi_-^0(x)$. Next, assume that $\min BR(\psi_-^{t-1}|x) \geq \psi_-^{t-1}(x)$. By the construction of $(\psi_-^t(x))_{t=0}^\infty$ in (A.1), $\psi_-^t(x)$ is equal to either $\psi_-^{t-1}(x)$ or $\min BR(\psi_-^{t-1}|x)$. In both cases, we have $\min BR(\psi_-^{t-1}|x) \geq \psi_-^t(x)$. Since $(\psi_-^t)_{t=0}^\infty$ is weakly increasing, we have $\min BR(\psi_-^t|x) \geq \min BR(\psi_-^{t-1}|x)$ by the supermodularity. Hence, $\min BR(\psi_-^t|x) \geq \psi_-^t(x)$.

Now let $\sigma_-^t = \psi_-^{t+T}$ for $t \geq 0$. Then $(\sigma_-^t)_{t=0}^\infty$ satisfies properties (1)–(5). In particular, along the sequential best response sequence $(\psi_-^t)_{t=0}^\infty$, at most T players change actions by period T, so that $\sigma_-^0(x) = \psi_-^T(x) = 0$ except for finitely many x. The construction of $(\sigma_+^t)_{t=0}^\infty$ is analogous.

For property (6), pick any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^\infty$ with $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$. For each $x \in \mathcal{X}$, let $\underline{\tilde{\sigma}}^t(x) = \inf_{\tau \geq t} \tilde{\sigma}^\tau(x)$, and $\tilde{\sigma}_-(x) = \liminf_{t\to\infty} \tilde{\sigma}^t(x)$ $(= \lim_{t\to\infty} \underline{\tilde{\sigma}}^t(x))$.

Claim 1. $\liminf_{t\to\infty} \min BR(\tilde{\sigma}^t|x) \geq \min BR(\tilde{\sigma}_-|x)$ *for all* $x \in \mathcal{X}$.

Proof. Fix any $x \in \mathcal{X}$. By the supermodularity, we have $\min BR(\tilde{\sigma}^t|x) \geq \min BR(\underline{\tilde{\sigma}}^t|x)$ for all $t \geq 0$. Therefore, we have

$$\begin{aligned}\liminf_{t\to\infty} \min BR(\tilde{\sigma}^t|x) &\geq \liminf_{t\to\infty} \min BR(\underline{\tilde{\sigma}}^t|x) \\ &\geq \min BR\left(\lim_{t\to\infty} \underline{\tilde{\sigma}}^t \;\middle|\; x\right) = \min BR(\tilde{\sigma}_-|x),\end{aligned}$$

where the second inequality follows from the lower semicontinuity of $\min BR(\cdot|x)$ in the product topology on $S^{\mathcal{X}}$. ▮

Claim 2. $\tilde{\sigma}_-(x) \geq \min BR(\tilde{\sigma}_-|x)$ *for all* $x \in \mathcal{X}$.

Proof. Fix any $x \in \mathcal{X}$. By Claim 1, there exists $T_1 \geq 0$ such that $\min BR(\tilde{\sigma}^t|x) \geq \min BR(\tilde{\sigma}_-|x)$ for all $t \geq T_1$. By (ii) and (iii) in Definition A.1, there exists $T_2 \geq T_1$ such that $\tilde{\sigma}^{T_2}(x) \geq \min BR(\tilde{\sigma}_-)$. By (ii) in Definition A.1, we also have $\tilde{\sigma}^t(x) \geq \tilde{\sigma}^{T_2}(x) \wedge \min_{T_2 \leq \tau < t} \min BR(\tilde{\sigma}^\tau|x)$ for all $t \geq T_2$. Therefore, by Claim 1 it follows that $\tilde{\sigma}^t(x) \geq \min BR(\tilde{\sigma}_-)$ for all $t \geq T_2$, and hence $\tilde{\sigma}_-(x) \geq \min BR(\tilde{\sigma}_-|x)$, as desired. ▮

Claim 3. $\sigma_-^t \leq \tilde{\sigma}_-$ *for all* $t \geq 0$.

Proof. We proceed by induction. First, we want to show $\sigma_-^0 \leq \tilde{\sigma}_-$. By assumption, $\sigma_-^0 \leq \tilde{\sigma}^0$. Assume that $\sigma_-^0 \leq \tilde{\sigma}^{t-1}$, and consider any $x \in \mathcal{X}$ such that $\tilde{\sigma}^t(x) \neq \tilde{\sigma}^{t-1}(x)$. Then by the property (5) and the supermodularity, $\sigma_-^0(x) \leq \min BR(\sigma_-^0|x) \leq \min BR(\tilde{\sigma}^{t-1}|x) \leq \tilde{\sigma}^t(x)$. Therefore, we have $\sigma_-^0 \leq \tilde{\sigma}^t$ for all $t \geq 0$, and hence $\sigma_-^0 \leq \tilde{\sigma}_-$.

Next, assume that $\sigma_-^{t-1} \leq \tilde{\sigma}_-$, and let $x \in \mathcal{X}$ be such that $\sigma_-^t(x) \neq \sigma_-^{t-1}(x)$. Then by the property (3), the induction hypothesis, the supermodularity, and Claim 2, we have $\sigma_-^t(x) = \min BR(\sigma_-^{t-1}|x) \leq \min BR(\tilde{\sigma}_-|x) \leq \tilde{\sigma}_-(x)$. Thus we have $\sigma_-^t \leq \tilde{\sigma}_-$. ▮

Symmetrically, defining $\tilde{\sigma}_+(x) = \limsup_{t\to\infty} \tilde{\sigma}^t(x)$, we can show that $\tilde{\sigma}_+ \leq \sigma_+^t$ for all $t \geq 0$. For each $x \in \mathcal{X}$, since $\lim_{t\to\infty} \sigma_-^t(x) =$

$\lim_{t\to\infty} \sigma_+^t(x) = s^*$, we have $\tilde{\sigma}_-(x) = \tilde{\sigma}_+(x) = s^*$, and hence $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$.

This completes the proof of Lemma A.1. ▮

Proposition A.1. *Fix a network $(\mathcal{X}, P)$ and a supermodular game u. Then s^* is contagious by sequential best responses in $(\mathcal{X}, P)$ if and only if it is contagious by generalized best responses in $(\mathcal{X}, P)$.*

Proof. The "if" part holds by definition. To show the "only if" part, suppose that s^* is contagious by sequential best responses in $(\mathcal{X}, P)$. Let $(\sigma_-^t)_{t=0}^\infty$ and $(\sigma_+^t)_{t=0}^\infty$ be sequential best response sequences as in Lemma A.1. Let Y be a finite subset of $\mathcal{X}$ such that $\sigma_-^0(x) = 0$ and $\sigma_+^0(x) = n$ for all $x \in \mathcal{X} \setminus Y$. Then for any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^\infty$ with $\tilde{\sigma}^0(x) = s^*$ for all $x \in Y$, we have $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$, and hence by Lemma A.1, $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$ for all $x \in \mathcal{X}$. Thus s^* is contagious by generalized best responses in $(\mathcal{X}, P)$. ▮

Similarly, we can prove the equivalence between contagion by simultaneous best responses and contagion by generalized best responses. Here we assume that the set of neighbors $\Gamma(x)$ is finite for each player $x \in \mathcal{X}$, which is satisfied in all of our examples.

Proposition A.2. *Fix a network $(\mathcal{X}, P)$ such that $\Gamma(x)$ is finite for each $x \in \mathcal{X}$ and a supermodular game u. Then s^* is contagious by simultaneous best responses in $(\mathcal{X}, P)$ if and only if it is contagious by generalized best responses in $(\mathcal{X}, P)$.*

Proof. The "if" part holds by definition. The proof of the "only if" part is to mimic the proofs of Lemma A.1 and the "only if" part of Proposition A.1. Indeed, as in the proof of Lemma A.1, we take a simultaneous best response sequence $(\phi_-^t)_{t=0}^\infty$, modify it to obtain a generalized (not necessarily simultaneous) best response sequence $(\psi_-^t)_{t=0}^\infty$, and then define $(\sigma_-^t)_{t=0}^\infty$ by $\sigma_-^t = \psi_-^{t+T}$ for sufficiently large T. The only difference lies here, where it is not the case in general that "at most T players change actions by period T". Instead, we first assume without loss of generality that action 0 (as well as action n) is a Nash equilibrium of u, and resort to the finiteness of $\Gamma(x)$ to show that in each step of $(\psi_-^t)_{t=0}^T$, only finitely many players have minimum best responses other than action 0. ▮

Another definition is to only require *some* sequential best response sequence to converge.

Definition A.3. Action s^* is *weakly contagious in network* $(\mathcal{X}, P)$ if there exists a finite subset Y of $\mathcal{X}$ such that for any initial action configuration σ^0 such that $\sigma^0(x) = s^*$ for any $x \in Y$, there exists a sequential best response sequence (σ^t) such that $\lim_{t\to\infty} \sigma^t(x) = s^*$ for any $x \in \mathcal{X}$.

By definition, contagion implies weak contagion. The converse does not always hold. A counterexample is given by the trivial payoff function $u \equiv 0$, where all actions are weakly contagious but none of them is contagious. Nevertheless, we can show that weak contagion is equivalent to contagion for generic supermodular games.

We say that a game u is *generic for* $(\mathcal{X}, P)$ if no player has multiple best responses to any action configuration on $(\mathcal{X}, P)$. If each player has finitely many neighbors, then genericity excludes at most countably many hyperplanes in the payoff parameter space.

Proposition A.3. *Fix a network* $(\mathcal{X}, P)$ *and a generic supermodular game* u *for* $(\mathcal{X}, P)$*. Then* s^* *is weakly contagious in* $(\mathcal{X}, P)$ *if and only if it is contagious by generalized best responses in* $(\mathcal{X}, P)$*.*

Proof. Once again, the proof is almost the same as the proofs of Lemma A.1 and Proposition A.1. We only need to make the following two changes.

First, in the first paragraph of the proof of Lemma A.1, given a finite set $Y \subset \mathcal{X}$ as in Definition A.3, let $(\phi_-^t)_{t=0}^{\infty}$ be a sequential best response sequence such that $\phi_-^0(x) = s^*$ for all $x \in Y$, $\phi_-^0(x) = 0$ for all $x \in \mathcal{X} \setminus Y$, and $\lim_{t \to \infty} \phi_-^t(x) = s^*$ for all $x \in \mathcal{X}$. Here it follows from the genericity of u that we have $\phi_-^t(x) \in \{\phi_-^{t-1}(x), BR(\phi_-^{t-1}|x)\}$ for any $x \in \mathcal{X}$ and $t \geq 1$, where with an abuse of notation, $BR(\phi_-^{t-1}|x)$ denotes the unique best response.

Second, a weakly contagious action is always a Nash equilibrium, but may not be a *strict* Nash equilibrium in all games. Here again, the genericity assumption guarantees that the weakly contagious action s^* is a strict Nash equilibrium. ∎

A.2 Proof of Lemma 2

Recall

$$e^* = \frac{(a-d)(d-b)}{2(c-b)},$$
$$e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)},$$
$$e^\sharp = \frac{(d-b)\{2(a-c)-(d-b)\}}{2(a-c)},$$

where $e^* \lesseqgtr e^{**} \lesseqgtr e^\sharp$ if $c - b \lesseqgtr a - c$.

Proof of Lemma 2. (1) We first note that for all $p \in (0, 1/2)$, $u(2, \pi^b) > u(0, \pi^b)$ and hence $0 \notin br(\pi^b)$.

We divide the argument into two cases: (α) $e > (a-c)/2$ and (β) $e \leq (a-c)/2$.

(α) $e > (a-c)/2$: In this case, if we let $p = 0$ (hence $\pi^a = \pi^b$), $br(\pi^a) = br(\pi^b) = \{2\}$, and thus condition (4.2) holds for some $p \in (0, 1/2)$ close to 0 due to the upper semi-continuity of br.

(β) $e \le (a-c)/2$: In this case, for all $p \in (0, 1/2)$, $u(1, \pi^a) > u(2, \pi^a)$ and hence $2 \notin br(\pi^a)$. Therefore, $\max br(\pi^a) = 0 \Leftrightarrow u(0, \pi^a) > u(1, \pi^a)$ and $\max br(\pi^b) \le 1 \Leftrightarrow u(1, \pi^b) > u(2, \pi^b)$, while $\min br(\pi^a) \ge 1 \Leftrightarrow u(1, \pi^a) > u(0, \pi^a)$ and $\min br(\pi^b) = 2 \Leftrightarrow u(2, \pi^b) > u(1, \pi^b)$.

Verify that $e^* \le (a-c)/2$ with the equality holding if and only if $c = d$. Consider first the case where $e^* < (a-c)/2$ (or $c < d$). Then, since

$$
\begin{aligned}
u(0, \pi^a) - u(1, \pi^a) &= (d-b)\left\{p - \frac{(d-b) - 2e}{2(d-b)}\right\}, \\
u(1, \pi^b) - u(2, \pi^b) &= (d-c)\left\{\frac{(a-c) - 2e}{2(d-c)} - p\right\},
\end{aligned}
$$

it follows that condition (4.1) holds for some $p \in (0, 1/2)$ if and only if

$$
\frac{(d-b) - 2e}{2(d-b)} < \frac{(a-c) - 2e}{2(d-c)} \iff e < e^*,
$$

while condition (4.2) holds for some $p \in (0, 1/2)$ if and only if

$$
\frac{(a-c) - 2e}{2(d-c)} < \frac{(d-b) - 2e}{2(d-b)} \iff e > e^*.
$$

If $e^* = (a-c)/2$ (or $c = d$), then $u(0, \pi^a) > u(1, \pi^a)$ and $u(1, \pi^b) > u(2, \pi^b)$ for some $p \in (0, 1/2)$ close to 1/2 whenever $e < e^*$. (The condition $e > e^*$ never holds in the current case of $e \le (a-c)/2$ $(= e^*)$.)

(2) We first note that $u(2, \pi^d) > u(0, \pi^d)$ and hence $0 \notin br(\pi^d)$ for all $r \in (0, 1)$. Therefore,

$$
\begin{aligned}
\max br(\pi^d) \le 1 &\iff u(1, \pi^d) > u(2, \pi^d) \\
&\iff r < \frac{(a-c) - 2e}{a-c}. \qquad \text{(A.2)}
\end{aligned}
$$

For the last inequality to hold, it is necessary that $e < (a-c)/2$.

Under the condition that $e < (a-c)/2$, note that $u(1, \pi^c) > u(2, \pi^c)$ and hence $2 \notin br(\pi^c)$ for all $q \in (0, 1)$. Therefore,

$$
\begin{aligned}
\max br(\pi^c) = 0 &\iff u(0, \pi^c) > u(1, \pi^c) \\
&\iff q > \frac{(d-b) - 2e}{d-b}. \qquad \text{(A.3)}
\end{aligned}
$$

Finally,

$$
\max br(\pi^e) = 0
$$

$$\iff u(0,\pi^e) > u(1,\pi^e) \text{ and } u(0,\pi^e) > u(2,\pi^e)$$

$$\iff r > \frac{(d-b)-2e}{d-b}q \text{ and } r > \frac{(d-b)-(a-d)}{a-b}q. \tag{A.4}$$

From (A.2)–(A.4), it follows that condition (4.3) holds for some $0 < r \leq q < 1$ if and only if

$$\begin{cases} \dfrac{(a-c)-2e}{a-c} > \left\{\dfrac{(d-b)-2e}{d-b}\right\}^2 \\ \dfrac{(a-c)-2e}{a-c} > \dfrac{(d-b)-(a-d)}{a-b} \cdot \dfrac{(d-b)-2e}{d-b}, \end{cases}$$

which reduces to $e < \min\{e^\sharp, e^{**}\}$. ▮

A.3 Proof of Lemma 3

Given a payoff function $f\colon S \times S \to \mathbb{R}$, we write BR_f for the best correspondence in the local interaction game $(\mathcal{X}, P, f)$:

$$BR_f(\sigma|x) = \bigl\{s \in S \bigm| \textstyle\sum_{y\in\Gamma(x)} P(y|x) f(s,\sigma(y)) \geq \sum_{y\in\Gamma(x)} P(y|x) f(s',\sigma(y)) \text{ for all } s' \in S\bigr\}.$$

(Thus the best response correspondence for u as defined in (2.2) is now denoted BR_u.) Recall that $BR_f(\sigma|x) = br_f(\pi(\sigma|x))$. We show a result stronger than Lemma 3, that a strict MP-maximizer is uninvadable with respect to sequences that are required only to satisfy properties (i) and (ii) in Definition 1. Such sequences do not necessarily satisfy property (iii) in Definition 1, so that some players may have no opportunity to revise their suboptimal actions. We call those sequences partial best response sequences.

Definition A.4. Given a local interaction system $(\mathcal{X}, P)$ and for a payoff function $f\colon S \times S \to \mathbb{R}$, a sequence of action configurations $(\sigma^t)_{t=0}^\infty$ is a *partial best response sequence* in f if it satisfies the following properties: (i) for all $t \geq 1$, there is at most one $x \in \mathcal{X}$ such that $\sigma^t(x) \neq \sigma^{t-1}(x)$; and (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR_f(\sigma^{t-1}|x)$.

Let s^* be a strict MP-maximizer of u with a strict MP-function v. Recall that v is a symmetric function (i.e., $v(h,k) = v(k,h)$). The game defined by a symmetric function v is called a *potential game*, and given that $\{(s^*,s^*)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, $s^* \in S$ is called a *potential maximizer* of v. The following result is due to Morris (1999, Proposition 6.1). We provide its proof for completeness.

Lemma A.2. *Suppose that s^* is a potential maximizer of a potential game v. For any unbounded local interaction system $(\mathcal{X}, P)$ and for any partial best response sequence $(\sigma^t)_{t=0}^\infty$ in v with $\sigma_P^0(S \setminus \{s^*\}) < \infty$, there exists $M < \infty$ such that $\sigma_P^t(S \setminus \{s^*\}) \leq M$ for all $t \geq 0$.*

Proof. Let s^* be a potential maximizer of a potential game v. Let $\bar{\gamma} = \max_{h,k}\big(v(s^*, s^*) - v(h, k)\big) < \infty$ and $\underline{\gamma} = \min_{(h,k)\neq(s^*,s^*)}\big(v(s^*, s^*) - v(h, k)\big) > 0$. Fix any local interaction system $(\mathcal{X}, P)$. Let $(\sigma^t)_{t=0}^{\infty}$ be any partial best response sequence in v such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$.

Let

$$V(t) = \frac{1}{2} \sum_{(x,y)\in\mathcal{X}\times\mathcal{X}} P(x, y)\big(v(\sigma^t(x), \sigma^t(y)) - v(s^*, s^*)\big).$$

Note that

$$-\bar{\gamma}\sigma_P^t(S \setminus \{s^*\}) \leq V(t) \leq -\underline{\gamma}\sigma_P^t(S \setminus \{s^*\}).$$

Since $\sigma_P^0(S \setminus \{s^*\}) < \infty$, we have $V(0) > -\infty$.

Consider any time $t \geq 1$ such that some player changes his action (there is at most one such player by property (i)), and call that player x^t. Then we have

$$\begin{aligned} &V(t) - V(t-1) \\ &\qquad = \sum_{y\in\Gamma(x^t)} P(x^t, y)\big(v(\sigma^t(x^t), \sigma^{t-1}(y)) - v(\sigma^{t-1}(x^t), \sigma^{t-1}(y))\big) \geq 0 \end{aligned}$$

by property (ii). It follows from the induction on t that V is nondecreasing, so that $V(t) \geq V(0)$ for all t.

Therefore, we have $\sigma_P^t(S \setminus \{s^*\}) \leq -V(t)/\underline{\gamma} \leq -V(0)/\underline{\gamma}$ for all t. ∎

Lemma 3 is a direct corollary of the following.

Lemma A.3. *Suppose that s^* is a strict MP-maximizer of u with a strict MP-function v. If u or v is supermodular, then for any unbounded local interaction system $(\mathcal{X}, P)$ and for any partial best response sequence $(\sigma^t)_{t=0}^{\infty}$ in u with $\sigma_P^0(S\setminus\{s^*\}) < \infty$, there exists $M < \infty$ such that $\sigma_P^t(S\setminus\{s^*\}) \leq M$ for all $t \geq 0$.*

Proof. Let $s^* = \underline{s}, \bar{s}$ be a strict MP-maximizer of u with a strict MP-function v. We only consider the case where $s^* = \underline{s}$. Fix any local interaction system $(\mathcal{X}, P)$. Let $(\sigma^t)_{t=0}^{\infty}$ be any partial best response sequence in u such that $\sigma_P^0(S \setminus \{\underline{s}\}) < \infty$.

Let $(\hat{\sigma}^t)_{t=0}^{\infty}$ be defined by $\hat{\sigma}^0 = \sigma^0$ and for $t \geq 1$,

$$\hat{\sigma}^t(x) = \begin{cases} \max BR_v(\hat{\sigma}^{t-1}|x) & \text{if } \sigma^t(x) \neq \sigma^{t-1}(x), \\ \hat{\sigma}^{t-1}(x) & \text{otherwise.} \end{cases}$$

Clearly, $(\hat{\sigma}^t)_{t=0}^{\infty}$ is a partial best response sequence in v. Therefore, by Lemma A.2, there exists M such that $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show that if u or v is supermodular, then

$$\sigma^t \leq \hat{\sigma}^t \qquad (\star_t)$$

for all $t \geq 0$. Then, $\sigma_P^t(S\backslash\{\underline{s}\}) \leq \hat{\sigma}_P^t(S\backslash\{\underline{s}\})$ for all t, and since $\hat{\sigma}_P^t(S\backslash\{\underline{s}\}) \leq M$ for all t, it follows that $\sigma_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show by induction that $(\star_t)$ holds for all $t \geq 0$. First, $(\star_0)$ trivially holds by the definition of $\hat{\sigma}^0$. Then, assume $(\star_{t-1})$. It implies that for all $x \in \mathcal{X}$, $\pi(\sigma^{t-1}|x) \precsim \pi(\hat{\sigma}^{t-1}|x)$. Let $x \in \mathcal{X}$ be such that $\sigma^t(x) \neq \sigma^{t-1}(x)$, and hence $\hat{\sigma}^t(x^t) = \max br_v(\pi(\hat{\sigma}^{t-1}|x^t))$ by construction. If u is supermodular, then

$$\begin{aligned}\sigma^t(x^t) &\leq \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_u(\pi(\hat{\sigma}^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t),\end{aligned}$$

where the second inequality follows from the supermodularity of u, and the third inequality follows from the MP condition (4.5). If v is supermodular, then

$$\begin{aligned}\sigma^t(x^t) &\leq \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t),\end{aligned}$$

where the second inequality follows from the MP condition (4.5), and the third inequality follows from the supermodularity of v. Therefore, in each case, $(\star_t)$ holds. ▌

We show in passing that Lemma 3 extends to generalized best response sequences (Definition A.1) in any network where each player has finitely many neighbors.

Definition A.5. Action s^* is *uninvadable by generalized best response sequences in* $(\mathcal{X}, P)$ if there exists no generalized best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$ and $\lim_{t\to\infty} \sigma_P^t(S \setminus \{s^*\}) = \infty$.

Proposition A.4. *Let u be any game. If $s^* = \underline{s}, \bar{s}$ is a strict MP-maximizer of u with a strict MP-function v and if u or v is supermodular, then s^* is uninvadable by generalized best responses in $(\mathcal{X}, P)$ such that $\Gamma(x)$ is finite for all $x \in \mathcal{X}$.*

Proof. Let $s^* = \underline{s}, \bar{s}$ be a strict MP-maximizer of u with a strict MP-function v. We only consider the case where $s^* = \underline{s}$. Fix any local interaction system $(\mathcal{X}, P)$ such that $\Gamma(x)$ is finite for all $x \in \mathcal{X}$. Let $(\sigma^t)_{t=0}^{\infty}$ be any generalized best response sequence such that $\sigma_P^0(S \setminus \{\underline{s}\}) < \infty$. We will construct a nondecreasing partial best response sequence $(\hat{\sigma}^\tau)_{\tau=0}^{\infty}$ in the potential game v such that

$$\sigma^t \leq \bar{\sigma} \qquad (\star\star_t)$$

for all $t \geq 0$, where $\bar{\sigma}$ is defined by $\bar{\sigma}(x) = \lim_{\tau\to\infty} \hat{\sigma}^\tau(x)$ for all $x \in \mathcal{X}$. Then, we have, for all $t \geq 0$,

$$\sigma_P^t(S \setminus \{\underline{s}\}) \leq \bar{\sigma}_P(S \setminus \{\underline{s}\}) = \lim_{\tau\to\infty} \hat{\sigma}_P^\tau(S \setminus \{\underline{s}\}) < \infty$$

as desired, where the last inequality (the finiteness of $\lim_{\tau\to\infty} \hat{\sigma}_P^\tau(S \setminus \{\underline{s}\})$) follows from Lemma A.2.

We construct such a sequence $(\hat{\sigma}^\tau)_{\tau=0}^\infty$ as follows. First, pick a sequence $(x^\tau)_{\tau=1}^\infty$ in $\mathcal{X}$ such that $\{\tau \geq 1 \mid x^\tau = x\}$ is infinite for each $x \in \mathcal{X}$.[28] Then, let $\hat{\sigma}^0 = \sigma^0$, and for each $\tau \geq 1$, let $\hat{\sigma}^\tau(x^\tau) = \max\{\max BR_v(\hat{\sigma}^{\tau-1}|x^\tau), \hat{\sigma}^{\tau-1}(x^\tau)\}$ and $\hat{\sigma}^\tau(x) = \hat{\sigma}^{\tau-1}(x)$ for $x \neq x^\tau$. By construction, $(\hat{\sigma}^\tau)_{\tau=0}^\infty$ is a partial best response sequence in v, and for each $x \in \mathcal{X}$, $(\hat{\sigma}^\tau(x))_{t=0}^\infty$ is nondecreasing. Denote $\bar{\sigma}(x) = \lim_{\tau\to\infty} \hat{\sigma}^\tau(x)$. Note that $\bar{\sigma} \geq \hat{\sigma}^\tau$ for all $\tau \geq 0$.

Claim 1. $\max BR_v(\bar{\sigma}|x) \leq \bar{\sigma}(x)$ *for all* $x \in \mathcal{X}$.

Proof. Fix any $x \in \mathcal{X}$. By the finiteness of $\Gamma(x)$, there exists T such that $\hat{\sigma}^\tau(y) = \bar{\sigma}(y)$ for all $y \in \Gamma(x)$ and all $\tau \geq T$. By the construction of $(\hat{\sigma}^\tau)_{\tau=0}^\infty$, there exists $\tau' > T$ such that $x^{\tau'} = x$, and with such a τ' we have $\max BR_v(\bar{\sigma}|x) = \max BR_v(\hat{\sigma}^{\tau'-1}|x) \leq \hat{\sigma}^{\tau'}(x) \leq \bar{\sigma}(x)$. ▮

Now we show by induction that $(\star\star_t)$ holds for all $t \geq 0$. First, $(\star\star_0)$ holds by the construction of $(\hat{\sigma}^\tau)_{\tau=0}^\infty$. Next, assume $(\star\star_{t-1})$. It implies that for all $x \in \mathcal{X}$, $\pi(\sigma^{t-1}|x) \precsim \pi(\bar{\sigma}|x)$. Let $x \in \mathcal{X}$ be such that $\sigma^t(x) \neq \sigma^{t-1}(x)$, and hence $\sigma^t(x) \in br_u(\pi(\sigma^{t-1}|x))$. If u is supermodular, then $\sigma^t(x) \leq \max br_u(\pi(\sigma^{t-1}|x)) \leq \max br_u(\pi(\bar{\sigma}|x)) \leq \max br_v(\pi(\bar{\sigma}|x)) \leq \bar{\sigma}(x)$, where the second inequality follows from the supermodularity of u, the third from the MP condition (4.5), and the fourth from Claim 1. If v is supermodular, then $\sigma^t(x) \leq \max br_u(\pi(\sigma^{t-1}|x)) \leq \max br_v(\pi(\sigma^{t-1}|x)) \leq \max br_v(\pi(\bar{\sigma}|x)) \leq \bar{\sigma}(x)$, where the second inequality follows from the MP condition (4.5), the third from the supermodularity of v, and the fourth from Claim 1. Therefore, in each case, $(\star\star_t)$ holds. ▮

A.4 Proof of Lemma 4

For $f\colon S \times S \to \mathbb{R}$ and $h \in S$, let

$$\Pi_h(f) = \{\pi \in \Delta(S) \mid h \in br_f(\pi)\}.$$

Note that 0 is a strict MP-maximizer of u with MP-function v if and only if $\{(0,0)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, and

$$\Pi_2(u) \subset \Pi_2(v) \text{ and } \Pi_1(u) \subset \Pi_1(v) \cup \Pi_2(v),$$

[28] For example, enumerate $\mathcal{X} = \{x_1, x_2, x_3, \ldots\}$, and for each $\tau \geq 1$, let $\ell(\tau)$ be the largest integer ℓ such that $\ell(\ell+1)/2 < \tau$, and let $k(\tau) = \tau - \ell(\tau)(\ell(\tau)+1)/2$ and $x^\tau = x_{k(\tau)}$.

while 2 is a strict MP-maximizer of u with MP-function v if and only if $\{(2,2)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, and

$$\Pi_0(u) \subset \Pi_0(v) \text{ and } \Pi_1(u) \subset \Pi_0(v) \cup \Pi_1(v).$$

Recall

$$e^* = \frac{(a-d)(d-b)}{2(c-b)},$$
$$e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)},$$

and denote

$$e^\flat = \frac{(a-d)(d-b)}{a-b}.$$

Verify that $e^\flat \lesseqgtr e^* \lesseqgtr e^{**}$ if $c-b \lesseqgtr a-c$.

Lemma 4 is proved by Lemmas A.4–A.7 which follow. Lemma A.4 considers the case in which $c = d$ and $e < e^*$ $(= (a-c)/2)$; Lemma A.5 considers the cases of $e < e^*$ and $e^* < e \leq e^\flat$ under the assumption that $c \neq d$; and Lemmas A.6 and A.7 cover the cases of $\max\{e^{**}, e^\flat\} < e \leq (a-c)/2$ and $e > (a-c)/2$, respectively; see Figure A.1.

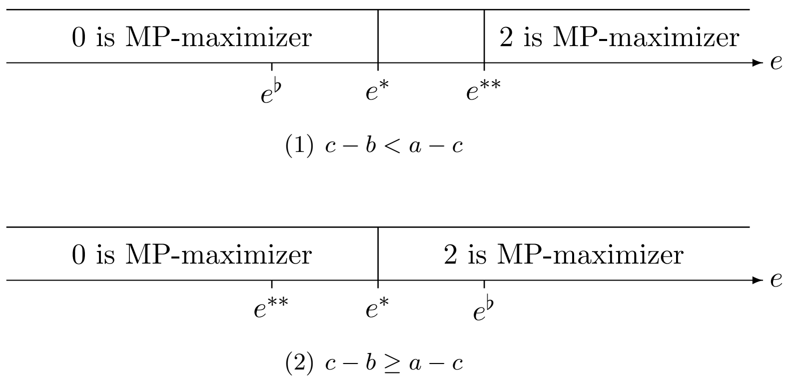


Figure A.1: MP-maximizer

Lemma A.4. *Suppose that $c = d$. If $e < e^*$ $(= (a-c)/2 = (a-d)/2)$, then 0 is a strict MP-maximizer.*

Proof. Observe first that, since $e < e^* = (a-c)/2 = (a-d)/2 < (d-b)/2$,

$$u(0,k) - u(1,k) < u(1,k) - u(2,k) \tag{A.5}$$

for all $k = 0, 1, 2$. Let v be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} e & 0 & -\lambda e-(d-b)+e \\ 0 & -e & -\lambda e \\ -\lambda e-(d-b)+e & -\lambda e & 0 \end{pmatrix} \end{array}, \tag{A.6}$$

where

$$\lambda = \frac{(d-b)-e}{(a-d)-2e} > 0.$$

This function is maximized at $(0, 0)$. Verify that

$$v(0,k) - v(1,k) = u(0,k) - u(1,k) \tag{A.7}$$

$$v(1,k) - v(2,k) \leq \lambda\big(u(1,k) - u(2,k)\big) \tag{A.8}$$

for all $k = 0, 1, 2$. Then, we have $\Pi_1(u) \subset \Pi_1(v) \cup \Pi_2(v)$ by (A.7), and $\Pi_2(u) \subset \Pi_2(v)$ by (A.5), (A.7), and (A.8). ∎

Lemma A.5. *Suppose that $c \neq d$.* (i) *If $e < e^*$, then 0 is a strict MP-maximizer.* (ii) *If $e^* < e \leq e^\flat$, then 2 is a strict MP-maximizer.*

Proof. Let v be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 2\lambda e & \lambda e & \lambda e-(a-c)+e \\ \lambda e & 0 & -(a-d)+e \\ \lambda e-(a-c)+e & -(a-d)+e & -(a-d)+2e \end{pmatrix} \end{array}, \tag{A.9}$$

where

$$\lambda = \frac{d-c}{d-b} > 0.$$

We show that this function v works as a strict MP-function if $e \leq \max\{e^*, e^\flat\}$ and $e \neq e^*$.

We first have the following.

Claim 2. (i) $\{(0,0)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$ *if $e < e^*$.* (ii) $\{(2,2)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$ *if $e > e^*$.*

Verify that

$$v(0,k) - v(1,k) = \lambda\big(u(0,k) - u(1,k)\big) \tag{A.10}$$

$$v(1,k) - v(2,k) = u(1,k) - u(2,k) \tag{A.11}$$

for all $k = 0, 1, 2$. These immediately imply the following.

Claim 3. $\Pi_1(u) = \Pi_1(v)$.

For $\pi = (\pi_0, \pi_1, \pi_2) \in \Delta(S)$, we have

$$u(0, \pi) - u(1, \pi) = (d - b)\left(\frac{e}{d - b} - \pi_2\right), \tag{A.12}$$

$$\begin{aligned} u(1, \pi) - u(2, \pi) = (d - c)&\left\{\pi_0 - \frac{(a - b)e - (a - d)(d - b)}{(d - b)(d - c)}\right\} \\ &+ (a - d)\left(\frac{e}{d - b} - \pi_2\right), \end{aligned} \tag{A.13}$$

and

$$v(2, \pi) - v(0, \pi) = \big(u(2, \pi) - u(0, \pi)\big) + (c - b)\left(\frac{e}{d - b} - \pi_2\right). \tag{A.14}$$

These imply the following.

Claim 4. $\Pi_2(u) \subset \Pi_2(v)$.

Proof. Assume that $\pi = (\pi_0, \pi_1, \pi_2) \in \Pi_2(u)$ ($\Leftrightarrow u(2, \pi) \geq u(0, \pi)$ and $u(2, \pi) \geq u(1, \pi)$). First, by (A.11), $u(2, \pi) \geq u(1, \pi)$ implies $v(2, \pi) \geq v(1, \pi)$. Second, if $\pi_2 \geq e/(d - b)$, then by (A.10) and (A.12), we have $v(1, \pi) \geq v(0, \pi)$ and therefore $v(2, \pi) \geq v(0, \pi)$, while if $\pi_2 < e/(d - b)$, then by (A.14), $u(2, \pi) \geq u(0, \pi)$ implies $v(2, \pi) > v(0, \pi)$. We thus have $\pi \in \Pi_2(v)$. ∎

Claim 5. *If $e \leq e^\flat$, then $br_u = br_v$.*

Proof. Suppose that $e \leq e^\flat$. In light of Claim 3, we want to show that $\Pi_0(u) = \Pi_0(v)$ and $\Pi_2(u) = \Pi_2(v)$.

Note in (A.13) that $e \leq e^\flat$ implies $\{(a - b)e - (a - d)(d - b)\}/\{(d - b)(d - c)\} \leq 0$. By (A.12) and (A.13), we therefore have $u(0, \pi) \geq u(1, \pi) \Rightarrow u(1, \pi) \geq u(2, \pi)$ and $u(2, \pi) \geq u(1, \pi) \Rightarrow u(1, \pi) \geq u(0, \pi)$. By (A.10) and (A.11), it thus follows that $\pi \in \Pi_0(u) \Leftrightarrow u(0, \pi) \geq u(1, \pi) \Leftrightarrow v(0, \pi) \geq v(1, \pi) \Leftrightarrow \pi \in \Pi_0(v)$ and $\pi \in \Pi_2(u) \Leftrightarrow u(2, \pi) \geq u(1, \pi) \Leftrightarrow v(2, \pi) \geq v(1, \pi) \Leftrightarrow \pi \in \Pi_2(v)$. ∎

We now complete the proof of Lemma A.5. (i) If $e < e^*$, Claims 2, 3, and 4 imply that 0 is a strict MP-maximizer. (ii) If $e^* < e \leq e^\flat$, Claims 2 and 5 imply that 2 is a strict MP-maximizer. ∎

Lemma A.6. *If $\max\{e^{**}, e^\flat\} < e \leq (a - c)/2$, then 2 is a strict MP-maximizer.*

Proof. Suppose that $\max\{e^{**}, e^{\flat}\} < e \leq (a-c)/2$. Let v be defined by

$$\begin{array}{c} \\ 0 \\ \\ 1 \\ \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\
\begin{pmatrix}
0 & -\lambda e & -\lambda e - \{(a-c)-e\} \\
-\lambda e & -2\lambda e & \lambda\{(d-b)-2e\} - \{(a-c)-e\} \\
-\lambda e - \{(a-c)-e\} & \lambda\{(d-b)-2e\} - \{(a-c)-e\} & \lambda\{(d-b)-2e\} - \{(a-c)-2e\}
\end{pmatrix}
\end{array}, \tag{A.15}$$

where

$$\lambda = \frac{(a-c)(d-b)-(a-b)e}{(a-b)\{(d-b)-e\}} > 0$$

($\lambda > 0$ follows from $e \leq (a-c)/2$). We show that this function v works as a strict MP-function.

First, the function (A.15) is maximized at $(2,2)$ (by $e > e^{**}$). Second, one can verify, for all $k = 0, 1, 2$,

$$v(0,k) - v(1,k) = \lambda\big(u(0,k) - u(1,k)\big) \tag{A.16}$$

$$v(0,k) - v(2,k) \geq \frac{a-c}{a-b}\big(u(0,k) - u(2,k)\big) \tag{A.17}$$

(by $e \leq (a-c)/2$), and

$$v(1,k) - v(2,k) \geq u(1,k) - u(2,k) \tag{A.18}$$

(since $\lambda < (d-c)/(d-b)$ by $e > e^{\flat}$). Therefore, $\pi \in \Pi_0(u) \Rightarrow \pi \in \Pi_0(v)$ by (A.16)–(A.17) and $\pi \in \Pi_1(u) \Rightarrow \pi \in \Pi_0(v) \cup \Pi_1(v)$ by (A.18). ▮

Lemma A.7. *If $e > (a-c)/2$, then 2 is a strict MP-maximizer.*

Proof. Action 2 is strictly p-dominant with

$$p = \max\left\{\frac{a-c-e}{a-c}, \frac{a-c}{(a-c)+(d-b)}\right\},$$

i.e., $\{2\} = br_u(\pi)$ for any $\pi = (\pi_0, \pi_1, \pi_2) \in \Delta(S)$ such that $\pi_2 > p$ (Morris et al. (1995), Kajii and Morris (1997)). If $e > (a-c)/2$, we have $p < 1/2$. Therefore, the function

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\begin{pmatrix}
0 & 0 & -p \\
0 & 0 & -p \\
-p & -p & 1-2p
\end{pmatrix}
\end{array} \tag{A.19}$$

is a strict MP-function for 2 (see Morris and Ui (2005) or Oyama et al. (2008, Lemma 4.1)). ▮

A.5 Proof of Theorem 2

Since the network used in the proof of Lemma 1(1) is linear, combined with Lemma 2(1) it follows that if $e < e^*$ (resp. $e > e^*$), then action 0 (resp. 2) is contagious in linear networks. Also, by Theorem 1(i), if $e < e^*$, then 0 is uninvadable, hence uninvadable in linear networks. Thus, we only need to show that 2 is uninvadable in linear networks if $e > e^*$.

By Lemma 2, there exists $p \in (0, 1/2)$ that satisfies (4.2). By the upper semi-continuity of br, there exists $\varepsilon \in (0, 1/2 - p)$ such that $\min br(\tilde{\pi}^a) \geq 1$ and $\min br(\tilde{\pi}^b) = 2$, where

$$\tilde{\pi}^a = \left(\tfrac{1}{2} + \varepsilon, p, \tfrac{1}{2} - p - \varepsilon\right), \quad \tilde{\pi}^b = \left(\tfrac{1}{2} - p + \varepsilon, p, \tfrac{1}{2} - \varepsilon\right).$$

Fix any linear network $(\mathbb{Z}, P)$. Since $P(0|0) = 0$ and $P(y|0) = P(-y|0)$ for all $y > 0$, we have $\sum_{y=1}^{\infty} P(y|0) = 1/2$. Let n_1 be the smallest integer such that $\sum_{y=1}^{n_1} P(y|0) \geq p$, and n_2 be a sufficiently large integer such that $\sum_{y>n_2} P(y|0) \leq \varepsilon$.

Consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(\{0,1\}) < \infty$. Let K be the set of all $k \in \mathbb{Z}$ such that $\sigma^0(x) = 2$ if $|x - k| \leq n_1 + n_2$. Then K is co-finite (i.e., $\mathbb{Z} \setminus K$ is finite), and so is $L = \bigcup_{k \in K} \{x \in \mathbb{Z} \mid |x - k| \leq n_2\}$. (Otherwise, $\sigma^0(x) \neq 2$ for infinitely many x, which contradicts the finiteness of $\sigma_P^0(\{0,1\})$.)

For each $k \in K$, we want to show that

$$\begin{aligned} &\sigma^t(x) = 2 \text{ if } |x - k| \leq n_2, \\ &\sigma^t(x) \geq 1 \text{ if } n_2 + 1 \leq |x - k| \leq n_1 + n_2 \end{aligned}$$

for all $t \geq 0$. First, this holds for $t = 0$ by construction. Next, assume that it holds for $t - 1$. Then,

- for players x such that $|x - k| \leq n_2$,

$$\pi(\sigma^{t-1}|x)(2) \geq \sum_{y=1}^{n_2} P(y|0) \geq \frac{1}{2} - \varepsilon,$$

$$\pi(\sigma^{t-1}|x)(1) + \pi(\sigma^{t-1}|x)(2) \geq \sum_{y=1}^{n_1} P(y|0) + \sum_{y=1}^{n_2} P(y|0) \geq \frac{1}{2} + p - \varepsilon,$$

  which implies that $\pi(\sigma^{t-1}|x) \succsim \tilde{\pi}^b$ and hence $\sigma^t(x) = 2$;

- for players x such that $n_2 + 1 \leq |x - k| \leq n_1 + n_2$,

$$\pi(\sigma^{t-1}|x)(2) \geq \sum_{y=1}^{n_2} P(y|0) - \sum_{y=1}^{n_1 - 1} P(y|0) > \frac{1}{2} - p - \varepsilon,$$

$$\pi(\sigma^{t-1}|x)(1) + \pi(\sigma^{t-1}|x)(2) \geq \sum_{y=1}^{n_1} P(y|0) \geq \frac{1}{2} - \varepsilon,$$

which implies that $\pi(\sigma^{t-1}|x) \succsim \tilde{\pi}^a$ and hence $\sigma^t(x) \geq 1$.

Therefore, $\{x \in \mathbb{Z} \mid \sigma^t(x) = 2\} \supset L$, and hence $\sigma^t_P(\{0,1\})$ is bounded from above. ▮

A.6 Proof of Theorem 4

We fix the dimension m. A sequence $(P_n)_{n=0}^{\infty}$ of interaction weights on the m-dimensional lattice $\mathbb{Z}^m$ is *well-behaved* if the following conditions are satisfied.

- For each n, P_n is invariant up to translation, i.e., $P_n(x, y) = P_n(x+z, y+z)$ for $x, y, z \in \mathbb{Z}^m$.
- There exist a pair of nonnegative integrable functions $f, g \colon \mathbb{R}^m \to \mathbb{R}_+$ such that for almost every $\nu = (\nu_1, \dots, \nu_m) \in \mathbb{R}^m$,

  $$n^m P_n(([n\nu_1], \dots, [n\nu_m])|0) \to f(\nu)$$

  as $n \to \infty$ (pointwise convergence), and

  $$n^m P_n(([n\nu_1], \dots, [n\nu_m])|0) \leq g(\nu)$$

  for every n.[29]
- f has connected support.

For example, the sequence of n-max distance interactions is well-behaved since $n^m P_n(([n\nu_1], \dots, [n\nu_m])|0)$ converges to 2^{-m} times the indicator function of $\{\nu \in \mathbb{R}^m \mid \max_i \nu_i \leq 1\}$.

The next result characterizes contagious and uninvadable actions in any well-behaved sequence of multidimensional lattice networks. Theorem 4 follows as an immediate corollary.

Theorem A.1. *Let u be the bilingual game given by* (3.1). *Fix the dimension m and a well-behaved sequence $(P_n)_{n=0}^{\infty}$ of interaction weights on $\mathbb{Z}^m$.*

(i) *If $e < e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$, 0 is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.* (ii) *If $e > e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$, 2 is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.*

[29] For $\eta \in \mathbb{R}$, $[\eta]$ denotes the largest integer that does not exceed η.

Proof. We will show (i) only. The proof for (ii) is analogous.

By Lemma 2, there exists $p \in (0, 1/2)$ that satisfies (4.1). By the upper semi-continuity of br, there exists $\varepsilon \in (0, 1/2 - p)$ such that $\max br(\hat{\pi}^a) = 0$ and $\max br(\hat{\pi}^b) \le 1$, where

$$\hat{\pi}^a = \left(\tfrac{1}{2} - \varepsilon, p, \tfrac{1}{2} - p + \varepsilon\right), \quad \hat{\pi}^b = \left(\tfrac{1}{2} - p - \varepsilon, p, \tfrac{1}{2} + \varepsilon\right).$$

Let $f(\nu)$ be the pointwise limit of $n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0)$ as $n \to \infty$. Since P_n is symmetric and translation invariant, f is symmetric, i.e., $f(\nu) = f(-\nu)$ for almost all ν. We also have $\int_{\mathbb{R}^m} f(\nu) d\nu = 1$.

By the symmetry of f and the connectedness of the support of f, for each $\lambda \in \mathbb{R}^m$ whose Euclidean norm $\|\lambda\|$ is 1, there exists a unique $\delta = \delta(\lambda) > 0$ that satisfies

$$\int_{0 \le \lambda \cdot x \le \delta} f(x) dx = p$$

and $\delta(\lambda)$ is continuous in λ, since the left hand side is continuous in λ and δ and strictly increasing in δ (whenever the left hand side is less than 1/2).

For each positive real number r, let $B_r = \{\nu \in \mathbb{R}^m \mid \|\nu\| \le r\}$ and $C_r = \{\nu \in \mathbb{R}^m \mid r < \|\nu\| \le r + \delta(\nu/\|\nu\|)\}$. Note that for large r and any boundary point ν of B_r, we have $\lambda \cdot \xi \approx r$ for any boundary point ξ of B_r near ν. By the continuity of $\delta(\cdot)$, the same is true for the boundary of C_r; i.e., for large r and any boundary point ν of B_r, we have $\lambda \cdot \xi \approx r + \delta(\nu/\|\nu\|)$ for any boundary point ξ of C_r near ν. Thus, there exists r_1 such that for any $r \ge r_1$,

$$\int_{B_r} f(\xi - \nu) d\xi \ge \frac{1}{2} - \frac{\varepsilon}{3}, \qquad \int_{B_r \cup C_r} f(\xi - \nu) d\xi \ge \frac{1}{2} + p - \frac{\varepsilon}{3}$$

if $\nu \in B_r$, and

$$\int_{B_r} f(\xi - \nu) d\xi \ge \frac{1}{2} - p - \frac{\varepsilon}{3}, \qquad \int_{B_r \cup C_r} f(\xi - \nu) d\xi \ge \frac{1}{2} - \frac{\varepsilon}{3}$$

if $\nu \in C_r$.

For each integer k, let $\hat{B}_k = \{x \in \mathbb{Z}^m \mid \|x\| \le k\}$ and $\hat{C}_{k,n} = \{x \in \mathbb{Z}^m \mid k < \|x\| \le k + n\delta(x/\|x\|)\}$. Since $(P_n)_{n=0}^\infty$ is well-behaved, one can apply the dominated convergence theorem to show that there exists n_1 such that for any $n \ge n_1$,

$$\left| \sum_{y \in \hat{B}_k} P_n(y - x|0) - \int_{B_{k/n}} f(\xi - x/n) d\xi \right| \le \frac{\varepsilon}{3},$$

$$\left| \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y - x|0) - \int_{B_{k/n} \cup C_{k/n}} f(\xi - x/n) d\xi \right| \le \frac{\varepsilon}{3}$$

for any $x \in \mathbb{Z}^m$ and k. Therefore, there exists $n_2 \geq n_1$ such that for any $n \geq n_2$ and any $k \geq r_1 n$,

$$\sum_{y \in \hat{B}_k} P_n(y|x) \geq \frac{1}{2} - \varepsilon, \qquad \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y|x) \geq \frac{1}{2} + p - \varepsilon$$

for any $x \in \hat{B}_{k+1}$, and

$$\sum_{y \in \hat{B}_k} P_n(y|x) \geq \frac{1}{2} - p - \varepsilon, \qquad \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y|x) \geq \frac{1}{2} - \varepsilon$$

for any $x \in \hat{C}_{k+1,n}$.

Now let $n \geq n_2$. We show that 0 is contagious in $(\mathbb{Z}^m, P_n)$. The proof is similar to that of Lemma 1(1). Pick an integer $K \geq r_1 n$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in \hat{B}_K \cup \hat{C}_{K,n}$. Then one can show by induction on k that for any $k \geq K$, there exists T_k such that for any $T \geq T_k$, we have $\sigma^t(x) = 0$ for all $x \in \hat{B}_k$ and $\sigma^0(x) \leq 1$ for all $x \in \hat{C}_{k,n}$.

This argument also shows that 0 is uninvadable in $(\mathbb{Z}^m, P_n)$ because for any initial configuration that satisfies $\sigma^0_{P_n}(\{1, 2\}) < \infty$, there exists a translation Y of $\hat{B}_K \cup \hat{C}_{K,n}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. ▮

A.7 Proof of Theorem 5

Let φ be a weight-preserving node identification from $(\mathcal{X}, P)$ to $(\hat{\mathcal{X}}, \hat{P})$ with a finite set E of exceptional nodes. Fix a supermodular game u, and assume that s^* is contagious in $(\mathcal{X}, P)$ (and hence a strict Nash equilibrium). We show that s^* is contagious in $(\hat{\mathcal{X}}, \hat{P})$.

Let $F \supset E$ be a sufficiently large finite subset of $\mathcal{X}$ such that for any $\hat{x} \in \varphi(E)$, the unique best response for player $\hat{x}$ is s^* if all players in $\varphi(F)$ play action s^*. (Since s^* is a strict Nash equilibrium, we can find such a finite set even if some player $\hat{x} \in \varphi(E)$ has infinitely many neighbors.)

Let $(\sigma^t_-)_{t=0}^{\infty}$ and $(\sigma^t_+)_{t=0}^{\infty}$ be sequential best response sequence in $(\mathcal{X}, P)$ that satisfy properties (1)–(5) in Lemma A.1. Pick a $T \geq 0$ such that $\sigma^T_-(x) = \sigma^T_+(x) = s^*$ for all $x \in F$, and let $Y = \{x \in \mathcal{X} \mid \sigma^T_-(x) \neq 0 \text{ or } \sigma^T_+(x) \neq n\}$. Note that Y is finite.

Define action configurations $\hat{\sigma}_-$ and $\hat{\sigma}_+$ in $(\hat{\mathcal{X}}, \hat{P})$ by

$$\hat{\sigma}_-(\hat{x}) = \max_{x \in \varphi^{-1}(\hat{x})} \sigma^T_-(x) \text{ and } \hat{\sigma}_+(\hat{x}) = \min_{x \in \varphi^{-1}(\hat{x})} \sigma^T_+(x)$$

for all $\hat{x} \in \hat{\mathcal{X}}$. Note that $\hat{\sigma}_-(\hat{x}) = \hat{\sigma}_+(\hat{x}) = s^*$ for all $\hat{x} \in \varphi(F)$, and $\hat{\sigma}_-(\hat{x}) = 0$ and $\hat{\sigma}_+(\hat{x}) = n$ for all $\hat{x} \in \hat{\mathcal{X}} \setminus \varphi(Y)$. Denote by $\widehat{BR}$ the set of best responses defined in $(\hat{\mathcal{X}}, \hat{P})$.

Claim 1. $\min \widehat{BR}(\hat{\sigma}_-|\hat{x}) \geq \hat{\sigma}_-(\hat{x})$ *and* $\hat{\sigma}_+(\hat{x}) \leq \max \widehat{BR}(\hat{\sigma}_+|\hat{x})$ *for all* $\hat{x} \in \hat{\mathcal{X}}$.

Proof. We only show the first inequality; the proof of the second is analogous. For any $\hat{x} \in \varphi(E)$, since $\hat{\sigma}_-(\hat{y}) = s^*$ for all $\hat{y} \in \varphi(F)$, we have $BR(\hat{\sigma}_-|\hat{x}) = \{s^*\}$ by the construction of F. Consider next any $\hat{x} \in \mathcal{X} \setminus \varphi(E)$. Write $\bar{\sigma}_-^T$ for the action configuration in $(\mathcal{X}, P)$ given by $\bar{\sigma}_-^T(y) = \hat{\sigma}_-(\hat{y})$ if $y \in \varphi^{-1}(\hat{y})$, and let $\bar{x} \in \arg\max_{x \in \varphi^{-1}(\hat{x})} \sigma_-^T(x)$. Then we have $\min \widehat{BR}(\hat{\sigma}_-|\hat{x}) = \min BR(\bar{\sigma}_-^T|\bar{x}) \geq BR(\sigma_-^T|\bar{x}) \geq \sigma_-^T(\bar{x}) = \hat{\sigma}_-(\hat{x})$, where the first equality follows from the weight preserving property of φ, the first inequality from the supermodularity, and the second inequality from property (5) in Lemma A.1. ∎

Let $\hat{Y} = \varphi(Y)$, which is finite. Pick any sequential best response sequence $(\hat{\sigma}^t)$ in $(\hat{\mathcal{X}}, \hat{P})$ such that $\hat{\sigma}^0(\hat{x}) = s^*$ for all $\hat{x} \in \hat{Y}$. We want to show that $\lim_{t \to \infty} \hat{\sigma}^t(\hat{x}) = s^*$ for all $\hat{x} \in \hat{\mathcal{X}}$.

Claim 2. $\hat{\sigma}_- \leq \hat{\sigma}^t \leq \hat{\sigma}_+$ *for all* $t \geq 0$.

Proof. We only show the first inequality; the proof of the second is analogous. First we have $\hat{\sigma}^0 \geq \hat{\sigma}_-$ by construction. Next assume $\hat{\sigma}^{t-1} \geq \hat{\sigma}_-$. If $\hat{\sigma}^t(\hat{x}) \neq \hat{\sigma}^{t-1}(\hat{x})$, then we have $\hat{\sigma}^t(\hat{x}) \geq \min \widehat{BR}(\hat{\sigma}^{t-1}|\hat{x}) \geq \min \widehat{BR}(\hat{\sigma}_-|\hat{x}) \geq \hat{\sigma}_-(\hat{x})$, where the first inequality follows from the supermodularity and the second from Claim 1. ∎

Claim 2 implies in particular that $\hat{\sigma}^t(\hat{x}) = s^*$ for all $\hat{x} \in \varphi(F)$ and all $t \geq 0$.

Given the sequence $(\hat{\sigma}^t)_{t=0}^\infty$ in $(\hat{\mathcal{X}}, \hat{P})$, let $(\tilde{\sigma}^t)_{t=0}^\infty$ be the corresponding sequence in $(\mathcal{X}, P)$ defined by

$$\tilde{\sigma}^t(x) = \hat{\sigma}^t(\varphi(x))$$

for all $x \in \mathcal{X}$ and $t \geq 0$. First, we have $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$ since by Claim 2, $\sigma_-^0(x) \leq \sigma_-^T(x) \leq \hat{\sigma}_-(\varphi(x)) \leq \hat{\sigma}^0(\varphi(x)) \leq \hat{\sigma}_+(\varphi(x)) \leq \sigma_+^T(x) \leq \sigma_+^0(x)$ for all $x \in \mathcal{X}$. Second, $(\tilde{\sigma}^t)_{t=0}^\infty$ is a generalized best response sequence in $(\mathcal{X}, P)$ as defined in Definition A.1. (Notice that players in $\varphi^{-1}(\hat{x})$ change actions simultaneously.) Indeed, for $x \in \mathcal{X} \setminus E$, we have $BR(\tilde{\sigma}^t|x) = \widehat{BR}(\hat{\sigma}^{t-1}|\varphi(x))$ for all $t \geq 0$ by the weight preserving property of φ, while for $x \in E$, by construction we have $\tilde{\sigma}^t(x) = \sigma_-^{t+T}(x) = \sigma_+^{t+T}(x) = s^*$ and $BR(\sigma_-^{t+T}|x) = BR(\tilde{\sigma}^t|x) = BR(\sigma_+^{t+T}|x) = \{s^*\}$ for all $t \geq 0$. Thus, by Lemma A.1(6), $(\sigma^t(x))_{t=0}^\infty$ converges to s^* for all $x \in \mathcal{X}$, and hence $(\hat{\sigma}^t(\hat{x}))_{t=0}^\infty$ also converges to s^* for all $\hat{x} \in \hat{\mathcal{X}}$. ∎

A.8 The Case Where Pareto-Dominance and Risk-Dominance Coincide

For completeness, we report the contagion and uninvadability result also for the case where action 0 is both Pareto-dominant and pairwise risk-dominant. The game u,

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} a & a & b \\ a-e & a-e & d-e \\ c & d & d \end{pmatrix} \end{array}, \tag{A.20a}$$

now satisfies

$$c \leq d < a, \quad d-b < a-c, \quad e > 0. \tag{A.20b}$$

Theorem A.2. *Let u be the bilingual game given by* (A.20). 0 *is always contagious and uninvadable.*

Proof. In light of Lemma 1(1-i) and Lemma 3, it suffices to show that condition (4.1) holds for some p and that 0 is a strict MP-maximizer. If $e \leq (d-b)/2$, we have $(c-b)e < (a-d)(d-b)/2$. Therefore, these follow from the argument in case (α) in the proof of Lemma 2(1) and Claims 2–4 in the proof of Lemma A.5. If $e > (d-b)/2$, they follow from the symmetric arguments for 0 in place of 2 as in case (β) in the proof of Lemma 2(1) and Lemma A.7. ▮

The contagion part of this theorem has been shown by Goyal and Janssen (1997, Theorem 3) in their circular network setting with a continuum of players.

Immorlica et al. (2007) consider the current case with a payoff parameter restriction $a = 1-q$, $b = c = 0$, and $d = q$, so the game is given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1-q & 1-q & 0 \\ 1-q-e & 1-q-e & q-e \\ 0 & q & q \end{pmatrix} \end{array}, \qquad 0 < q < \frac{1}{2}. \tag{A.21}$$

This game is a potential game with action 2 being the potential maximizer (more generally, the bilingual game is a potential game whenever $b = c$). Immorlica et al. (2007) focus on the class $\mathcal{G}_\Delta$ of Δ-regular networks; for a natural number Δ, a Δ-regular network is a constant-weight local interaction system where each player has Δ neighbors. They consider the "epidemic region" $\Omega(G) \subset (0, 1/2) \times \mathbb{R}_{++}$, the set of points (q, e) for which action 0 spreads contagiously in a network G, and show that for any fixed Δ, there exists a point $(q,e) \notin \Omega_\Delta := \bigcup_{G \in \mathcal{G}_\Delta} \Omega(G)$, and in particular, Ω_Δ is not convex. On the other hand, since the linear network constructed in Lemma 1(1-i) (with a choice of a rational number p; see Footnote 6) can be replicated by a Δ-regular network, our Theorem A.2 implies that for any $(q,e) \in (0,1/2) \times \mathbb{R}_{++}$, there exists Δ such that $(q,e) \in \Omega_\Delta$, so that $\bigcup_\Delta \Omega_\Delta$ covers the whole space $(0,1/2) \times \mathbb{R}_{++}$ and is convex.

References

Basteck, C. and T. R. Daniëls (2011). "Every Symmetric 3×3 Global Game of Strategic Complementarities Has Noise-Independent Selection," *Journal of Mathematical Economics* **47**, 749-754.

Berninghaus, S. and H. Haller (2010). "Local Interaction on Random Graphs," *Games* **1**, 262-285.

Blume, L. E. (1993). "The Statistical Mechanics of Strategic Interaction," *Games and Economic Behavior* **5**, 387-424.

Blume, L. E. (1995). "The Statistical Mechanics of Best-Response Strategy Revision," *Games and Economic Behavior* **11**, 111-145.

Carlsson, H. and E. van Damme (1993). "Global Games and Equilibrium Selection," *Econometrica* **61**, 989-1018.

Easley, D. and J. Kleinberg (2010). *Networks, Crowds, and Markets: Reasoning about a Highly Connected World*, Cambridge University Press, Cambridge.

Ellison, G. (1993). "Learning, Local Interaction, and Coordination," *Econometrica* **61**, 1047-1071.

Ellison, G. (2000). "Basins of Attraction, Long-Run Stochastic Stability, and the Speed of Step-by-Step Evolution," *Review of Economic Studies* **67**, 17-45.

Frankel, D. M., S. Morris, and A. Pauzner (2003). "Equilibrium Selection in Global Games with Strategic Complementarities," *Journal of Economic Theory* **108**, 1-44.

Fudenberg, D. and D. K. Levine (1998). *The Theory of Learning in Games*, MIT Press, Cambridge.

Galeotti, A., S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv (2010). "Network Games," *Review of Economic Studies* **77**, 218-244.

Galesloot, B. M. and S. Goyal (1997). "Costs of Flexibility and Equilibrium Selection," *Journal of Mathematical Economics* **28**, 249-264.

Goyal, S. (2007). *Connections: An Introduction to the Economics of Networks*, Princeton University Press, Princeton.

Goyal, S. and M. C. W. Janssen (1997). "Non-Exclusive Conventions and Social Coordination," *Journal of Economic Theory* **77**, 34-57.

Granovetter, M. S. (1973). "The Strength of Weak Ties," *American Journal of Sociology* **78**, 1360-1380.

Honda, J. (2011). "Noise-Independent Selection in Global Games and Monotone Potential Maximizer: A Symmetric 3×3 Example," *Journal of Mathematical Economics* **47**, 663-669.

Immorlica, N., J. Kleinberg, M. Mahdian, and T. Wexler (2007). "The Role of Compatibility in the Diffusion of Technologies through Social Networks," in *Proceedings of the 8th ACM Conference on Electronic Commerce*, 75-83.

Jackson, M. O. (2008). *Social and Economic Networks*, Princeton University Press, Princeton.

Kajii, A. and S. Morris (1997). "The Robustness of Equilibria to Incomplete Information," *Econometrica* **65**, 1283-1309.

Kreindler, G. E. and H. P. Young (2012). "Rapid Innovation Diffusion with Local Interaction," University of Oxford Economics Discussion Paper.

Lee, I. H. and A. Valentinyi (2000). "Noisy Contagion without Mutation," *Review of Economic Studies* **67**, 47-56.

Lelarge, M. (2012). "Diffusion and Cascading Behavior in Random Networks," *Games and Economic Behavior* **75**, 752-775.

López-Pintado, D. (2006). "Contagion and Coordination in Random Networks," *International Journal of Game Theory* **34**, 371-381.

López-Pintado, D. (2008). "Diffusion in Complex Social Networks," *Games and Economic Behavior* **62**, 573-590.

Manski, C. F. (1993). "Identification of Endogenous Social Effects: The Reflection Problem," *Review of Economic Studies* **60**, 531-542.

Matsui, A. (1991). "Cheap-Talk and Cooperation in a Society," *Journal of Economic Theory* **54**, 245-258.

Matsui, A. and K. Matsuyama (1995). "An Approach to Equilibrium Selection," *Journal of Economic Theory* **65**, 415-434.

Monderer, D. and L. Shapley (1996). "Potential Games," *Games and Economic Behavior* **14**, 124-143.

Montanari, A. and A. Saberi (2010). "The Spread of Innovations in Social Networks," *Proceedings of the National Academy of Sciences of the United States of America* **107**, 20196-20201.

Morris, S. (1997). "Interaction Games: A Unified Analysis of Incomplete Information, Local Interaction and Random Matching," CARESS Working Paper No.97-02, University of Pennsylvania.

Morris, S. (1999). "Potential Methods in Interaction Games," mimeo.

Morris, S. (2000). "Contagion," *Review of Economic Studies* **67**, 57-78.

Morris, S., R. Rob, and H. S. Shin (1995). "p-Dominance and Belief Potential," *Econometrica* **63**, 145-157.

Morris, S. and H. S. Shin (2003). "Global Games: Theory and Applications," in M. Dewatripont, L. P. Hansen, and S. J. Turnovsky, eds., *Advances in Economics and Econometrics: Theory and Applications: Eighth World Congress, Volume 1*, Cambridge University Press, Cambridge.

Morris, S. and T. Ui (2005). "Generalized Potentials and Robust Sets of Equilibria," *Journal of Economic Theory* **124**, 45-78.

Oury, M. (2012). "Noise-Independent Selection in Multidimensional Global Games," mimeo.

Oyama, D. and S. Takahashi (2009). "Monotone and Local Potential Maximizers in Symmetric 3×3 Supermodular Games," *Economics Bulletin* **29**, 2132-2144.

Oyama, D. and S. Takahashi (2011). "On the Relationship between Robustness to Incomplete Information and Noise-Independent Selection in Global Games," *Journal of Mathematical Economics* **47**, 683-688.

Oyama, D., S. Takahashi, and J. Hofbauer (2008). "Monotone Methods for Equilibrium Selection under Perfect Foresight Dynamics," *Theoretical Economics* **3**, 155-192.

Oyama, D. and O. Tercieux (2010). "Robust Equilibria under Non-Common Priors," *Journal of Economic Theory* **145**, 752-784.

Oyama, D. and O. Tercieux (2012). "On the Strategic Impact of an Event under Non-Common Priors," *Games and Economic Behavior* **74**, 321-331.

Pastor-Satorras, R. and A. Vespignani (2001). “Epidemic Spreading in Scale-Free Networks,” *Physical Review Letters* **86**, 3200-3203.

Rubinstein, A. (1989). “The Electronic Mail Game: Strategic Behavior under ‘Almost Common Knowledge’,” *American Economic Review* **79**, 385-391.

Takahashi, S. (2008). “Perfect Foresight Dynamics in Games with Linear Incentives and Time Symmetry,” *International Journal of Game Theory* **37**, 15-38.

Vega-Redondo, F. (2007). *Complex Social Networks*, Cambridge University Press, Cambridge.

Watts, D. J. (2002). “A Simple Model of Global Cascades on Random Networks,” *Proceedings of the National Academy of Sciences of the United States of America* **99**, 5766-5771.

Wortman, J. (2008). “Viral Marketing and the Diffusion of Trends on Social Networks,” Technical Report MS-CIS-08-19, University of Pennsylvania.

Young, H. P. (1998). *Individual Strategy and Social Structure*, Princeton University Press, Princeton.

Young, H. P. (2011). “The Dynamics of Social Innovation,” *Proceedings of the National Academy of Sciences of the United States of America* **108**, 21285-21291.