Generalized Belief Operator and the Impact of Small Probability Events on Higher Order Beliefs

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Abstract

This paper studies the robustness of an equilibrium to incomplete information in binary-action supermodular games. Using a generalized version of belief operator, we explore the restrictions that prior beliefs impose on higher order beliefs. In particular, we obtain a non-trivial lower bound on the probability of a common belief event, uniform over type spaces, when the underlying game has a monotone potential. Conversely, when the game has no monotone potential, we construct a type space with an arbitrarily high probability event in which players never have common belief about that event. As an implication of these results, we show for generic binary-action supermodular games that an action profile is robust to incomplete information if and only if it is a monotone potential maximizer. Our study offers new methodology and insight to the analysis of global game equilibrium selection.

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1 Introduction

Consider a situation in which many agents make binary decisions, to be “active” (action 1) or “inactive” (action 0), with a motive to coordinate their actions. Examples of such situations include those of currency attacks, bank runs, liquidity crises, and policy changes, among many others. Understanding the equilibrium outcomes in these games entails understanding the uncertainty that players face, not only about payoff relevant parameters, but also about the other players’ behavior, which in turn entails understanding higher order beliefs about payoff relevant parameters.

In the presence of both payoff uncertainty and strategic uncertainty, equilibrium predictions may be fragile to small perturbations in information, as demonstrated, e.g., through the “contagion” arguments by Rubinstein (1989) and Carlsson and van Damme (1993a). Indeed, for a large class of global games with strategic complementarity, a unique strategy profile survives the iterated elimination of strictly dominated strategies in the limit as the noise vanishes (Frankel et al. (2003)). In 2 × 2 coordination games and symmetric binary-action supermodular games, for example, the risk-dominant equilibrium and the “Laplacian” equilibrium, respectively, are selected by the global game approach independently of the noise structure (Carlsson and van Damme (1993a), Morris and Shin (1998, 2003)). On the other hand, in a certain class of asymmetric games, Corsetti et al. (2004) demonstrate that different equilibria are selected depending on the noise structure.

In this paper, within the class of binary-action supermodular games, we aim at identifying when action 1 survives in all information perturbations, and conversely, when action 0 is played contagiously in some information perturbation. We emphasize two forms of generality in our analysis: we allow for general payoffs beyond two-player games or symmetric games, and examine robustness against general information perturbations, including global-game perturbations with various noise structures as particular instances. More specifically, we study the notion of robustness introduced by Kajii and Morris (1997a): a Nash equilibrium of a complete information game is robust to incomplete information if in any incomplete information game where with high probability, all players know that their payoff functions are given by those in the original complete information game, the equilibrium action profile continues to be played with high probability in some Bayesian Nash equilibrium. Our main theorems characterize the robustness and non-robustness of equilibria in this sense.

For our study, we employ a generalized belief operator (Morris and Shin (2007)), which generalizes the p-belief operator of Monderer and Samet (1989), to accommodate the class of all binary-action supermodular games. Given any binary-action game $f = (f_i)_{i \in I}$ with a set $I$ of players and any information structure, our $f_i$-belief operator $B^f_i$ associates player $i$’s beliefs with his incentives in the game $f$. In Section 3, after stating the definition and some basic properties of the operator, we establish formal connections between common $f$-beliefs $CB^f$ and equilibria of incomplete information games that embed the game $f$ (Proposition 3). It turns out that examining the robustness of action profile “all 1” amounts to examining the behavior of the probability $P(CB^f(E))$ of common $f$-belief of an event profile $E = (E_i)_{i \in I}$ as the probability $P(E)$ of the event $E = \prod_{i \in I} E_i$ becomes close to one (Proposition 4).

In Section 4, we state and prove our main theorems. First, our Theorem 1 extends the Critical Path Theorem of Kajii and Morris (1997a, Proposition 4.2) from p-belief (with a vector $p = (p_i)_{i \in I}$) to the generalized belief. It establishes a relationship between the probability $P(CB^f(E))$ of common $f$-belief and the probability $P(E)$ when the underlying game $f$ has a monotone potential as introduced by Morris and Ui (2005). Formally, it shows that if “all 1” is a monotone potential maximizer in $f$, then the probability $P(CB^f(E))$ of common $f$-belief converges to 1 as the probability $P(E)$ converges to 1, where the convergence is uniform over all
information structures and all event profiles. We provide two proofs of this theorem. The first proof significantly simplifies the original proof by Kajii and Morris (1997a) even in the case of $p$-belief; the second proof adopts the potential maximization approach of Ui (2001) and Morris and Ui (2005).

Second, Theorem 2 shows a generic converse of our Critical Path Theorem. That is, for a generic game $f$, if “all 1” is not a monotone potential maximizer in $f$, then there exist an information structure and an arbitrarily high probability event profile such that players never have common $f$-belief about that event profile. For the proof of this theorem, by exploiting the duality between payoffs and probabilities, we construct a desired incomplete information perturbation based on a novel application of a duality theorem to the system of linear inequalities that characterizes the existence of a monotone potential. Specifically, the duality theorem gives us a distribution over sequences, or rankings, of players, which in our construction, determines the posterior beliefs about the rankings of types among the players.

Third, combining the results above, Theorem 3 and Corollary A.1 establish a complete characterization of robust equilibrium in generic binary-action supermodular games: an equilibrium is robust to incomplete information if and generically only if it is a monotone potential maximizer. Note that results by Morris and Ui (2005) prove the robustness of a monotone potential maximizer in supermodular games with many actions, but are silent about the non-existence of a robust equilibrium when there is no monotone potential. For illustration of our results, we determine the robust equilibrium in two classes of games, unanimity games and games with cyclic symmetry.

Our study offers new methodology and insight to the analysis of global games beyond the Laplacian criterion (Morris and Shin (2003)) for the case of homogeneous players. In Section 5, as a minimal departure from symmetric games, we revisit the asymmetric currency attack game of Corsetti et al. (2004) with a large trader and small traders and characterize its robust equilibrium. In particular, we observe that the condition for the noise-dependent selection in the global game of Corsetti et al. (2004) precisely corresponds to that for the non-existence of a monotone potential in the underlying complete information game, which boils down to the solvability of a system of two linear inequalities in one variable. When no monotone potential exists, we discuss how the distribution over the rankings among players, which is obtained from the duality, corresponds to the relative precision of the signals of the large and the small traders.

1.1 Related Literature

The notion of robust equilibrium by Kajii and Morris (1997a) allows for a much richer set of payoff perturbations than the classical refinement concepts, such as Kohlberg and Mertens’ (1986) strategic stability. Indeed, Kajii and Morris (1997a, Example 3.1) present an example of a three-player three-action game (without supermodularity) that has a unique Nash equilibrium, which is strict, but has no robust equilibrium. Yet, several sufficient conditions for robustness have been identified. For example, Kajii and Morris (1997a) show that if the game has a $p$-dominant equilibrium with $\sum_{i \in I} p_i < 1$, then it is robust. Using the technique of potential functions, Ui (2001) shows that if the game admits a potential, then the potential maximizing action profile is robust. Subsequently, Morris and Ui (2005) extend Ui’s approach to gen-

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1See Kajii and Morris (1997b) for the relationship between robustness and the existing refinement concepts.
2For $p = (p_i)_{i \in I}$, an action profile $a^*$ is a $p$-dominant equilibrium if for each player $i \in I$, $a_i$ is a best response to any belief that assigns at least probability $p_i$ to the opponents’ playing $a_{-i}$. This notion generalizes the notion of risk-dominance in $2 \times 2$ coordination games.
3Precisely, Ui (2001) uses the notion of robustness to canonical elaborations, introduced in Kajii and Morris (1997b), which is, in principle, weaker than the original notion of Kajii and Morris (1997a). For binary-action supermodular games, which are focus of the present paper, the two notions of robustness coincide. See
eralized/monotone potential games. Their sufficient condition for robustness subsumes both Kajii-Morris’ and Ui’s conditions. Our paper provides an alternative proof for Morris-Ui’s result via the Critical Path Theorem and proves its (generic) converse for binary-action supermodular games.

Global games, as introduced by Carlsson and van Damme (1993a), offer a tractable modelling device that leads to equilibrium selection, thus allowing comparative statics exercises for policy implications, and have found applications where coordination of actions plays an important role. Examples include Morris and Shin (1998) for currency attacks, Morris and Shin (2004) for debt pricing, Goldstein and Pauzner (2005) for bank runs, and Bueno de Mesquita (2010), Edmond (2013), and Chen and Suen (2017) for policy changes. For the class of symmetric binary-action supermodular games, Morris and Shin (2003) show that the limit equilibrium (either under the uniform improper prior or in the limit with vanishingly small noise) is independent of the noise structure, and characterized by the “Laplacian” action, i.e., the best response to the uniform belief over the rankings of the players. This characterization has been extended to accommodate some forms of preference heterogeneity (Guimaraes and Morris (2008, Section 6.1), Sákovic and Steiner (2012)). On the other hand, in the context of currency attacks, Corsetti et al. (2004) incorporate a large atomic player with a continuum of small players, and show that the selected equilibrium is sensitive to the relative precision of the signals between the large and the small players.

Global games also offer a tractable class of incomplete information perturbations for examining the robustness of an equilibrium. Formally, Basteck et al. (2013) and Oury and Tercieux (2007) show that for a generic supermodular game, if a robust equilibrium exists, then every global game that embeds the given game, regardless of the noise structure, must select the robust equilibrium. This result implies, in particular, that a generic supermodular game has at most one robust equilibrium, and that it has no robust equilibrium if the global game equilibrium selection depends on the noise structure. The converse of the result of Basteck et al. and Oury-Tercieux, however, does not hold for games with more than two actions (Basteck and Daniëls (2011), Oyama and Takahashi (2011)). Our Theorem 3, along with the construction in the proof of Theorem 2, in fact implies that the converse does hold for generic binary-action supermodular games, i.e., for these games, an equilibrium is robust to incomplete information if and only if it is noise-independent global game selection (Proposition 5).

The literature has provided sporadic examples of games with no robust equilibrium, in addition to the example by Kajii and Morris (1997a, Example 3.1) as mentioned earlier, such as those in Carlsson (1989), Frankel et al. (2003), Corsetti et al. (2004), Basteck et al. (2013), Oury (2013), and Oyama and Takahashi (2011, 2015). In demonstrating non-existence of robust equilibria, these papers rely on ad hoc constructions of incomplete information perturbations. The present paper (the proof of Theorem 2), in contrast, offers a systematic construction of perturbations for binary-action supermodular games based on a duality theorem.

Equilibrium behavior in incomplete information games can be analyzed by belief operators and the associated notions of common beliefs. For example, using the belief operator of Monderer and Samet (1989) (with a generalization to vector $p$), Kajii and Morris (1997a, Lemma 5.2) show that if $a^*$ is $p$-dominant in a complete information game, then any incomplete informa-

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4For further developments in the literature, see e.g., Oyama and Tercieux (2009), Nora and Uno (2014), and Haimanko and Kajii (2016).

5Other forms of preference heterogeneity are considered in Leister et al. (2018) and Serrano-Padial (2018), where the equilibrium is characterized by a (weighted) potential maximization problem (Frankel et al. (2003)).

6The first five papers in this list demonstrate noise-dependent global game selection, which implies non-existence of a robust equilibrium by Basteck et al. (2013) and Oury and Tercieux (2007); see Basteck et al. (2013, Table 1) for a list of games with noise-dependent global game selection.
tion game has a Bayesian Nash equilibrium in which $a^*$ is played wherever there is common $p$-belief that the players know that their payoffs are given by those in the complete information game. Kajii and Morris (1997a, Proposition 5.3) combine this lemma and their Critical Path Theorem to obtain the sufficient condition for robustness in terms of $p$-dominance. Morris and Shin (2007) generalize the $p$-belief of Monderer and Samet (1989) to accommodate general, state-dependent, supermodular payoff functions with binary actions. Using their generalized belief operator, Morris and Shin (2007) give a common belief foundation for global game selection by identifying conditions on rank beliefs under which the game has a unique rationalizable strategy for the case of separable-symmetric payoffs. The present paper follows Kajii and Morris’ (1997a) approach by establishing the Critical Path Theorem for the generalized belief operator and characterizing the robustness of equilibria for general binary-action supermodular games, without making any symmetry assumption.

We follow Kajii and Morris (1997a) to adopt an ex ante perspective on robustness. Weinstein and Yildiz (2004, 2007) instead consider the robustness to perturbations in interim beliefs. They show that under a richness assumption, for any type $t_i$ in the universal type space and any rationalizable action $a_i$ for type $t_i$, there exists a type arbitrarily close to $t_i$ (in the product topology on the universal type space) for which $a_i$ is uniquely rationalizable. Thus, according to Weinstein and Yildiz (2004), an action profile is interim robust if and only if it is uniquely rationalizable. Note that Weinstein-Yildiz’ analysis does not impose any restriction on the ex ante probability of the event that the payoffs are given by those of the underlying game, and that due to Lipman (2003), their result holds irrespective of whether the common prior assumption holds or not. In our ex ante approach, in contrast, the Critical Path Theorem (of Kajii and Morris (1997a) or ours) quantifies non-trivial implications that the common prior assumption imposes on the ex ante probabilities of higher order belief events.

2 Preliminaries

2.1 Complete Information Games

Let $I = \{1, \ldots, |I|\}$ be the set of players with $|I| \geq 2$. We write $\mathcal{I} = 2^I$ for the collection of all subsets of $I$ (including the empty set), and for each $i \in I$, $\mathcal{I}_{-i} = 2^{I \setminus \{i\}}$ for the collection of all subsets of $I \setminus \{i\}$.

The finite set of actions available to each player $i \in I$ is denoted by $A_i$, where we write $A = \prod_{i \in I} A_i$ and $A_{-i} = \prod_{j \neq i} A_j$ as usual. A complete information game is then represented by a profile $g = (g_i)_{i \in I}$ of payoff functions $g_i : A \rightarrow \mathbb{R}$, $i \in I$.

2.2 Type Spaces and Incomplete Information Elaborations

A type space $(T, P)$ consists of a countable set $T_i$ of each player $i$’s types and a common prior distribution $P \in \Delta(T)$, where $T = \prod_{i \in I} T_i$. We assume $P(\{t_i\} \times T_{-i}) > 0$ for each $i \in I$ and $t_i \in T_i$, where $T_{-i} = \prod_{j \neq i} T_j$. For any $i \in I$ and $t_i \in T_i$, the posterior of type $t_i$ is given by

$$P(E_{-i}|t_i) = \frac{P(\{t_i\} \times E_{-i})}{P(\{t_i\} \times T_{-i})}$$

for $E_{-i} \subset T_{-i}$.
With $I$ and $(A_i)_{i \in I}$ fixed, an incomplete information elaboration consists of a type space $(T, P)$ and a profile $u = (u_i)_{i \in I}$ of payoff functions $u_i : A \times T \to \mathbb{R}, i \in I$. A (behavioral) strategy for player $i \in I$ is a function $\sigma_i : T_i \to \Delta(A_i)$. Denote by $\Sigma_i$ the set of all strategies for player $i$, and write $\Sigma = \prod_{i \in I} \Sigma_i$ and $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$. The expected payoff to player $i$ of type $t_i \in T_i$ from playing $a_i \in A_i$ against opponents’ strategy profile $\sigma_{-i} = (\sigma_j)_{j \neq i} \in \Sigma_{-i}$ is

$$\mathbb{E}[u_i((a_i, \sigma_{-i}(.)), (t_i,.)) | t_i] = \sum_{t_{-i} \in T_{-i}} P(t_{-i} | t_i) u_i((a_i, \sigma_{-i}(t_{-i})), (t_i, t_{-i})),$$

where $u_i((a_i, .), t)$ is extended to $\prod_{j \neq i} \Delta(A_j)$ in the usual manner. Let the correspondence $BR_i : \Sigma_{-i} \times T_i \to A_i$ for each $i$ be defined by

$$BR_i(\sigma_{-i})(t_i) = \arg \max_{a_i \in A_i} \mathbb{E}[u_i((a_i, \sigma_{-i}(.)), (t_i,.)) | t_i].$$

A strategy profile $\sigma^* = (\sigma^*_i)_{i \in I} \in \Sigma$ is a Bayesian Nash equilibrium of $(T, P, u)$ if for all $i \in I$, all $t_i \in T_i$, and all $a_i \in A_i$,

$$\sigma^*_i(a_i | t_i) > 0 \Rightarrow a_i \in BR_i(\sigma^*_{-i})(t_i).$$

### 2.3 Robust Equilibria

Given a complete information game $g$ and an incomplete information elaboration $(T, P, u)$, we denote

$$T^g_i = \{ t_i \in T_i | u_i(a, (t_i, t_{-i})) = g_i(a) \text{ for all } a \in A$$

$$\text{and for all } t_{-i} \in T_{-i} \text{ with } P(t_{-i} | t_i) > 0 \}.$$

For $\varepsilon \in [0,1]$, an incomplete information elaboration $(T, P, u)$ is an $\varepsilon$-elaboration of $g$ if $P(\prod_{i \in I} T^g_i) \geq 1 - \varepsilon$. The notion of information robustness of equilibrium is due to Kajii and Morris (1997a).

**Definition 1.** A Nash equilibrium $a^* = (a^*_i)_{i \in I} \in A$ of a complete information game $g$ is robust (to incomplete information) in $g$ if for any $\delta > 0$, there exists $\varepsilon > 0$ such that for any $\varepsilon$-elaboration $(T, P, u)$ of $g$, there exists a Bayesian Nash equilibrium $\sigma^* = (\sigma^*_i)_{i \in I} \in \Sigma$ such that

$$\sum_{t = (t_i)_{i \in I} \in T} P(t) \prod_{i \in I} \sigma^*_i(a^*_i | t_i) \geq 1 - \delta.$$ 

Not all (strict) Nash equilibria are robust to incomplete information (Rubinstein (1989)), and even a unique equilibrium, which is strict, may not be robust (Kajii and Morris (1997a)). For $p = (p_i)_{i \in I} \in [0,1]^I$, an action profile $a^* \in A$ is a $p$-dominant equilibrium of $g$ if for all $i \in I$ and all $\nu_i \in \Delta(A_{-i})$,

$$\nu_i(a^*_{-i}) \geq p_i \Rightarrow a^*_i \in \arg \max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} g_i(a_i, a_{-i}) \nu_i(a_{-i}).$$

Kajii and Morris (1997a) show that a $p$-dominant equilibrium with $\sum_{i \in I} p_i < 1$ is robust to incomplete information.
3 Generalized Belief Operators

3.1 Binary-Action Supermodular Games

In what follows, we consider binary-action games, where \( A_1 = \cdots = A_{|I|} = \{0, 1\} \). For \( S \in I \), we denote by \( 1_S \) the action profile such that all players in \( S \) play action 1, and the others play action 0; by convention, we write \( 1 \) for \( 1_I \). Given a binary-action complete information game \( g \), for each \( i \in I \), we define the “payoff increment” function \( f_i : I_{-i} \to \mathbb{R} \) by

\[
  f_i(S) = g_i(1_{S \cup \{i\}}) - g_i(1_S)
\]

for \( S \in I_{-i} \). That is, \( f_i(S) \) is the payoff increment for player \( i \) from playing action 1 over playing action 0 when the set of opponent players playing action 1 is \( S \). Without loss of generality, we identify the binary-action complete information game with the profile \( f = (f_i)_{i \in I} \) of payoff increment functions. Throughout our analysis, we focus on supermodular games: we assume that every player has weakly increasing payoff increments, i.e., for all \( i \in I \) and all \( S, S' \in I_{-i} \), \( f_i(S) \leq f_i(S') \) whenever \( S \subseteq S' \).

3.2 Generalized Beliefs

Let a binary-action supermodular game \( f = (f_i)_{i \in I} \) and a type space \((T, P)\) be given as above. We denote \( T_i = 2^{T_i} \), \( T = \prod_{i \in I} T_i \), and \( T_{-i} = \prod_{j \neq i} T_j \) for \( i \in I \).

We define \( f_i \)-belief for each player \( i \) as follows. This notion has been introduced in Morris and Shin (2007) and generalizes that of \( p \)-belief by Monderer and Samet (1989). The reader should bear in mind that an event \( E_i \in T_i \) can be thought as player \( i \)'s (pure) strategy, in a binary-action incomplete information elaboration, that plays action 1 on, and only on, \( E_i \), while \( E_{-i} = (E_j)_{j \neq i} \in T_{-i} \) as a profile of opponent players’ strategies, and that the \( f_i \)-belief operator is closely related to the best response function for player \( i \)'s types whose payoffs are given by \( f_i \), as discussed formally in Section 3.4.

For \( i \in I \) and \( E_{-i} \in T_{-i} \), define the function \( S_{E_{-i}} : T_{-i} \to I_{-i} \) by

\[
  S_{E_{-i}}(t_{-i}) = \{ j \neq i \mid t_j \in E_j \}.
\]

Then the expectation of the random variable \( t_{-i} \mapsto f_i(S_{E_{-i}}(t_{-i})) \) conditional on \( t_i \in T_i \) is

\[
  \mathbb{E}[f_i(S_{E_{-i}}(\cdot)|t_i)] = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)f_i(\{j \neq i \mid t_j \in E_j\})
\]

\[
  = \sum_{S \in I_{-i}} P \left( \prod_{j \in S} E_j \times \prod_{j \not\in S \cup \{i\}} (T_j \setminus E_j) | t_i \right) f_i(S).
\]

**Definition 2.** For \( i \in I \), \( t_i \in T_i \), and \( E = (E_i)_{i \in I} \in T \), a type \( t_i \) of player \( i \) is said to have \( f_i \)-belief about \( E \) if \( t_i \in E_i \) and \( \mathbb{E}[f_i(S_{E_{-i}}(\cdot)|t_i)] \geq 0 \). Player \( i \)'s \( f_i \)-belief operator \( B_i^{f_i} : T \to T_i \) is defined by

\[
  B_i^{f_i}(E) = \{ t_i \in E_i \mid \mathbb{E}[f_i(S_{E_{-i}}(\cdot)|t_i)] \geq 0 \}
\]

for \( E = (E_i)_{i \in I} \in T \), i.e., \( B_i^{f_i}(E) \) is the set of player \( i \)'s types that have \( f_i \)-belief about \( E \).

As we explain below, the notion of \( f_i \)-belief generalizes the notion of \( p_i \)-belief and accommodates some other interesting cases.
Example 1 (p_i-Belief). The notion of f_i-belief generalizes that of p_i-belief. For p_i \in [0, 1], player i's p_i-belief operator B^{p_i}_i : \mathcal{T} \rightarrow \mathcal{T}_i is defined by

\[ B^{p_i}_i(E) = \{ t_i \in E_i \mid P(E_{-i}|t_i) \geq p_i \}, \]

where E_{-i} = \prod_{j \neq i} E_j. To see that the notion of p_i-belief is a special case of that of f_i-belief, given p_i, define the payoff increment function f^{p_i}_i by

\[ f^{p_i}_i(S) = \begin{cases} 1 - p_i & \text{if } S = I \setminus \{i\}, \\ -p_i & \text{otherwise} \end{cases} \]

for S \in \mathcal{I}_{-i}. Then one can verify that for any E \in \mathcal{T},

\[ E[f^{p_i}_i(S|_{E_{-i}()},|t_i)] = (1 - p_i)P(E_{-i}|t_i) - p_i(1 - P(E_{-i}|t_i)) \]

= P(E_{-i}|t_i) - p_i,

and hence B^{f^{p_i}_i}_i(E) = B^{p_i}_i(E).

The payoff increment function f^{p_i}_i is constant for all S \subseteq I \setminus \{i\}, and hence B^{f^{p_i}_i}_i(E) is determined only by E_i and the posterior probability that all opponents belong to E_{-i}. In contrast, if |I| \geq 3, then the generalized belief operator may depend on other statistics such as the number of opponents belonging to E_j.

Example 2 (Aggregation). For a weakly increasing function h : \{0, \ldots, |I| - 1\} \rightarrow \mathbb{R} and a threshold value c_i \in \mathbb{R}, define the payoff increment function f^{h,c}_i by

\[ f^{h,c}_i(S) = h(|S|) - c_i \]

for S \in \mathcal{I}_{-i}. Then a type has f^{h,c}_i-belief about E if and only if the type lies in E_i and the expected value of the number of players j \neq i whose types belong to E_j, transformed by the h function, is at least c_i, i.e.,

\[ \sum_{k=0}^{[|I| - 1]} P(\{t_{-i} \in \mathcal{T}_{-i} \mid \{j \neq i \mid t_j \in E_j\} = k\}|t_i)h(k) \geq c_i. \]

This class of belief operators subsumes the p_i-belief operator in Example 1 with

\[ h(k) = \begin{cases} 1 & \text{if } k = n - 1, \\ 0 & \text{otherwise} \end{cases} \]

and c_i = p_i for all i. Coordinated attack models with symmetric players, often studied in the global game literature (e.g., Morris and Shin (1998)), also belong to this class, where h is of the form

\[ h(k) = \begin{cases} 1 & \text{if } k \geq \bar{k}, \\ 0 & \text{otherwise}, \end{cases} \]

and a common threshold value, i.e., c_i = c for all i. More generally, all symmetric binary-action supermodular games can be written as f^{h,c}_i with a common threshold value c.\(^{10}\)

\(^{10}\)For this class of games, Carlsson and van Damme (1993b) and Kim (1996) investigate various approaches to equilibrium selection, including the global game approach.
Let a binary-action game

Proposition 2. We say that

Following Monderer and Samet (1989) and Morris and Shin (2007), we use the

Example 3 (Unanimity). For \( y_i, z_i > 0 \), define the payoff increment function \( f_i^{y_i,z_i} \) by

for \( S \in \mathcal{I}_i \). Then a type has \( f_i^{y_i,z_i} \)-belief about \( E \) if and only if the type lies in \( E_i \) and the ratio between the posterior probabilities of \( \prod_{j \neq i} E_j \) and of \( \prod_{j \neq i} (T_j \setminus E_j) \) is at least \( y_i/z_i \).

Similarly to the \( p \)-belief operator by Monderer and Samet (1989), the \( f_i \)-belief operator satisfies the following properties. For \( E = (E_i)_{i \in I}, E' = (E'_i)_{i \in I} \in \mathcal{T} \), we write \( E \subset E' \) if \( E_i \subset E'_i \) for all \( i \in I \); for a sequence \( (E^n)_{n=0}^\infty = ((E^n_i)_{i \in I})_{n=0}^\infty \) in \( \mathcal{T} \), we write \( \bigcap_{n=0}^\infty E^n \) for \( \bigcap_{n=0}^\infty E^n_i \).

Proposition 1. Let a payoff increment function \( f_i \) be weakly increasing.

(1) \( B_i^f(E) \subset E_i \).

(2) If \( E \subset E' \), then \( B_i^f(E) \subset B_i^f(E') \).

(3) If \( (E^n)_{n=0}^\infty \) is a weakly decreasing sequence, then \( B_i^f(\bigcap_{n=0}^\infty E^n) = \bigcap_{n=0}^\infty B_i^f(E^n) \).

Property (1) holds by definition, while property (2) by the monotonicity of \( f_i \). In property (3), the inclusion \( \supseteq \) follows from the continuity of the probability measure for monotone sequences, while the reverse inclusion \( \subset \) from property (2).

3.3 Common Beliefs

Following Monderer and Samet (1989) and Morris and Shin (2007), we use the \( f_i \)-belief operators to define common \( f \)-belief.

First, a profile \( F = (F_j)_{j \in I} \in \mathcal{T} \) is \( f \)-evident if \( F_i \subset B_i^f(F) \) for all \( i \in I \). By property (1) in Proposition 1, the condition is equivalent to that \( F_i = B_i^f(F) \) for all \( i \in I \), i.e., that \( F \) is a fixed point of \( B_i^f \).

We next define common \( f \)-belief by iteration of the \( f_i \)-belief operators. Let

The sequence \( (B_i^{f,n}(E))_{n=0}^\infty \) is weakly decreasing by properties (1) and (2). Now define

We say that \( t_i \) has common \( f \)-belief about \( E \) if \( t_i \in CB_i^f(E) \).

By property (3), \( (CB_i^f(E))_{i \in I} \) is \( f \)-evident. By property (2), if \( F \subset E \), and \( F \) is \( f \)-evident, then \( F_i \subset CB_i^f(E) \) for all \( i \in I \). Thus we have:

Proposition 2. Let a binary-action game \( f \) be supermodular. For \( E \in \mathcal{T} \), \( (CB_i^f(E))_{i \in I} \) is the largest \( f \)-evident event profile contained in \( E \).
This is a straightforward generalization of the corresponding result for common \( p \)-belief (Monderer and Samet (1989), Kajii and Morris (1997a)), where for \( p = (p_i)_{i \in I} \in [0, 1]^I \), \( CB_i^p \) and \( p \)-evidence are defined similarly to the above with \( (B_i^p)_{i \in I} \) (Example 1) in place of \( (B_i^T)_{i \in I} \).

### 3.4 Connections to Incomplete Information Elaborations

Given a binary-action supermodular game \( f \), consider an incomplete information elaboration \((T, P, u)\) of \( f \). Denote

\[
T_i^f = \{ t_i \in T_i \mid u_i(1, (t_i, t_{-i})) - u_i(1, (t_i, t_{-i})) = f_i(S) \text{ for all } S \in \mathcal{I}_{-i} \text{ and for all } t_{-i} \in T_{-i} \text{ with } P(t_i, t_{-i}) > 0 \},
\]

and \( T^f = (T_i^f)_{i \in I} \in \mathcal{T} \).

We identify an event \( E_i \in \mathcal{T}_i \) with player \( i \)'s (pure) strategy \( \sigma_i \) such that \( \sigma_i(1|t_i) = 1 \) if and only if \( t_i \in E_i \). Through this identification, type \( t_i \in T_i^f \) has \( f_i \)-belief about \((\{t_i\}, E_{-i})\) if and only if action 1 is a best response to \((\text{the strategy profile identified with } E_{-i})\) for type \( t_i \).

Common \( f \)-belief in type space \((T, P)\) is closely related to the iterated elimination procedure of strictly dominated strategies and Bayesian Nash equilibria in elaboration \((T, P, u)\). To see this, take a “canonical” elaboration where 0 is a dominant action for all types outside \( T_i^f \). In this elaboration, \( B_i^T(T_i^f, E_{-i}) \) corresponds to the largest best response to \( E_{-i} \), i.e., \( t_i \in B_i^T(T_i^f, E_{-i}) \) if and only if \( 1 \in BR_i(\sigma_{-i}|t_i) \), where \( \sigma_{-i} \) is the strategy profile such that for all \( j \neq i \), \( \sigma_j(1|t_j) = 1 \) if and only if \( t_j \in E_j \). The \( n \)-times iteration \( B_i^T|E^f(T^f) \) corresponds to the largest strategy that survives the \( n \)-times iterated elimination of strictly dominated strategies, and the limit \( CB_i^f(T^f) \) corresponds to the largest strategy that survives the iterated elimination of strictly dominated strategies. In fact, the profile \((CB_i^f(T^f))_{i \in I} \) is the largest Bayesian Nash equilibrium in \((T, P, u)\).

In a general elaboration, where the payoff functions of the types outside \( T_i^f \) are arbitrary, the largest best response to \( E_{-i} \) may be strictly larger than \( B_i^T(T_i^f, E_{-i}) \). As shown in the proposition below, there exists a Bayesian Nash equilibrium at least as large as the largest \( f \)-evident event profile \((CB_i^f(T^f))_{i \in I} \), i.e., a Bayesian Nash equilibrium \( \sigma^* \) such that for each \( i \in I \), \( \sigma^*_i(1|t_i) = 1 \) whenever \( t_i \in CB_i^f(T^f) \).

The next proposition formally states these relations.

**Proposition 3.** Let a binary-action game \( f \) be supermodular.

1. For any elaboration \((P, T, u)\) of \( f \), there exists a Bayesian Nash equilibrium \( \sigma^* \) such that for each \( i \in I \), \( \sigma^*_i(1|t_i) = 1 \) whenever \( t_i \in CB_i^f(T^f) \).

2. For any type space \((T, P)\) and any event profile \( E \in \mathcal{T} \), there exists a profile \( u \) of payoff functions such that in elaboration \((T, P, u)\), \( T^f = E \) and for each \( i \in I \), the largest strategy \( \sigma_i \) that survives the iterated elimination of strictly dominated strategies is such that \( \sigma_i(1|t_i) = 1 \) if and only if \( t_i \in CB_i^f(T^f) \).

**Proof.** For part (1), let \( \Sigma_i^* \subset \Sigma_i \) be the set of all strategies \( \sigma_i \) such that \( \sigma_i(1|t_i) = 1 \) whenever \( t_i \in CB_i^f(T^f) \). Write \( \Sigma^* = \prod_{i \in I} \Sigma_i^* \) and \( \Sigma_{-i}^* = \prod_{j \neq i} \Sigma_j^* \). Note that \( \Sigma^* \) is a nonempty, convex, and compact subset of a Banach space. We define the correspondence \( \beta : \Sigma^* \to \Sigma_i^* \) by

\[
\beta(\sigma) = \{ \sigma' \in \Sigma^* \mid \text{for all } i \in I: \sigma'_i(a_i|t_i) > 0 \Rightarrow a_i \in BR_i(\sigma_{-i}|t_i) \}.
\]

By Proposition 2, \((CB_i^f(T^f))_{i \in I} \) is \( f \)-evident and contained in \( T^f \). Thus, if \( t_i \in CB_i^f(T^f) \) and \( \sigma_{-i} \in \Sigma_{-i}^* \), then \( 1 \in BR_i(\sigma_{-i}|t_i) \), implying the nonempty-valuedness of \( \beta \). We can also
verify that $\beta$ is convex- and compact-valued and upper semi-continuous. Hence, it follows from Kakutani’s fixed-point theorem that $\beta$ has a fixed point $\sigma^*$ in $\Sigma^*$, which is a Bayesian Nash equilibrium of the elaboration $(P, T, u)$.

For part (2), let the payoff functions $u$ be such that for each $i \in I$, $T_i = E_i$, and action $0$ is a dominant action for all types $t_i \notin E_i$. The conclusion then follows as in the discussion above.

By Proposition 3, we obtain the following characterization of the robustness of $1 = (1, \ldots, 1) \in A$ in terms of common $f$-belief: the “if” and the “only if” parts follow from parts (1) and (2) of Proposition 3, respectively.

**Proposition 4.** Let a binary-action game $f$ be supermodular. Then $1$ is robust to incomplete information in $f$ if and only if for any $\delta > 0$, there exists $\varepsilon > 0$ such that for any type space $(T, P)$ and any event profile $E = (E_i)_{i \in I} \in \mathcal{T}$, we have

$$P(E) \geq 1 - \varepsilon \Rightarrow P\left(CB^f(E)\right) \geq 1 - \delta,$$

where $E = \prod_{i \in I} E_i$ and $CB^f(E) = \prod_{i \in I} CB^f_i(E)$.

Our next task is to explore the relationship between $P(E)$ and $P(CB^f(E))$. For each $n$, if $P(E)$ is close to 1, then $P(\prod_i B_i^{f,n}(E))$ is necessarily close to 1 (as long as $f_i(I \setminus \{i\}) > 0$ for all $i$), but, in general, this implication does not hold for $P(CB^f(E))$. Our main theorems characterize a (generic) necessary and sufficient condition for $f$ under which this implication holds, uniformly over all type spaces and event profiles thereof.

# 4 Robust Equilibria in Binary-Action Supermodular Games

## 4.1 Potentials

Monderer and Shapley (1996) introduce the notion of potential for general normal form games. In our notation for binary-action games, the definition is written as follows:

**Definition 3.** A function $v: \mathcal{I} \to \mathbb{R}$ is a potential of a binary-action game $f = (f_i)_{i \in I}$ if

$$f_i(S) = v(S \cup \{i\}) - v(S)$$

for all $i \in I$ and $S \in \mathcal{I}_{-i}$.

$S^* \in \mathcal{I}$ is a potential maximizer in $f$ if $f$ admits a potential $v$ that is strictly maximized at $S^*$, i.e., $v(S^*) > v(S)$ for all $S \in \mathcal{I}$ with $S \neq S^*$.

Not every game admits a potential. It is easy to see that a binary-action game $f$ admits a potential if and only if

$$f_i(S \cup \{j\}) - f_i(S) = f_j(S \cup \{i\}) - f_j(S)$$

for any $i \neq j$ and $S \subset I \setminus \{i, j\}$. If $f$ admits a potential, then the potential is determined uniquely up to constants:

$$v(S) = v(\emptyset) + \sum_{\ell=1}^{k} f_{i_{\ell}}(\{i_1, \ldots, i_{\ell-1}\})$$

for $S = \{i_1, \ldots, i_k\} \in \mathcal{I}$, where the summation is independent of the order of players in $S$. 10
Example 4 (\(p_i\)-Belief). Suppose that each player \(i \in I\) has the payoff increment function \(f^{p_i}_i\) as in Example 1. Then the binary-action game \((f^{p_i}_i)_{i \in I}\) admits the following potential:

\[
v(S) = \begin{cases} 
1 - \sum_{i \in I} p_i & \text{if } S = I, \\
- \sum_{i \in S} p_i & \text{otherwise.}
\end{cases}
\]

Note that this function \(v\) is strictly maximized at \(I\) if and only if \(\sum_{i \in I} p_i < 1\).

Example 5 (Aggregation). Suppose that each player \(i \in I\) has the payoff increment function \(f^{h,c_i}_i\) as in Example 2, where we assume that \(c_1 \leq c_2 \leq \cdots \leq c_{|I|}\) without loss of generality. Then the binary-action game \((f^{h,c_i}_i)_{i \in I}\) admits the following potential:

\[
v(S) = \sum_{n=0}^{[|S|-1]} h(n) - \sum_{i \in S} c_i,
\]

which is strictly maximized at \(I\) if and only if \(\sum_{n=k}^{[|I|-1]} h(n) > \sum_{i=k+1}^{[|I|]} c_i\) for any \(0 \leq k \leq |I| - 1\).

Example 6 (Unanimity). Suppose that each player \(i \in I\) has the payoff increment function \(f^{y_i,z_i}_i\) as in Example 3. Then the binary-action game \((f^{y_i,z_i}_i)_{i \in I}\) admits a potential if and only if \(y_1 = \cdots = y_{|I|}\) and \(z_1 = \cdots = z_{|I|}\).

However, the unanimity payoff increment function \(f^{y_i,z_i}_{1,\zeta}\) is “more permissive” than another unanimity payoff increment function \(f^{1,\zeta}_{1,\zeta}\) with

\[
\zeta = \min_{i \neq j} \frac{z_i z_j}{y_i y_j}
\]

in the sense that for any \(E \in \mathcal{T}\), a type has \(f^{y_i,z_i}_{1,\zeta}\)-belief about \(E\) whenever the type has \(f^{1,\zeta}_{1,\zeta}\)-belief about \(E\), i.e., \(B^{f^{y_i,z_i}_{1,\zeta}}_i(E) \supset B^{f^{1,\zeta}_{1,\zeta}}_i(E)\). Moreover, the profile \((f^{1,\zeta}_{1,\zeta})_{i \in I}\) admits the following potential:

\[
v(S) = \begin{cases} 
\zeta & \text{if } S = I, \\
1 & \text{if } S = \emptyset, \\
0 & \text{otherwise},
\end{cases}
\]

which is strictly maximized at \(I\) if and only if \(\zeta > 1\), i.e., \(z_i z_j > y_i y_j\) for all \(i \neq j\).

In order to accommodate the last example, we relax the definition of potential as follows, which is in parallel to characteristic/monotone potentials for normal form games introduced by Morris and Ui (2005).

Definition 4. \(S^* \in \mathcal{I}\) is a monotone potential maximizer in a binary-action game \(f = (f_i)_{i \in I}\) if there exist a function \(v: \mathcal{I} \rightarrow \mathbb{R}\) and \(\lambda = (\lambda_i)_{i \in I}\) with \(\lambda_i > 0\) such that

\[
\lambda_i f_i(S) \geq v(S \cup \{i\}) - v(S)
\]

for all \(i \in S^*\) and \(S \in \mathcal{I}_{-i}\),

\[
\lambda_i f_i(S) \leq v(S \cup \{i\}) - v(S)
\]

for all \(i \in I \setminus S^*\) and \(S \in \mathcal{I}_{-i}\), and \(v(S^*) > v(S)\) for all \(S \in \mathcal{I}\) with \(S \neq S^*\).

Such a function \(v\) is called a monotone potential of \(f\) for \(S^*\).
Clearly, if \( S^\ast \) is a potential maximizer in \( f \), then it is a monotone potential maximizer in \( f \). In the definition of a monotone potential, the equality in the definition of a potential is replaced with inequalities, while the multiplier \( \lambda_i > 0 \) is for normalization.\(^\text{11}\)

In the unanimity game \((f_i^{p_i,z_i})_{i \in I}\) considered in Examples 3 and 6, \( I \) is a monotone potential maximizer if \( z_iz_j > y_iy_j \) for all \( i \neq j \), where the function \( v \) given by (4.1) is a monotone potential for \( I \); symmetrically, \( \emptyset \) is a monotone potential maximizer if \( z_iz_j < y_iy_j \) for all \( i \neq j \).\(^\text{12}\)

**Remark 1.** As a corollary of their main results, Oyama et al. (2008, p. 176) show that a supermodular game has at most one monotone potential maximizer. That is, if \( f \) is supermodular, and \( v \) is a monotone potential of \( f \) for \( S^\ast \), then there is no monotone potential of \( f \) for any \( S \neq S^\ast \). On the other hand, there is a nonempty open set of supermodular games that have no monotone potential maximizer. See, e.g., Morris and Ui (2005, Section 7.2) and Examples 7 and 8 below.

In what follows, we focus on the case of \( S^\ast = I \) for expositional ease. In particular, our main theorems (Theorems 1–3) will be stated in terms of the (non-)existence of a monotone potential for \( I \). Nevertheless, we demonstrate in Appendix A.6 that Theorem 3 extends to general \( S^\ast \) (Corollary A.1).

### 4.2 The Critical Path Theorem

Using the \( p \)-belief operator, Kajii and Morris (1997a, Proposition 4.2) show the following “Critical Path Theorem”: For \( p = (p_i)_{i \in I} \in [0,1]^I \), if \( \sum_{i \in I} p_i < 1 \), then for any type space \((T,P)\) and any event profile \( E \in \mathcal{T} \), we have

\[
P(CBP(E)) \geq 1 - \kappa^{KM}(p) (1 - P(E)),
\]

with

\[
\kappa^{KM}(p) = \frac{1 - \min_{i \in I} p_i}{1 - \sum_{i \in I} p_i},
\]

where \( CBP(E) = \prod_{i \in I} CBP_i(E) \) and \( E = \prod_{i \in I} E_i \). Note that the coefficient \( \kappa^{KM}(p) \) depends only on \( p \) and is independent of the type space \((T,P)\) or the event profile \( E \). Thus, the inequality implies that, with a fixed \( p \) satisfying \( \sum_{i \in I} p_i < 1 \), we have \( P(CBP(E)) \to 1 \) as \( P(E) \to 1 \). Combining this theorem with an argument similar to our Propositions 3 and 4, Kajii and Morris (1997a, Proposition 5.3) establish the robustness of a \( p \)-dominant equilibrium with \( \sum_{i \in I} p_i < 1 \).

We extend the Critical Path Theorem to the \( f \)-belief operator, where the assumption “\( \sum_{i \in I} p_i < 1 \)” is replaced with the existence of a monotone potential of \( f \) for \( I \). Let \( \nu: \mathcal{I} \to \mathbb{R} \) be a function that is strictly maximized at \( I \). Define

\[
\kappa(\nu) = 1 + \frac{M}{\nu(I) - \nu'},
\]

where

\[
\nu' = \max_{S \subseteq I} \nu(S),
\]

\(^\text{11}\)Our definition is in fact equivalent to strict monotone potential introduced by Oyama et al. (2008), and slightly stronger than monotone potential in the sense of Morris and Ui (2005), which requires the existence of \( \lambda' = (\lambda'_i)_{i \in I} \) with \( \lambda'_i \geq 0 \) such that \( f_i(S) \geq (\text{resp. } \leq) \lambda'_i (\nu(S) - \nu(S \cup \{i\})) \) for all \( i \in S^\ast \) (resp. \( i \in I \setminus S^\ast \)) and \( S \in \mathcal{I}_I \).

\(^\text{12}\)In fact, in each case, the statement holds also for the “only if” direction; see Morris and Ui (2005, Section 7.2).
\[ M = \max_{S \subseteq S'} (v(S) - v(S')). \]

Similarly to \( \kappa^\text{KM}(p) \), \( \kappa(v) \) depends only on the function \( v \).

**Theorem 1.** Let a binary-action game \( f \) be supermodular. Suppose that \( I \) is a monotone potential maximizer in \( f \) with a monotone potential \( v \). Then for any type space \( (T, P) \) and any event profile \( E = (E_i)_{i \in I} \in \mathcal{T} \), we have

\[
P \left( CB^f(E) \right) \geq 1 - \kappa(v) \left( 1 - P(E) \right),
\]

where \( CB^f(E) = \prod_{i \in I} CB^f_i(E) \) and \( E = \prod_{i \in I} E_i \).

Theorem 1 reduces to Kajii and Morris (1997a, Proposition 4.2) in the case of common \( p \)-belief with \( \sum_{i \in I} p_i < 1 \). To verify this, given \( p \) let the binary-game \( (f^p_i)_{i \in I} \) be as given in Example 1,

\[
f^p_i(S) = \begin{cases} 1 - p_i & \text{if } S = I \setminus \{i\}, \\ -p_i & \text{otherwise}, \end{cases}
\]

and the potential \( v \) be as given in Example 4,

\[
v(S) = \begin{cases} 1 - \sum_{i \in I} p_i & \text{if } S = I, \\ -\sum_{i \in S} p_i & \text{otherwise}, \end{cases}
\]

which is strictly maximized at \( I \) if and only if \( \sum_{i \in I} p_i < 1 \). Then, we have \( v' = 0 \) and \( M = \max_{i \in I} \sum_{j \neq i} p_j \), and thus

\[
\kappa(v) = 1 + \frac{\max_{i \in I} \sum_{j \neq i} p_j}{1 - \sum_{i \in I} p_i} = \frac{1 - \min_{i \in I} p_i}{1 - \sum_{i \in I} p_i} = \kappa^\text{KM}(p).
\]

We provide two proofs for Theorem 1. Our first proof, given in Appendix A.1, proceeds along the same line as the higher order beliefs approach of Kajii and Morris (1997a) in that we apply the \( f_i \)-belief operators to \( E \) iteratively, and evaluate the probability of the event that survives in each step. Our proof is, however, significantly simpler and more transparent, in our opinion, than the proof by Kajii and Morris (1997a), even in the case of common \( p \)-belief, i.e., when \( f = (f^p_i)_{i \in I} \). Technically, the iteration steps give us inequalities involving the prior probabilities of relevant events weighted with the values of \( f_i \)’s. We sum these inequalities across different players and different steps, utilize the common function \( v \) to “aggregate” the player-specific weights, and thus obtain the estimate of \( P \left( CB^f(E) \right) \).

Our second proof, given in Appendix A.2, adopts the potential maximization approach of Ui (2001) and Morris and Ui (2005). Fix a type space \( (T, P) \) and an event profile \( E \in \mathcal{T} \), and consider the maximization problem of the function

\[
V(F) = \sum_{t \in T} P(t)v(\{i \in I \mid t_i \in F_i\})
\]

with respect to \( F \in \mathcal{T} \) with \( F \subseteq E \). In the proof, we show that any solution \( F^\ast \) to this problem is \( f \)-evident, and the inequality \( V(F^\ast) \geq V(E) \) gives an estimate of \( P(F^\ast) \) in terms of \( P(E) \) and \( v \) only. Combined with Proposition 2, this gives the desired estimate of \( P \left( CB^f(E) \right) \).
4.3 Contagion

Theorem 1 implies that if $I$ is a monotone potential maximizer in $f$, then for any type space and any high-probability event, players have common $f$-belief about that event with high probability. In this section, we prove the generic converse (Theorem 2): if $I$ is not a monotone potential maximizer in $f$, then there generically exist a type space and a high-probability event such that players never have common $f$-belief about that event.

Our proof exploits the duality between payoffs and probabilities. We first give a characterization for the existence of a monotone potential with a fixed $\lambda$ (Lemma 1). Then, by a duality theorem, we eliminate $\lambda$ to obtain a dual condition in terms of expected payoffs, where the dual variable is a probability distribution, denoted $\mu$, over the set of finite sequences of distinct players (Lemma 2). Finally, we construct a type space à la global games (Carlsson and van Damme (1993a)) with the desired property, where the “noise” structure is determined by $\mu$ obtained above (as sketched after the statement of Theorem 2).

Let $\Gamma$ be the set of all finite sequences $\gamma = (i_1, \ldots, i_k)$ of distinct players in $I$, where $1 \leq k \leq |I|$. For each $\gamma = (i_1, \ldots, i_k) \in \Gamma$ and $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i > 0$, we define

$$F(\gamma, \lambda) = \sum_{\ell=1}^{k} \lambda_{i_\ell} f_{i_\ell}(I \setminus \{i_1, \ldots, i_\ell\}).$$

In the following lemma, we characterize the existence of a monotone potential with a fixed $\lambda$, which is a special case of Okada and Tercieux (2012, Proposition 1).

**Lemma 1.** For a binary-action game $f = (f_i)_{i \in I}$ and $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i > 0$, $f$ admits a monotone potential with $\lambda$ for $I$ if and only if $F(\gamma, \lambda) > 0$ for all $\gamma \in \Gamma$.

**Proof.** Suppose that $v : I \to \mathbb{R}$ is a monotone potential of $f$ with $\lambda$ for $I$. Fix any $\gamma = (i_1, \ldots, i_k) \in \Gamma$. Then we have

$$\lambda_{i_\ell} f_{i_\ell}(I \setminus \{i_1, \ldots, i_{\ell-1}\}) \geq v(I \setminus \{i_1, \ldots, i_{\ell-1}\}) - v(I \setminus \{i_1, \ldots, i_{\ell}\})$$

for any $\ell = 1, \ldots, k$. Summing up over all $\ell$, we have $F(\gamma, \lambda) \geq v(I) - v(I \setminus \{i_1, \ldots, i_k\}) > 0$.

Conversely, if $F(\gamma, \lambda) > 0$ for every $\gamma \in \Gamma$, then it is easy to check that

$$v(S) = \begin{cases} 0 & \text{if } S = I, \\ -\min_{\gamma=(i_1, \ldots, i_k) : v(I \setminus \{i_1, \ldots, i_k\}) = F(\gamma, \lambda) I \setminus S} F(\gamma, \lambda) & \text{otherwise} \end{cases}$$

is a monotone potential of $f$ with $\lambda$ for $I$.

Notice that the condition “$F(\gamma, \lambda) > 0$ for all $\gamma \in \Gamma$” constitutes a system of linear inequalities in $\lambda$. Thus, we can use the duality theorem, namely Farkas’ lemma, to characterize whether this system has a solution or not.

For $i \in I$, let $\Gamma_i$ be the set of sequences in $\Gamma$ that contain $i$, and for $\gamma \in \Gamma_i$, let $S(i, \gamma) \in \mathcal{I}_{-i}$ be the set of player $i$’s opponents who are not listed in $\gamma$ earlier than $i$, i.e., the set of players who are either listed in $\gamma$ later than $i$ or not listed in $\gamma$. Fix $\mu \in \Delta(\Gamma)$, let $I(\mu) = \{i \in I \mid \mu(\Gamma_i) > 0\}$, i.e., the set of players who are listed in some $\gamma$ such that $\mu(\gamma) > 0$. Let $\Delta^*(\Gamma) = \{\mu \in \Delta(\Gamma) \mid \mu(\Gamma_i) = 1 \text{ for all } i \in I(\mu)\}$. Note that $\mu \in \Delta^*(\Gamma)$ assigns positive probability only to permutations of $I(\mu)$.

**Lemma 2.** For a binary-action game $f = (f_i)_{i \in I}$,
(1) either $I$ is a monotone potential maximizer in $f$, or there exists $\mu \in \Delta(\Gamma)$ such that
\[
\sum_{S \in \mathcal{I}_i} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\}) f_i(S) \leq 0
\] (4.2)
for all $i \in I$, but not both; and

(2) in the latter case, if $f$ is supermodular, then there exists $\mu \in \Delta^*(\Gamma)$ such that (4.2) holds for all $i \in I$.

Proof. By Lemma 1, the existence of a monotone potential for $I$ is equivalent to the existence of $\lambda$ with $\lambda_i > 0$ such that $F(\gamma, \lambda) > 0$ for any $\gamma \in \Gamma$. By a variant of Farkas’ lemma, either this condition holds, or there exists $\mu \in \Delta(\Gamma)$ such that
\[
\sum_{\gamma \in \Gamma_i} \mu(\gamma) f_i(S(i, \gamma)) \leq 0
\] for all $i \in I$, but not both. The desired expression (4.2) in part (1) follows by summing up $\mu(\gamma)$ over all $\gamma$ with the same $S(i, \gamma)$. The proof of part (2) is relegated to Appendix A.3.

For generic supermodular $f$, the condition (4.2) can be strengthened to its strict version: there exists $\mu \in \Delta^*(\Gamma)$ such that
\[
\sum_{S \in \mathcal{I}_i} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\}) f_i(S) < 0
\] (4.3)
for all $i \in I$. More precisely, the set of all binary-action supermodular games that satisfy (4.3) for some $\mu \in \Delta^*(\Gamma)$ is open and dense in the set of those that satisfy (4.2) for some $\mu \in \Delta^*(\Gamma)$. The former set is open because for each $\mu$, (4.3) imposes finitely many strict inequalities; it is dense in the latter set because, for all $i \in I(\mu)$, perturbing $f_i$ by $f_i'(S) = f_i(S) - \varepsilon$ for all $S \in \mathcal{I}_i$ would make the corresponding inequality strict.

The following theorem shows that the existence of a monotone potential for $I$ is not only sufficient, but also necessary for the Critical Path Theorem for a generic choice of $f$.

**Theorem 2.** Let a binary-action game $f$ be supermodular. For generic $f$, if $I$ is not a monotone potential maximizer in $f$, then for any $\varepsilon \in (0, 1]$, there exist a type space $(T, P)$ and an event profile $E = (E_i)_{i \in I} \in T$ such that $P(E) = 1 - \varepsilon$ and $P(CB(E)) = 0$, where $E = \prod_{i \in I} E_i$ and $CB(E) = \prod_{i \in I} CB_i(E)$.

**Remark 2.** Theorem 2 in fact holds under the following condition: for any sufficiently small $\eta > 0$, there exists $\mu \in \Delta^*(\Gamma)$ such that
\[
\sum_{S \in \mathcal{I}_i} (1 - \eta)|S| \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\}) f_i(S) < 0
\] (4.4)
for all $i \in I(\mu)$. Clearly, condition (4.4) is satisfied if $f$ is generic in the sense of condition (4.3). The converse does not hold. For example, consider the two-player game

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\[\text{We need to impose some genericity condition to exclude the trivial game } f \equiv 0, \text{ where } I \text{ is not a monotone potential maximizer, and yet the Critical Path Theorem holds with } \kappa = 1.\]
The induced profile of payoff increment functions,
\[ f_i(\emptyset) = -1, \quad f_i(\{3-i\}) = 1, \]  
(4.5)
satisfies condition (4.4), but violates condition (4.3).

**Remark 3.** If (4.3) or (4.4) holds with \( \mu \in \Delta^*(\Gamma) \), then, as can be seen in the proof of Theorem 2, the conclusion of the theorem holds with “\( CB^\mu_i(\mathbf{E}) = \emptyset \) for all \( i \in I(\mu) \)” in place of “\( P(CB^\mu(\mathbf{E})) = 0 \)”.

The proof of Theorem 2, provided in Appendix A.4, is by construction: given (a family of) \( \mu \in \Delta^*(\Gamma) \) satisfying condition (4.4), we construct a desired type space \((T, P)\) along with event profile \( \mathbf{E} \). Here, we describe how it is generated by a signal structure à la global games.

Let \( \mu \in \Delta^*(\Gamma) \) be given. For simplicity, we assume that \( I(\mu) = I \). The “state of the world” \( m \) is drawn according to the geometric distribution on nonnegative integers with parameter \( \eta > 0 \). Independently of \( m \), a permutation \( \gamma = (i_1, \ldots, i_{|I|}) \) of \( I \) is drawn according to \( \mu \), and the “noise” profile \((\xi_i)_{i \in I}\) is determined by \( \xi_i = \ell \) if and only if \( i = i_\ell \). Given the realization of \((m, \gamma)\), each player \( i \) observes the “signal”
\[ t_i = m + \xi_i. \]

Thus, \( T_i = \{1, 2, \ldots\} \), and \( P \in \Delta(T) \) is the law of the random variable \( t = (t_i)_{i \in I} \). Then let \( E_i = \{|I|, |I| + 1, \ldots\} \) for all \( i \in I \). Note that \( P(E) \) is close to 1 for small \( \eta > 0 \).

The key property of this type space is that the posterior beliefs about the rankings of signals among the players are given by \( \mu \). Note that if \( m \) followed the improper uniform distribution on the nonnegative integers, then for player \( i \) with signal \( t_i \), the posterior beliefs about opponent players who receive signals no smaller than \( t_i \) would be given by
\[ P(\{j \neq i \mid t_j \geq t_i\} = S|t_i) = \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\} \]for each \( S \in \mathcal{L}_i \). As \( \eta > 0 \), the actual posterior beliefs about \( t_j \)’s are slightly skewed toward smaller values, so that \( P(\{j \neq i \mid t_j \geq t_i\} = S|t_i) \) is proportional to \((1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\}) \). Therefore, if \( \mu \) satisfies condition (4.4), then the condition for \( t_i = \tau \notin B^\mu_i(\mathbf{E}^\tau) \), where \( E_i^\tau = \{\tau, \tau + 1, \ldots\} \), is precisely (4.4). By iteration, we have \( CB^\mu_i(\mathbf{E}) = \emptyset \) for all \( i \).

Rubinstein’s (1989) email game is perhaps the earliest example of Theorem 2 for symmetric \( 2 \times 2 \) coordination games. In our terminology, he constructs a two-player type space \((T, P)\) and an event profile \( \mathbf{E} \) such that \( P(E) \) is bounded away from 0, and \( CB^\mu_i(\mathbf{E}) = CB^\mu_2(\mathbf{E}) = \emptyset \) with \( \mathbf{p} = (1/2, 1/2) \). For many-player games, in the special case where \( \mathbf{f} \) is given by
\[ f_i(S) = -f_i^p(I \setminus (S \cup \{i\})) = \begin{cases} p_i - 1 & \text{if } S = \emptyset, \\ p_i & \text{otherwise} \end{cases} \]
for some \( \mathbf{p} = (p_i)_{i \in I} \) such that \( \sum_{i \in I} p_i \leq 1 \) (in which \( \mathbf{0} \) is a \( \mathbf{p} \)-dominant equilibrium), Kajii and Morris (1997a, Lemma 5.5) present a similar construction of type space in which the conclusion of Theorem 2 holds. Also, if there exists a monotone potential of \( \mathbf{f} \) for some \( S^* \neq I \), then \( 1_S \) is selected by global games regardless of the noise structure (Frankel et al. (2003, Theorem 4)), and thus an appropriate discretization of the global-game perturbations can be used to obtain the conclusion of Theorem 2.

Note that we prove Theorem 2 under the non-existence of a monotone potential for \( I \). Recall that any supermodular game has at most one monotone potential maximizer, and a nonempty open set of games have no monotone potential maximizer (Remark 1 to Definition 4). Therefore, if some \( S^* \neq I \) is a monotone potential maximizer in \( \mathbf{f} \), then \( I \) is not a monotone potential
maximizer in \( f \), while the converse does not hold in general. That is, our assumption is strictly weaker than the existence of a monotone potential for some \( S^* \neq I \). In particular, when \( f \) has no monotone potential maximizer, our proof uses the fine details of the payoff structure of \( f \) through condition (4.4) to specify the noise structure in the construction there.

### 4.4 A Generic Characterization of Robust Equilibria

Combining Theorems 1 and 2 with Proposition 4, we provide a generic characterization of robust equilibria in binary-action supermodular games. Here, it is stated in terms of (non-)robustness of the action profile \( 1 = (1, \ldots, 1) \); it applies also to \( 0 = (0, \ldots, 0) \) by reversing the action labels 0 and 1.

**Theorem 3.** Let a binary-action game \( f \) be supermodular. Then the following results hold.

1. If \( I \) is a monotone potential maximizer in \( f \), then \( 1 \) is robust to incomplete information in \( f \).
2. For generic \( f \), if \( I \) is not a monotone potential maximizer in \( f \), then \( 1 \) is not robust to incomplete information in \( f \).

Part (1) of Theorem 3 is a special case of Proposition 2 of Morris and Ui (2005), which applies to many-action games; our contribution is to provide a new proof, the first proof of Theorem 1, as discussed in Section 4.2. The generic converse, part (2), as well as its proof is new in the literature; see Section 4.3.

The qualification “for generic \( f \)” in Theorem 3(2) is inherited from Theorem 2. It can be dispensed with at least in the two subclasses of binary-action supermodular games to be discussed in the next subsection, where we verify condition (4.4) in Remark 2 whenever \( I \) is not a monotone potential maximizer.

**Remark 4.** Haimanko and Kajii (2016) introduce the notion of approximate robustness, a weakening of the Kajii-Morris robustness. Under the stronger genericity condition (4.3), we can strengthen the conclusion of Theorem 3(2) with “\( 1 \) is not approximately robust” in place of “\( 1 \) is not approximately robust”; see Appendix A.5.

**Remark 5.** The characterization extends to action profiles other than \( 1 \) and \( 0 \); see Appendix A.6.

Our results also have an implication about global game selection. In a global game, a state of the world \( \theta \) is drawn from the real line and determines the payoffs \( u_i(a, \theta) \) of the players. Each player observes a noisy signal \( \theta + \nu \varepsilon_i \), where \( (\varepsilon_1, \ldots, \varepsilon_{|I|}) \) is a noise profile that is independent of the state \( \theta \), and \( \nu > 0 \) is a scale parameter.\(^{14}\) Under supermodularity and state-monotonicity in payoffs and the existence of dominance regions, Frankel et al. (2003) show, for many-player many-action games, that an essentially unique equilibrium survives iterative deletion of dominated strategies as \( \nu \to 0 \) (limit uniqueness), while the limit equilibrium may depend on the joint distribution of noise terms (noise dependence). An action profile \( a^* \) is noise-independent global game selection at \( u(\cdot, \theta) \) if the limit equilibrium plays \( a^* \) at \( \theta \) independent of the noise distribution.

By appropriately modifying the discrete type spaces in the proof of Theorem 2 to the corresponding global games with continuous states and noises, we can show that if \( I \) is not a monotone potential maximizer, then \( 1 \) is not noise-independent global game selection. Together with Theorem 3 (and Corollary A.1 in Appendix A.6), this leads to the following characterization.

---

\(^{14}\) We follow Carlsson and van Damme (1993a) to allow correlation in noise terms among players. The results of Frankel et al. (2003) continue to hold even without their assumption of independence among noise terms.
Proposition 5. Let a binary-action game \( f \) be supermodular. For generic \( f \), \( a^* \) is noise-independent global game selection at \( f \) if and only if \( a^* \) is robust to incomplete information in \( f \).

The “if” direction holds for many-player many-action supermodular games (Basteck et al. (2013), Oury and Terceux (2007)). The “only if” direction, on the other hand, fails with more than two actions (Basteck and Daniëls (2011), Oyama and Takahashi (2011)).

4.5 Examples

To illustrate the results in this section, we consider two classes of binary-action supermodular games, unanimity games and games with cyclic symmetry. In particular, for each class, we determine the robust equilibrium in terms of the payoff parameters.

Example 7 (Unanimity). Recall the class of binary-action unanimity games discussed in Examples 3 and 6. In this class of games, the existence of a monotone potential and its implications have been studied in Morris and Ui (2005, Section 7.2), Oyama et al. (2008, Section 4.3.4), Oyama et al. (2011, Section 4), and Okada and Terceux (2012, Example 1). The following proposition establishes a full characterization of robust equilibria, where the condition is given by the pairwise Nash products.\(^{15}\) The proof is provided in Appendix A.7.

Proposition 6. In the binary-action unanimity game \( f = (f_i^{y,z_i})_{i \in I} \),

1. if \( z_iz_j > y_iz_j \) for all \( i, j \in I \) with \( i \neq j \), then \( 1 \) is a unique robust equilibrium;

2. if \( z_iz_j < y_iz_j \) for all \( i, j \in I \) with \( i \neq j \), then \( 0 \) is a unique robust equilibrium; and

3. otherwise, there is no robust equilibrium.

As a parametric illustration, consider a three-player example with payoffs \( y_1 = 6 + c, y_2 = y_3 = 1 \), and \( z_1 = z_2 = z_3 = 2 \), where \( c > -6 \). In this case, by Proposition 6, if \( c < -2 \), then \( 1 \) is a monotone potential maximizer, hence a robust equilibrium, and if \( c \geq -2 \), then the game has no monotone potential maximizer, hence no robust equilibrium.\(^{16}\)

Carlsson (1989) has shown that the global game selection may depend on the noise structure for the class of three-player binary-action unanimity games. Namely, he constructed a noise structure for which \( 1 \) (resp. \( 0 \)) is selected if \( z_1z_2z_3 > (\text{resp. } <) y_1y_2(y_3) \) and another noise structure for which \( 1 \) (resp. \( 0 \)) is selected if \( (z_1)^2z_2z_3 > (\text{resp. } <) (y_1)^2y_2y_3 \), and therefore, the selected equilibrium depends on the noise structure if these inequalities hold with opposite directions. Note that this last condition is stronger than that for Proposition 6(3). With the above parameterization, for example, Carlsson’s sufficient condition for noise dependence is that \( -2 < c < 10 \), while there is noise dependence also for \( c \geq 10 \) according to our (generic) characterization by Propositions 5 and 6.

Example 8 (Cyclic Symmetry). We consider three-player games with cyclic symmetry, studied in Oyama et al. (2011, Example 2) in a context of population game dynamics.\(^{17}\) Let \( I = \{1, 2, 3\} \), and let the payoffs be given by

\[
\begin{array}{c|ccc}
0 & 0 & 1 & 0 \\
\hline
0 & a, a, a & 0, 0, b & 0, 0, c, 0 \\
0 & 0, 0, 0 & c, 0, 0 & 0, 0, c, d, d \\
0 & 0, 0 & 1 & 1 \\
\end{array}
\]

\(^{15}\)This condition differs from the comparison of the \( |I| \)-player Nash products, \( \prod_{i \in I} z_i \geq \prod_{i \in I} y_i \). Namely, the condition in (1) (resp. (2)) in Proposition 6 implies \( \prod_{i \in I} z_i > (\text{resp. } <) \prod_{i \in I} y_i \), but not vice versa.

\(^{16}\)Morris and Ui (2005, Section 7.2) have used these payoff values with \( c = 0 \) as an example where no monotone potential maximizer exists, but they did not discuss its implication to the non-existence of robust equilibria.

\(^{17}\)For a similar game, Iijima (2015, Example 4) demonstrates noise dependence of the global game selection.
with \(a > b > 0\) and \(d > c > 0\). The induced payoff increment function \(f_i\) is then given by

\[
 f_i(\emptyset) = -a, \quad f_i(\{i - 1\}) = -b, \quad f_i(\{i + 1\}) = c, \quad f_i(\{i - 1, i + 1\}) = d, \quad (4.6)
\]

where \(i - 1\) and \(i + 1\) are understood modulo 3. This game \(f = (f_1, f_2, f_3)\) does not admit a potential since there is a better response cycle. We claim that \(I\) is a monotone potential maximizer in \(f\) if and only if \(a + b - d < 0\). To verify this, suppose first that \(a + b - d < 0\). Then the condition in Lemma 1 is satisfied with \(\lambda_1 = \lambda_2 = \lambda_3 = 1\), so that \(I\) is a monotone potential in \(f\), where the function \(v\) given by

\[
 v(\emptyset) = a + b - d, \quad v(\{1\}) = v(\{2\}) = v(\{3\}) = b - d,
\]

\[
 v(\{1, 2\}) = v(\{2, 3\}) = v(\{3, 1\}) = -d, \quad v(I) = 0
\]

is a monotone potential for \(I\). Therefore, Theorem 1 applies, and hence 1 is robust in \(f\) by Proposition 4.

Next suppose conversely that \(a + b - d \geq 0\). Then the probability distribution \(\mu \in \Delta(\Gamma)\) given by \(\mu(3, 2, 1) = \mu(2, 1, 3) = \mu(1, 3, 2) = 1/3\) is a solution to the system of inequalities in Lemma 2, where the left side of the inequality equals \(-(a + b - d)/3\) for each \(i\). This implies that \(I\) is not a monotone potential maximizer in \(f\). Indeed, this \(\mu\) satisfies condition (4.4) in Remark 2, i.e., for any \(\eta \in (0, 1]\), we have

\[
 \frac{1}{3}(-a) + (1 - \eta)\frac{1}{3}(-b) + (1 - \eta)^2\frac{1}{3}d = \frac{(1 - \eta)(-a - b + d)}{3} - \eta[a + (1 - \eta)d] < 0 \quad (4.7)
\]

since \(-a - b + d \leq 0\) and \(a, d > 0\). Therefore, by Remark 3 to Theorem 2, for any \(\epsilon \in (0, 1]\), there exist a type space \((T, P)\) and an event profile \(E \in T\) such that \(P(E) = 1 - \epsilon\) and \(CB^f_i(E) = \emptyset\) for all \(i \in I\). By Proposition 3(2), one can attach a profile \(u\) of payoff functions so that \((T, P, u)\) is dominance-solvable with action profile \(0\) played everywhere, and hence, 1 is not robust in \(f\).

Here, let us use this example to elaborate the proof of Theorem 2 by explicitly describing the constructed type space \((T, P)\) and event profile \(E\). Let \(\mu \in \Delta(\Gamma)\) be as above. Fix any \(\epsilon \in (0, 1]\), and let \(\eta = 1 - \sqrt{1 - \epsilon} \in (0, 1)\). The random variable \(m\) is drawn from \(\{0, 1, 2, \ldots\}\) with probability \(\eta(1 - \eta)^m\), while the profile \((\xi_1, \xi_2, \xi_3)\) takes values \((3, 2, 1), (2, 1, 3), (1, 3, 2)\) with probability \(1/3\) each. The type of each player is then \(t_i = m + \xi_i\). In the type space \((T, P)\) so generated, \(T_1 = \{1, 2, \ldots\}\), and \(P((m + 3, m + 2, m + 1)) = P((m + 2, m + 1, m + 3)) = P((m + 1, m + 3, m + 2)) = (1 - \eta)\eta^m/3, m = 0, 1, 2, \ldots\).

Now let \(E_i = \{3, 4, \ldots\}\) for each \(i\), where \(P(E) = \sum_{m \geq 2}(1 - \eta)\eta^m/3 = (1 - \eta)^2 = 1 - \epsilon\). We claim that \(B^f_i(E) \subset \{4, 5, \ldots\}\). To see this, consider player 1 and let \(t_1 = 3\). This type assigns posterior probabilities proportional to \(1, 1 - \eta\), and \((1 - \eta)^2\) to the opponents’ types \((t_2, t_3) = (2, 1), (2, 4), (5, 4)\), respectively. Therefore, we have

\[
 E[f_1(S_{E_{-1}}(\cdot))|t_1 = 3] = P((2, 1)|3)f_1(\emptyset) + P((2, 4)|3)f_1(\{3\}) + P((5, 4)|3)f_1(\{2, 3\}) \neq -a + (1 - \eta)(-b) + (1 - \eta)^2d < 0,
\]

where the last inequality follows from (4.7). This implies that \(3 \notin B^f_i(E)\). Similarly, we inductively have \(B^f_{i,n}(E) \subset \{n + 3, n + 4, \ldots\}\), and hence, \(CB^f_i(E) = \cap_{n=0}^{\infty} B^f_{i,n}(E) = \emptyset\).

By symmetry, we have the following.

\[\text{The stronger sufficient condition (4.3) holds if and only if } a + b - d > 0.\]

\[\text{This type space is essentially the same as the information structure constructed for the cyclic matching pennies game in Kajii and Morris (1997a, Example 3.1). More precisely, our type space consists of three belief closed subspaces (such as } (T', P') \text{ with } T'_1 = \{3, 6, \ldots\}, T'_2 = \{2, 5, \ldots\}, T'_3 = \{1, 4, \ldots\}, \text{ and } P' = P((T')), \text{ each of which is isomorphic to Kajii and Morris' information structure.}\]
Proposition 7. In the binary-action game $f$ given by (4.6),

(1) if $d - a > b$, then 1 is a unique robust equilibrium;
(2) if $d - a < -c$, then 0 is a unique robust equilibrium; and
(3) if $-c \leq d - a \leq b$, then there is no robust equilibrium.

5 Application: Asymmetric Currency Attack Game

In this section, we consider a model of speculative currency attacks with possibly asymmetric traders, as studied in the global games of Morris and Shin (1998) (symmetric traders) and Corsetti et al. (2004) (one large trader and symmetric small traders). We are interested in characterizing when this game has a robust equilibrium, which, in light of Theorem 3, amounts to identifying the condition for the existence of a monotone potential. As we studied in Section 4.3, this in turn reduces to examining the solvability of a finite system of linear inequalities. In the case of one large trader and symmetric small traders, we explicitly express the solvability condition in terms of the exogenous parameters, which coincides with the limit thresholds as the noise vanishes in the analysis of Corsetti et al. (2004). With this result in hand, we discuss how the noise-(in)dependent selection results by Corsetti et al. (2004) can be understood from our perspective, as well as compare the methodology between theirs and ours.

There are finitely many heterogeneous traders $i \in I$. These traders simultaneously choose whether to attack the currency of a certain country by short-selling it for dollars ($a_i = 1$) or not ($a_i = 0$), where each trader $i$ has a limit of the amount $L_i > 0$ of credit available to take a short position. Then the central bank abandons the currency peg if and only if the total amount of the currency short-sold is larger than or equal to $\Theta$, where $\Theta$ represents the strength of fundamentals. We denote by $S$ the collection of sets $S \in I$ of traders such that $\sum_{i \in S} L_i \geq \Theta$. Thus an action profile $a = (a_i)_{i \in I}$ leads to a successful attack if and only if $\{i \in I \mid a_i = 1\} \in S$.

Note that $S$ is monotone, i.e., if $S \subseteq S'$, then $S' \in S$. Short-selling costs $c_i$, $0 < c_i < 1$, for each trader $i$. If the attack is successful, each trader who joins the currency attack receives a payoff normalized to 1. Thus trader $i$’s net payoff from attack ($a_i = 1$) is given by

$$f_i(S) = \begin{cases} 1 - c_i & \text{if } S \cup \{i\} \in S, \\ -c_i & \text{if } S \cup \{i\} \notin S, \end{cases}$$

where $S = \{j \in I \setminus \{i\} \mid a_j = 1\}$, while that from not attack ($a_i = 0$) is normalized to 0.

By the monotonicity of $S$, the currency attack game $f = (f_i)_{i \in I}$ is supermodular. Note that if $\max_{i \in I} L_i < \Theta \leq \sum_{i \in I} L_i$, then this game has two pure-action Nash equilibria: “all attack” ($S = I$) and “no one attacks” ($S = \emptyset$). Otherwise, if $\Theta = \max_{i \in I} L_i$, then attack is iteratively dominant for all traders, and if $\Theta > \sum_{i \in I} L_i$, then not attack is dominant for all traders.

Let $F$ be the collection of sets $F \in I$ such that $F \notin S$ and $F \cup \{i\} \in S$ for some $i \in I \setminus F$. That is, $F$ is the collection of sets of traders whose attacks fail by “narrow margins.”

The following characterizes the condition under which the game $f$ has a monotone potential.\(^{20}\)

\(^{20}\)If $c_i = c$ for all $i \in I$, then the inequalities (5.1) in Lemma 3 can be interpreted as $\lambda$ belonging to the strict core of a coalitional game: “all attack” is a monotone potential maximizer if and only if the strict core is nonempty in the coalitional game $w^A : I \rightarrow \mathbb{R}$ given by $w^A(I) = 1$, $w^A(S) = c$ for all $S \in I$ such that $I \setminus S \in F$, and $w^A(S) = 0$ otherwise; “no one attacks” is a monotone potential maximizer if and only if the strict core is nonempty in the coalitional game $w^N : I \rightarrow \mathbb{R}$ given by $w^N(I) = 1$, $w^N(S) = 1 - c$ for all $S \in F$, and $w^N(S) = 0$ otherwise.
Lemma 3. Suppose that $\max_{i \in I} L_i < \Theta \leq \sum_{i \in I} L_i$. The currency attack game admits a monotone potential with $\lambda = (\lambda_i)_{i \in I}$ for “all attack” (resp. “no one attacks”) if and only if

$$\sum_{i \in S} \lambda_i > (\text{resp. } <) \sum_{i \in I} c_i \lambda_i$$  \hspace{1cm} (5.1)

for all $S \in \mathcal{I}$ such that $I \setminus S \in \mathcal{F}$.

Proof. Follows immediately from Lemma 1.

In the rest of this section, as in Corsetti et al. (2004), we focus on the case where there is a large trader (player 1), and the others (players 2, . . . , $|I|$) are homogeneous small traders. We set $L_1 = L \geq 1$, $L_2 = \cdots = L_{|I|} = 1$, and $c_1 = c_2 = \cdots = c_{|I|} = c$. In this case, “all attack” and “no one attacks” are Nash equilibria when $L < \Theta \leq L + |I| - 1$. To ease our computation, we assume that $L$ and $\Theta$ are positive integers.

The condition in Lemma 3 defines a system of linear inequalities in $\lambda$. In the next lemma, by examining the solvability of this system, we identify the conditions for the existence of a monotone potential in terms of the parameter $\Theta$. Define $\Theta^A$ and $\Theta^N$ by the following:

$$\Theta^A = \begin{cases} L + (|I| - 1) \left(1 - \frac{c}{L + |I| - 1}\right) & \text{if } \frac{L}{L + |I| - 1} > c, \\ (L + |I| - 1)(1 - c) & \text{if } \frac{L}{L + |I| - 1} \leq c \end{cases}$$ \hspace{1cm} (5.2)

if $c \leq 1/2$, and

$$\Theta^A = \begin{cases} L & \text{if } \frac{L}{L + |I| - 1} > 1 - c, \\ (L + |I| - 1)(1 - c) & \text{if } \frac{L}{L + |I| - 1} \leq 1 - c \end{cases}$$ \hspace{1cm} (5.3)

if $c > 1/2$, and

$$\Theta^N = L + |I|(1 - c) + 1.$$  \hspace{1cm} (5.4)

One can verify that $L \leq \lfloor \Theta^A \rfloor < \lfloor \Theta^N \rfloor \leq L + |I|$. The following lemma shows that $\lfloor \Theta^A \rfloor$ (resp. $\lfloor \Theta^N \rfloor$) is the largest (resp. smallest) value of $\Theta$ such that “all attack” (resp. “no one attacks”) is a monotone potential maximizer.

Lemma 4. In the currency attack game with one large trader and $|I| - 1$ small traders, “all attack” (resp. “no one attacks”) is a monotone potential maximizer if and only if $\Theta \leq \lfloor \Theta^A \rfloor$ (resp. $\Theta \geq \lfloor \Theta^N \rfloor$).

Proof. If $\Theta \leq L$, then “all attack” is an iteratively dominant equilibrium and hence a monotone potential maximizer; if $\Theta > L + |I| - 1$, then “no one attacks” is a dominant equilibrium and hence a monotone potential maximizer.

For the rest of the proof, we consider the case where $L < \Theta \leq L + |I| - 1$. In this case, the currency attack game admits a monotone potential if and only if there exists $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i > 0$ that satisfies the condition (5.1) in Lemma 3. We want to eliminate $\lambda$ from (5.1).

The symmetry of small traders allows us to simplify the condition (5.1). First, $\mathcal{F}$ is the collection of two kinds of sets: sets consisting of $k$ small traders with $k = \Theta - L, \ldots, \min(|I|, \Theta) - 1$, and sets consisting of the large trader and $\Theta - L - 1$ small traders. Second, we can assume without loss of generality that the weights $\lambda_2, \ldots, \lambda_{|I|}$ among small traders are symmetric. We normalize weights so that $\lambda_1 = \lambda$ and $\lambda_2 = \cdots = \lambda_{|I|} = 1$. Thus, the system of linear inequalities defined by (5.1) reduces to the following simpler system with one variable $\lambda$:

$$\lambda + |I| - 1 - k \geq c(\lambda + |I| - 1) \quad (k = \Theta - L, \ldots, \min(|I|, \Theta) - 1),$$  \hspace{1cm} (5.5)

\footnote{For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer that is smaller than or equal to $x$; $\lceil x \rceil$ denotes the smallest integer that is larger than or equal to $x$.}
Figure 1: Robustness and contagion in the currency attack game

\[ L + |I| - \Theta \geq c(\lambda + |I| - 1). \] (5.6)

Consider the condition for the existence of a monotone potential for “all attack”. This is equivalent to the existence of \( \lambda > 0 \) that satisfies the system (5.5) with \( k = \min(|I|, \Theta) - 1 \) and (5.6) (and with “\( > \)”), which in turn is equivalent to

\[
\max\left(0, \frac{\min(|I|, \Theta) - c - |I|}{1-c}\right) < \frac{L + |I| - \Theta - |I| + 1}{c}.
\]

Thus we have either \( |I| \leq \Theta \) and \( \Theta < L + (|I| - 1) \left(1 - \frac{c}{1-c}\right) + 1\) or \( \Theta < |I| \) and \( \Theta < (L + |I| - 1)(1-c) + 1\). Combining two cases, we obtain the desired condition.

Similarly, consider the condition for the existence of a monotone potential for “no one attacks”. This is equivalent to the existence of \( \lambda > 0 \) that satisfies the system (5.5) with \( k = \Theta - L \) and (5.6) (and with “\( < \)”), which in turn is equivalent to

\[
\max\left(0, \frac{L + |I| - \Theta - |I| + 1}{c}\right) < \frac{\Theta - L}{1-c} - |I| + 1.
\]

Thus we have \( \Theta > L + |I|(1-c) \).

We thus have the following.

**Proposition 8.** In the currency attack game with one large trader and \(|I| - 1\) small traders,

1. if \( \Theta \leq |\Theta^A| \), then “all attack” is a unique robust equilibrium;
2. if \( \Theta \geq |\Theta^N| \), then “no one attacks” is a unique robust equilibrium; and
3. if \( |\Theta^A| < \Theta < |\Theta^N| \), then there is no robust equilibrium.

Given Lemma 4, it follows from Theorem 3(1) that if \( \Theta \leq |\Theta^A| \), then “all attack” \( (a = 1) \) is robust while if \( \Theta \geq |\Theta^N| \), then “no one attacks” \( (a = 0) \) is robust. On the other hand, if \( \Theta < |\Theta^N| \), then one can verify that condition (4.4) in Remark 2 is satisfied, and hence by Theorem 2 and Proposition 3(2) together with Remark 3, “all attack” is contagious, i.e., for any \( \varepsilon > 0 \), there exists a dominance-solvable \( \varepsilon \)-elaboration in which “attack” is played by every type of every player. Similarly, if \( \Theta > |\Theta^A| \), then “no one attack” is contagious. The proposition thus follows; see Figure 1.

Here we elucidate the last point above. First, we consider the case where \( \Theta < |\Theta^N| \). To simplify the argument, suppose that \( \Theta^N \) is not an integer (so that \( |\Theta^N| < \Theta^N \)). As in the proof of Theorem 2 (as well as those of Propositions 6 and 7), we consider the following \( \varepsilon \)-elaboration. Each trader \( i \) receives a signal \( t_i = m + \xi_i \), where \( m \) follows the geometric distribution over nonnegative integers with parameter \( \eta = 1 - (1-\varepsilon)^{1/(2|I|-2)} \). Independently of \( m \),

- \( \text{N1} \) the noise term \( \xi_1 \) for the large trader is either \( |\Theta^N| - L - 1 \) or \( |\Theta^N| - L \), and
- \( \text{N2} \) given \( \xi_1 = k \), the noise profile \( \{\xi_2, \ldots, \xi_{|I|}\} \) for the small traders is uniformly drawn from all possible permutations of \( \{1, \ldots, k-1, k+1, \ldots, |I|\} \),
where
\[ \mu^N = \Pr(\xi_1 = \lfloor \Theta^N \rfloor - L) \]
is to be specified below to guarantee "all attack" to be contagious. If \( m \leq |I| - 2 \), then economic fundamentals are very weak, and the currency peg is abandoned even if it is attacked by one trader; if \( m \geq |I| - 1 \), then the payoffs are given by the original currency attack game \( f \), i.e., the peg is abandoned if and only if the amount of the currency short-sold exceeds \( \Theta \). In this elaboration, a trader for whom \( t_i \geq 2|I| - 1 \) is certain that \( m \geq |I| - 1 \). Therefore, every trader is certain that the payoffs are given by \( f \) with probability \( \Pr(t_i \geq 2|I| - 1 \text{ for all } i \in I) = \Pr(m \geq 2|I| - 2) = 1 - \varepsilon \).

We demonstrate that with an appropriate choice of \( \mu^N \), this elaboration is dominance-solvable, and the surviving strategy plays "attack" everywhere. First, if \( t_i \leq |I| - 1 \), then trader \( i \) is certain that \( m \leq |I| - 2 \), and hence plays "attack" as a dominant action. Then, as an induction hypothesis, for each \( n \geq |I| \), assume that every trader \( i \) attacks if \( t_i \leq n \). Suppose that the large trader receives \( t_1 = n + 1 \). If \( \xi_1 = \lfloor \Theta^N \rfloor - L \), then at least \( \lfloor \Theta^N \rfloor - L - 1 \) small traders attack, so that the large trader's attack will succeed (since \( \lfloor \Theta^N \rfloor - L - 1 + L \geq \Theta \)). The large trader believes this event to occur with probability
\[ \Pr(\xi_1 = \lfloor \Theta^N \rfloor - L | t_1 = n + 1) = \frac{\mu^N}{\mu^N + (1 - \eta)(1 - \mu^N)} \approx \mu^N \text{ if } \eta \approx 0. \]
Therefore, for small enough \( \eta \), he has a strict incentive to attack if
\[ \mu^N > c. \tag{5.7} \]
Suppose next that a small trader \( i \neq 1 \) receives \( t_i = n + 1 \). If \( \xi_i = \xi_1 \geq \lfloor \Theta^N \rfloor - L - 1 \), then the large trader and at least \( \lfloor \Theta^N \rfloor - L - 2 \) other small traders attack, so that trader \( i \)'s attack will succeed (since \( L + (\lfloor \Theta^N \rfloor - L - 2) + 1 \geq \Theta \)). Since trader \( i \) believes this event to occur with probability
\[ \Pr(\xi_i > \xi_1 | t_i = n + 1) = \frac{\sum_{k > \lfloor \Theta^N \rfloor - L}(1 - \eta)^{-k}\mu^N + \sum_{k > \lfloor \Theta^N \rfloor - L - 1}(1 - \eta)^{-k}(1 - \mu^N)}{\sum_{k > \lfloor \Theta^N \rfloor - L}(1 - \eta)^{-k}\mu^N + \sum_{k > \lfloor \Theta^N \rfloor - L - 1}(1 - \eta)^{-k}(1 - \mu^N)} \approx \frac{L + |I| - \lfloor \Theta^N \rfloor}{|I| - 1} \mu^N + \frac{L + |I| - \lfloor \Theta^N \rfloor + 1}{|I| - 1} (1 - \mu^N) \text{ if } \eta \approx 0, \]
it follows that he has a strict incentive to attack for small enough \( \eta \) if
\[ \frac{L + |I| - \lfloor \Theta^N \rfloor}{|I| - 1} \mu^N + \frac{L + |I| - \lfloor \Theta^N \rfloor + 1}{|I| - 1} (1 - \mu^N) > c. \tag{5.8} \]
By induction, all the traders of all types play "attack" if the conditions (5.7)-(5.8) are satisfied, which holds true if and only if
\[ c < \mu^N < L + |I| - \lfloor \Theta^N \rfloor - (|I| - 1)c + 1. \]
Such a \( \mu^N \) exists in \([0, 1]\) since \( 0 < c < 1 \) and \( \lfloor \Theta^N \rfloor < \Theta^N = L + |I|(1 - c) + 1 \).

Second, for the case of \( \Theta > \lfloor \Theta^A \rfloor \), consider the following \( \varepsilon \)-elaboration. Again, each trader \( i \) receives a signal \( t_i = m + \xi_i \), where \( m \) follows the geometric distribution over nonnegative integers with parameter \( \eta = 1 - (1 - \varepsilon)^{1/(2|I| - 2)} \), while independently of \( m \),
(A1) the noise term \( \xi_1 \) for the large trader is either 1 or \(|I|\), and
(A2) given \( \xi_1 = k \), the noise profile \( (\xi_2, \ldots, \xi_{|I|}) \) for the small traders is uniformly drawn from all possible permutations of \( \{1, \ldots, k - 1, k + 1, \ldots, |I|\} \).
If \( m \leq |I| - 2 \), then fundamentals are strong enough that the currency peg is kept even when attacked by all the traders; if \( m \geq |I| - 1 \), then the peg is abandoned if and only if the short sales exceed \( \Theta \). With this construction, a similar argument as above shows that with an appropriate choice of the value of \( \mu^A = \Pr(\xi_1 = 1) \), the game is dominance-solvable, and the surviving strategy plays “not attack” everywhere.

Let us discuss the analysis of Corsetti et al. (2004) from our perspective. They consider global game perturbations with a large trader of size \( \ell \) and a continuum of small traders of total size \( 1 - \ell \). The cost of attack is \( c \), and the attack is successful if and only if the total size of attackers is larger than or equal to \( \Theta \), in which case each attacker gains payoff 1.\(^{22}\) The large and small traders observe \( \theta + r \eta \) and \( \theta + \sigma \epsilon_i \), respectively, where \( \theta \) is distributed according to the (improper) uniform prior on \( \mathbb{R} \), and \( \eta \) and \( \epsilon_i \) are random variables with smooth symmetric densities, independent across all traders and identical among small traders. They show that the game has a unique equilibrium for each pair \((\sigma, \tau)\) of precision levels, which consists of threshold strategies of the large and the small traders, characterize the common limit \( \bar{\theta}(r) \) of the thresholds as \( \sigma, \tau \to 0 \) with \( \sigma/\tau \to r \), and show that \( \bar{\theta}(r) \) is increasing in \( r \). Therefore, for sufficiently small \( \sigma \) and \( \tau \), (1) if \( \theta < \bar{\theta}(0) := \lim_{r \to 0} \bar{\theta}(r) \) (resp. (2) if \( \theta > \bar{\theta}(\infty) := \lim_{r \to \infty} \bar{\theta}(r) \)), then all traders attack (resp. no trader attacks) independently of the noise structure within the class of global game perturbations; and (3) if \( \bar{\theta}(0) < \theta < \bar{\theta}(\infty) \), then whether all traders attack or no trader attacks depends on the noise structure through the relative precision \( r \).

Note that our setting is an appropriate discretization of the continuous setting of Corsetti et al. (2004) up to normalization. In particular, our thresholds \( \Theta^A \) and \( \Theta^N \) correspond to \( \bar{\theta}(0) \) and \( \bar{\theta}(\infty) \), respectively. Indeed, given \( \Theta^A \) and \( \Theta^N \) as defined by (5.2)–(5.3) and (5.4), one can check that as \( L, |I| \to \infty \) with \( L/(L + |I| - 1) \to \ell \), the normalized values \( \Theta^A/(L + |I| - 1) \) and \( \Theta^N/(L + |I| - 1) \) converge to \( \bar{\theta}(0) \) and \( \bar{\theta}(\infty) \), respectively.\(^{23}\) Figure 2 depicts \( \Theta^A/(L + |I| - 1) \) and \( \Theta^N/(L + |I| - 1) \) as functions of the relative size \( L/(L + |I| - 1) \) of the large trader, where \( L \) and \( |I| \) are varied with the total size \( L + |I| - 1 \) held fixed to 1000, for \( c = 0.4 \) (panel (a)) and \( c = 0.6 \) (panel (b)). Observe that panel (a) well approximates Figure 3 in Corsetti et al. (2004).

Our robustness result strengthens Corsetti et al.’s (2004) noise independence result. In light of our Proposition 8, (1) if \( \Theta \leq [\Theta^A] \) (resp. (2) if \( \Theta \geq [\Theta^N] \)), then “all attack” (resp. “no one attacks”) is a monotone potential maximizer (Lemma 4) and hence is robust, not only to the specific class of global game perturbations as considered in Corsetti et al. (2004), but also to all incomplete information elaborations.

Our non-robustness result, on the other hand, corresponds to Corsetti et al.’s (2004) noise dependence result, and in particular, our approach offers new insight to understand the role played by the relative precision \( \sigma/\tau \) in Corsetti et al. (2004). Specifically, in the case (3) where \( [\Theta^A] < \Theta < [\Theta^N] \), neither of “all attack” and “no one attacks” is a monotone potential

\(^{22}\)Corsetti et al. (2004) use the notations \( \lambda \) and \( t \) for the size of the large trader and the cost of attack, respectively.

\(^{23}\)There is an error in the expression of \( \bar{\theta}(0) \) in Corsetti et al. (2004). On page 100, they claim (in our notation) that regardless of \( c \leq 1/2 \) or \( c > 1/2 \),

\[
\bar{\theta}(0) = \begin{cases} 
\ell + (1 - \ell) \left( 1 - \frac{c}{1 + c} \right) & \text{if } \ell > c, \\
1 - c & \text{if } \ell \leq c.
\end{cases}
\]

This would imply \( \bar{\theta}(0) < \ell \) if \( c > 1/2 \) and \( \ell > 1 - c \), contradicting \( \bar{\theta}(r) \geq \ell \) for all \( r \) (which follows from their equation (4.2)). The correct expression of \( \bar{\theta}(0) \) is

\[
\bar{\theta}(0) = \begin{cases} 
\ell & \text{if } \ell > 1 - c, \\
1 - c & \text{if } \ell \leq 1 - c.
\end{cases}
\]

for \( c > 1/2 \). Compare our equation (5.3).
maximizer, and thus, there are dominance-solvable elaborations, one in which “attack” is played everywhere and another in which “not attack” is played everywhere. Indeed, following the proof of Theorem 2, we have explicitly constructed such elaborations as in (N1)–(N2) and (A1)–(A2), respectively. While there are apparent differences between our and Corsetti et al.’s perturbations (such as discrete or continuous player and type spaces), there is an essential parallelism: in the perturbation given by (N1)–(N2), the ranking of the large trader’s signal among all traders varies only by 1, which corresponds to Corsetti et al.’s perturbation with $\sigma/\tau \rightarrow \infty$ (i.e., the large trader’s signal is arbitrarily more precise than the small traders’), whereas in the perturbation given by (A1)–(A2), it takes the extreme values, 1 or $|I|$, which corresponds to Corsetti et al.’s with $\sigma/\tau \rightarrow 0$ (i.e., the large trader’s signal is arbitrarily less precise than the small traders’).

Finally, we emphasize the simplicity of our methodology. Corsetti et al. (2004) consider their global game for fixed precision levels $(\sigma, \tau)$, characterize the equilibrium threshold strategies by a system of nonlinear equations, and then take limit operations as $\sigma, \tau \rightarrow 0$ or $\infty$ to examine the noise dependence of the global game selection. In contrast, for each $\Theta$, we directly analyze the robustness of equilibria to small perturbations. By our Theorem 3, it is equivalent to the existence of a monotone potential, which boils down to the solvability of a system of two linear inequalities in one variable $\lambda$; recall the argument around (5.5)–(5.6) in the proof of Lemma 4.

6 Conclusion

For the class of binary-action supermodular games, the present paper has studied the robustness of an equilibrium to incomplete information in the sense of Kajii and Morris (1997a). Using the generalized belief operator (Morris and Shin (2007), Morris et al. (2016)), we proved a generalized version of the Critical Path Theorem of Kajii and Morris (1997a), which provides a non-trivial lower bound on the prior probability of a common belief event, uniform over type spaces, when the underlying game has a monotone potential (Morris and Ui (2005)). Conversely, when the game has no monotone potential, we constructed a type space with an arbitrarily high probability event in which players never have common belief about that event. Our construction is based on a novel application of a duality theorem applied to the system of linear inequalities that characterizes the existence of a monotone potential. Combining these results, we established a generic equivalence between robustness and monotone potential maximization for
binary-action supermodular games. Finally, we discussed how the methodology developed in this paper enriches our understanding of global game equilibrium selection. In particular, for the asymmetric global game of Corsetti et al. (2004), we demonstrated that the solvability condition for a simple system of linear inequalities characterizes the noise-(in)dependent selection.

The generalized belief operator and the Critical Path Theorem allow us to analyze the properties of equilibria across all common prior type spaces at once. In this study, we used these tools to address the particular question of robustness to incomplete information. Nonetheless, our theory has broader applicability. An example is information design with adversarial equilibrium selection (see Bergemann and Morris (2019, Section 7) and references therein). Specifically, consider the information design problem where the information designer chooses an \(\varepsilon\)-elaboration of \(f\) (as \(\varepsilon \to 0\)) to minimize the probability of action profile \(1\) to be played in the worst-case equilibrium thereof, i.e.,

\[
W^\ast = \lim_{\varepsilon \to 0} \inf_{(T, P, u)} \max_{\sigma^\ast} \sum_{t \in T} P(t) \prod_{i \in I} \sigma^\ast_i(1|t_i),
\]

where the \(\inf\) is taken over all \(\varepsilon\)-elaborations \((T, P, u)\) of \(f\) and the \(\max\) over all Bayesian Nash equilibria \(\sigma^\ast\) of \((T, P, u)\). By the definition of robustness, \(W^\ast = 1\) if and only if \(1\) is robust in \(f\). Our results imply that for generic \(f\), \(W^\ast\) equals either 1 or 0, and more specifically, we have \(W = 1\) (resp. \(W = 0\)) if \(I\) is (resp. is not) a monotone potential maximizer in \(f\). Extending this information design problem with state dependent payoffs and general objective functions is yet to be seen in the future research.

Appendix

A.1 Proof of Theorem 1: Higher Order Beliefs Approach

Fix a binary-action supermodular game \(f\) that admits a monotone potential \(v\). Fix a type space \((T, P)\) and an event profile \(E = (E_i)_{i \in I} \in T\) with \(E = \prod_{i \in I} E_i\). We consider a sequence of events obtained by applying the \(B_i^{f_i}\) operators to \(E\) sequentially for one player at a time. That is, we define \(E_i^1 = E_i\), \(E_i^n = (E_i^n)_{i \in I}\), and

\[
E_i^{n+1} = \begin{cases} B_i^{f_i}(E_i^n) & \text{if } n \equiv i \pmod{|I|} \\ E_i^n & \text{if } n \not\equiv i \pmod{|I|} \end{cases}
\]

for \(n = 1, 2, \ldots\). By convention, let \(E_i^0 = T_i\). Observe that for all \(n \in \mathbb{N} = \{0, 1, \ldots\}\),

\[
B_i^{f_i, n|I|}(E) \subset E_i^{n|I|+1} \subset B_i^{f_i, n}(E)
\]

for all \(i \in I\) by the construction of \(E_i^n\) and Proposition 1.

Then \(T_i\) is partitioned into \(\{D_i^n\}_{n \in \mathbb{N} \cup \{\infty\}}\), where \(D_i^n = E_i^n \setminus E_i^{n+1}\) and \(D_i^\infty = \bigcap_{n=1}^\infty E_i^n = CB_i^f(E)\). (Note that \(D_i^n \neq \emptyset\) only if \(n = 0, \infty\), or \(n \equiv i \pmod{|I|}\).)

For each \(n = (n_i)_{i \in I} \in (\mathbb{N} \cup \{\infty\})^I\), we denote \(\min(n) = \min\{n_1, \ldots, n_{|I|}\}\), \(\pi(n) = P(\prod_{i \in I} D_i^{n_i})\), and

\[
\varepsilon = 1 - P(E) = \sum_{n: \min(n) = 0} \pi(n).
\]

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24 We do not know whether our equivalence result extends to supermodular games with more than two actions. Note that one direction, the robustness of a monotone potential maximizer, is known by Morris and Ui (2005).

25 See also Bergemann and Morris (2019, Section 7) and Hoshino (2018) for discussions on the relationship between robustness and information design with adversarial equilibrium selection.
Note that $\pi(n) > 0$ only if for all $i \in I$, $n_i = 0, \infty$, or $n_i \equiv i \pmod{|I|}$. For each $n \in (\mathbb{N} \cup \{\infty\})^I$, let $S(k, n) = \{i \in I \mid n_i > k\}$ for $0 \leq k < \infty$ and $S(\infty, n) = \{i \in I \mid n_i = \infty\}$.

**Claim A.1.** For $i \in I$ and $1 \leq k < \infty$,
\[
\sum_{n : n_i = k} \pi(n)(v(S(n_i, n) \cup \{i\}) - v(S(n_i, n))) \leq 0.
\]

**Proof.** Fix $i \in I$ and $1 \leq k < \infty$. If $k \not\equiv i \pmod{|I|}$, then each term in the left hand side is equal to 0. If $k \equiv i \pmod{|I|}$, then we have
\[
\sum_{n : n_i = k} \pi(n)f_i(S(n_i, n)) = \sum_{t_i \in D_k^1} P(\{t_i\} \times T_{-i})E[f_i(S_{E_{-i}}^k(\cdot))t_i] \leq 0,
\]
where the inequality follows since $E[f_i(S_{E_{-i}}^k(\cdot))t_i] < 0$ for any $t_i \in D_k^1 = E_k^k \times B_t^k(E^k)$. Thus the claim follows from Definition 4. \qed

Let
\[
x = \sum_{n : 1 \leq \min(n) < \infty} \pi(n).
\]

We will find an upper bound for $x$.

We will use the following identity: for any $n$ with $\pi(n) > 0$,
\[
\sum_{i : 1 \leq n_i < \infty} (v(S(n_i, n) \cup \{i\}) - v(S(n_i, n))) = v(S(0, n)) - v(S(\infty, n)). \tag{A.1}
\]
This identity obtains from the fact that the function $v$ is not indexed by $i$ and, combined with Claim A.1, plays an important role in proving the following Claim.

**Claim A.2.** $x \leq \frac{M}{v(I)} - v' \varepsilon$.

**Proof.** We have
\[
0 \geq \sum_{i \in I} \sum_{n : 1 \leq n_i < \infty} \pi(n)(v(S(n_i, n) \cup \{i\}) - v(S(n_i, n)))
= \sum_{n : \min(n) < \infty} \sum_{i : 1 \leq n_i < \infty} \pi(n)(v(S(n_i, n) \cup \{i\}) - v(S(n_i, n)))
= \sum_{n : \min(n) < \infty} \pi(n)(v(S(0, n)) - v(S(\infty, n)))
= \sum_{n : 1 \leq \min(n) < \infty} \pi(n)(v(I) - v(S(\infty, n)))
+ \sum_{n : \min(n) = 0} \pi(n)(v(S(0, n)) - v(S(\infty, n)))
\geq x(v(I) - v') - \varepsilon M,
\]
where the inequality in the first line follows from Claim A.1, the equality in the second line is obtained by sorting all terms according to $n$, the equality in the third line follows from the identity (A.1), and the inequality in the last line follows from the definitions of $v'$, $M$, $x$, and $\varepsilon$. Then the claim follows because $v(I) - v' > 0$. \qed
By Claim A.2, we have
\[ 1 - P\left(CB^I(E)\right) = \sum_{n : \min(n) < \infty} \pi(n) = \varepsilon + x \leq \left(1 + \frac{M}{v(I) - v'}\right) \varepsilon = \kappa(v)\varepsilon, \]
as desired.

### A.2 Proof of Theorem 1: Potential Maximization Approach

Fix a binary-action supermodular game $f$ where $I$ is a monotone potential maximizer with $v$ and $\lambda = (\lambda_i)_{i \in I}$. Fix a type space $(T, P)$ and an event profile $E = (E_i)_{i \in I} \in T$ with $E = \prod_{i \in I} E_i$.

We consider the function $V : T \to \mathbb{R}$ given by
\[ V(F) = \sum_{t \in T} P(t)v(\{i \in I \mid t_i \in F_i\}) \]
for $F = (F_i)_{i \in I} \in T$.

**Lemma A.1.** For any $F = (F_i)_{i \in I}$ and for any $i \in I$ and $t_i \in F_i$, if $V(F) \geq V(F_i \setminus \{t_i\}, F_{-i})$, then $t_i \in B^F_{\lambda_i}(F)$.

**Proof.** Fix $F \in T$, $i \in I$, and $t_i \in F_i$. Then we have
\[
V(F) - V(F_i \setminus \{t_i\}, F_{-i}) = \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i})(v(\{j \neq i \mid t_j \in F_j\} \cup \{i\}) - v(\{j \neq i \mid t_j \in F_j\}))
\leq \lambda_i \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i})f_i(\{j \neq i \mid t_j \in F_j\})
= \lambda_i P(\{t_i\} \times T_{-i})E[f_i(S_{F_{-i}})\mid t_i].
\]
Since $\lambda_i > 0$ and $P(\{t_i\} \times T_{-i}) > 0$, it therefore follows that if $V(F) \geq V(F_i \setminus \{t_i\}, F_{-i})$, then we have $E[f_i(S_{F_{-i}})\mid t_i] \geq 0$, that is, $t_i \in B^F_{\lambda_i}(F)$. \qed

Consider the maximization problem:
\[
\max_{F \in T \setminus \{F_i\} \subset E} V(F). \tag{A.2}
\]

**Lemma A.2.** The problem (A.2) has a solution, and any solution to (A.2) is $f$-evident.

**Proof.** A solution exists because if each $F$ is identified with the profile of indicator functions for $F_i$ in $E_i$, the domain $\prod_{i \in I}\{0, 1\}^{E_i}$ (which is obviously nonempty) is compact and $V$ is continuous in the product topology.

The second claim, that any solution is $f$-evident, follows from Lemma A.1. \qed

We are ready to prove Theorem 1. Let $F^* = (F^*_i)_{i \in I}$ be any solution of (A.2). Denote $F^* = \prod_{i \in I} F^*_i$ and $\varepsilon = 1 - P(E)$. Then we have
\[
0 \leq V(F^*) - V(E)
= \sum_{t \in T} P(t)(v(\{i \in I \mid t_i \in F^*_i\}) - v(\{i \in I \mid t_i \in E_i\}))
= \sum_{t \in F^*} P(t) \times 0 + \sum_{t \in E \setminus F^*} P(t)(v(\{i \in I \mid t_i \in F^*_i\}) - v(I))
\]
\[ + \sum_{t \notin E} P(t)(v(i | t_i \in F^*_i)) - v(v(i | t_i \in E_i)) \]
\[ \leq (1 - \varepsilon - P(F^*)) \varepsilon = 1 - \kappa(v) \varepsilon, \]

where the inequality in the first line follows from the optimality of \( F^* \), the equality in the third line follows from splitting \( T \) into three disjoint events \( F^*, E \setminus F^*, \) and \( T \setminus E \), and the inequality in the last line follows from the definitions of \( \varepsilon, v' \), and \( M \) (see Section 4.2). Rearranging the terms, we have
\[ P(F^*) \geq 1 - \left( 1 + \frac{M}{v(I) - v'} \right) \varepsilon = 1 - \kappa(v) \varepsilon. \]

since \( v(I) - v' > 0 \). By Lemma A.2 and Proposition 2, we have
\[ P(CB^f(E)) \geq P(F^*) \geq 1 - \kappa(v) \varepsilon. \]

### A.3 The Proof of Lemma 2(2)

For \( \mu \in \Delta(\Gamma) \), let \( \ell(\mu) \) denote the length of the shortest sequences in \( \text{supp}(\mu) = \{ \gamma \in \Gamma | \mu(\gamma) > 0 \} \) and \( \#(\mu) \) denote the number of the shortest sequences in \( \text{supp}(\mu) \). Note that \( \ell(\mu) \leq |I(\mu)| \) in general and that \( \ell(\mu) = |I(\mu)| \) if and only if \( \mu \in \Delta^*(\Gamma) \).

For \( \mu \in \Delta(\Gamma) \) such that \( \ell(\mu) < |I(\mu)| \), define \( \varphi(\mu) \in \Delta(\Gamma) \) as follows. Let \( \gamma_0 \in \text{supp}(\mu) \) be any shortest sequence in \( \text{supp}(\mu) \). Let \( \tilde{S} \neq \emptyset \) be the set of players in \( I(\mu) \) who are not listed in \( \gamma_0 \), and let \( \bar{\Gamma} = \bigcup_{\tilde{i} \in \tilde{S}} \Gamma_{\tilde{i}} \), i.e., the set of all sequences in which at least one player in \( \tilde{S} \) is listed. For each \( \tilde{\gamma} \in \bar{\Gamma} \), let \( \tilde{\gamma}|_{\tilde{S}} \) denote the subsequence of \( \tilde{\gamma} \) consisting of the players in \( \tilde{S} \), and let \( (\gamma_0, \tilde{\gamma}|_{\tilde{S}}) \) denote the concatenation of \( \gamma_0 \) and \( \tilde{\gamma}|_{\tilde{S}} \). Then let \( \varphi(\mu) \) be given by
\[ \varphi(\mu)(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_0, \\ \mu(\gamma) + \frac{\mu(\gamma)}{\mu(\Gamma)} \mu((\gamma_{\tilde{\gamma}} \in \bar{\Gamma} | (\gamma_0, \tilde{\gamma}|_{\tilde{S}}) = \gamma)), & \text{otherwise} \end{cases} \]
for each \( \gamma \in \Gamma \).

**Claim A.3.** If \( \mu \) satisfies (4.2) and \( \ell(\mu) < |I(\mu)| \), then \( \varphi(\mu) \) satisfies (4.2) and

(i) if \( \#(\mu) > 1 \), then \( \ell(\varphi(\mu)) = \ell(\mu) \) and \( \#(\varphi(\mu)) = \#(\mu) - 1 \), and

(ii) if \( \#(\mu) = 1 \), then \( \ell(\varphi(\mu)) > \ell(\mu) \).

**Proof.** The properties (i) and (ii) hold by construction.

We show that \( \varphi(\mu) \) satisfies (4.2) for all \( i \in I \). First, for \( i \in I(\mu) \setminus \tilde{S} \), since \( S(i, (\gamma_0, \tilde{\gamma}|_{\tilde{S}})) = S(i, \gamma_0) \) for any \( \tilde{\gamma} \in \bar{\Gamma} \), we have
\[ \sum_{\gamma \in \Gamma_i} \varphi(\mu)(\gamma) f_i(S(i, \gamma)) = \sum_{\gamma \in \Gamma_i, \gamma \neq \gamma_0} \mu(\gamma) f_i(S(i, \gamma)) + \frac{\mu(\gamma_0)}{\mu(\Gamma)} \sum_{\tilde{\gamma} \in \bar{\Gamma}} \mu(\tilde{\gamma}) f_i(S(i, (\gamma_0, \tilde{\gamma}|_{\tilde{S}}))) \]
\[ = \sum_{\gamma \in \Gamma_i, \gamma \neq \gamma_0} \mu(\gamma) f_i(S(i, \gamma)) + \frac{\mu(\gamma_0)}{\mu(\Gamma)} \sum_{\tilde{\gamma} \in \bar{\Gamma}} \mu(\tilde{\gamma}) f_i(S(i, \gamma_0)) \]
\[ = \sum_{\gamma \in \Gamma_i} \mu(\gamma) f_i(S(i, \gamma)) \leq 0, \]
where the inequality follows from (4.2) for \( \mu \). Second, for \( i \in \tilde{S} \), since \( S(i, (\gamma_0, \tilde{\gamma}|_{\tilde{S}})) \subset S(i, \tilde{\gamma}) \) for any \( \tilde{\gamma} \in \Gamma_i \subset \bar{\Gamma} \), we have
\[ \sum_{\gamma \in \Gamma_i} \varphi(\mu)(\gamma) f_i(S(i, \gamma)) = \sum_{\gamma \in \Gamma_i} \mu(\gamma) f_i(S(i, \gamma)) + \frac{\mu(\gamma_0)}{\mu(\Gamma)} \sum_{\gamma \in \Gamma_i} \mu(\tilde{\gamma}) f_i(S(i, (\gamma_0, \tilde{\gamma}|_{\tilde{S}}))) \]
where the first inequality follows from the supermodularity of \( f_i \), while the second inequality from (4.2) for \( \mu \). Finally, for \( i \in I \backslash I(\mu) \), since \( \varphi(\mu)(\Gamma_i) = 0 \), we have \( \sum_{\gamma \in \Gamma_i} \varphi(\mu)(\gamma) f_i(S(i, \gamma)) = 0 \).

Now, we conclude the proof of Lemma 2(2). Take any \( \mu_0 \in \Delta(\Gamma) \) that satisfies (4.2). Then define the sequence \( \mu_1, \mu_2, \ldots \in \Delta(\Gamma) \) inductively as follows. If \( \ell(\mu_n) = |I(\mu_n)| \), then terminate the sequence. If \( \ell(\mu_n) < |I(\mu_n)| \), then let \( \mu_{n+1} = \varphi(\mu_n) \) with the operator \( \varphi \) defined above. Since \( \ell(\cdot) \) takes finitely many values, it follows from Claim A.3 that the procedure stops in finitely many steps \( N \), and \( \mu_N \) satisfies (4.2) and \( \ell(\mu_N) = |I(\mu_N)| \). That is, (4.2) holds with \( \mu = \mu_N \in \Delta^*(\Gamma) \).

### A.4 Proof of Theorem 2

Fix any \( \varepsilon \in (0, 1] \), and let \( \eta = 1 - (1 - \varepsilon)^{1/(|I|-1)} \in (0, 1] \). It suffices to consider the case where \( \varepsilon \) is sufficiently small, and so is \( \eta \).\(^{26}\) Take a \( \mu \in \Delta^*(\Gamma) \) that satisfies condition (4.4) in Remark 2.

We construct the type space \((T, P)\) as follows. For each \( i \in I \), let

\[
T_i = \begin{cases} 
\{1, 2, \ldots\} & \text{if } i \in I(\mu), \\
\{\infty\} & \text{otherwise.}
\end{cases}
\]

Let \( P \in \Delta(T) \) be given by

\[
P(t) = \begin{cases} 
\eta(1 - \eta)^m \mu(\gamma) & \text{if there exist } m \in \mathbb{N} \text{ and } \gamma = (i_1, \ldots, i_k) \in \Gamma \\
0 & \text{otherwise}
\end{cases}
\]

for each \( t = (t_i)_{i \in I} \in T \), where

\[
\ell(i, \gamma) = \begin{cases} 
\ell & \text{if there exists } \ell \in \{1, \ldots, k\} \text{ such that } i_\ell = i, \\
\infty & \text{otherwise}
\end{cases}
\]

for each \( i \in I \) and \( \gamma = (i_1, \ldots, i_k) \in \Gamma \).

For \( \tau \geq 1 \), we write \( E^\tau_i = \{\tau, \tau + 1, \ldots\} \) for \( i \in I(\mu) \) and \( E^\tau_i = \{\infty\} \) otherwise, and write \( E^\tau = (E^\tau_i)_{i \in I} \) and \( E^\tau_{-i} = (E^\tau_j)_{j \neq i} \). Let \( E = E^{|I|} \). Then

\[
P(E) = \sum_{m=|I|-1}^{\infty} \eta(1 - \eta)^m = (1 - \eta)^{|I|-1} = 1 - \varepsilon.
\]

**Claim A.4.** For any \( i \in I(\mu) \) and any \( \tau \geq |I| \),

\[
P(S_{E^\tau_{-i}, (t_{-i})} = S|_{t_i = \tau}) = (1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})/C_{i, \eta}
\]

for all \( S \in \mathcal{I}_{-i} \), where \( C_{i, \eta} = \sum_{\ell=1}^{|I|} (1 - \eta)^{|I|-\ell} \mu(\{\gamma = (i_1, \ldots, i_k) \in \Gamma_i \mid i_\ell = i\}) > 0 \).

\(^{26}\)Once we construct a pair \((T, P)\) and \( \mathbf{E} \) such that \( P(CB^f(\mathbf{E})) = 0 \), it is easy to construct another pair \((T', P')\) and \( \mathbf{E}' \) such that \( P'(E') \) takes an arbitrary value in \([0, P(E)]\) while \( P'(CB^f(\mathbf{E}')) = 0 \).
Proof. For each \( S \in \mathcal{I}_i \),
\[
P(S_{E^r_i}(t_{-i}) = S | t_i = \tau) = P(t_i = \tau, S_{E^r_i}(t_{-i}) = S) / P(t_i = \tau)
\]
\[
= \eta (1-\eta)^{|I|-|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) / P(t_i = \tau)
\]
\[
= (1-\eta)^{|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) / C_{i, \eta},
\]
as claimed. 

Claim A.5. For any \( i \in I(\mu) \) and any \( \tau \geq |I| \), \( B_{t}^{f_i}(E^r_i) \subset E_{t}^{r+1} \).

Proof. Consider type \( t_i = \tau \). Then
\[
E\left[f_i(S_{E^r_i}(\cdot)) \mid t_i = \tau \right] = \sum_{S \in \mathcal{I}_i} (1-\eta)^{|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) f_i(S) / C_{i, \eta} < 0,
\]
where the equality follows from Claim A.4 and the inequality from condition (4.4). Thus, \( \tau \notin B_{t}^{f_i}(E^r_i) \).

By applying Claim A.5 and Proposition 1 inductively, we have \( B_{t}^{f_i,n}(E) \subset E_{t}^{i|I|+n} \) for all \( i \in I(\mu) \) and all \( n \geq 0 \), and hence \( CB_{t}^{f_i}(E) = \emptyset \) for all \( i \in I(\mu) \). Thus, \( P(CB_{t}^{f_i}(E)) = 0 \).

A.5 Approximate Robustness

In this section, we prove a version of Theorem 3(2) with a stronger conclusion in terms of approximate robustness (Haimanko and Kajii (2016)) by assuming the stronger genericity condition (4.3) (see Remark 4).

Given an elaboration \((T, P, u)\) and \( \varepsilon > 0 \), we say that a strategy profile \( \sigma^* = (\sigma^*_i)_{i \in I} \in \Sigma \) is an interim \( \varepsilon \)-Bayesian Nash equilibrium of \((T, P, u)\) if for all \( i \in I \), all \( t_i \in T_i \), and all \( a_i, a'_i \in A_i \),
\[
\sigma_i^*(a_i | t_i) > 0 \Rightarrow E[u_i((a_i, \sigma^*_{-i}(\cdot), (t_i, \cdot)) | t_i) \geq E[u((a'_i, \sigma^*_{-i}(\cdot)), (t_i, \cdot)) | t_i] - \varepsilon.
\]

Definition A.1. A Nash equilibrium \( \sigma^* = (\sigma^*_i)_{i \in I} \in A \) of a complete information game \( g \) is approximately robust (to incomplete information) in \( g \) if for any \( \varepsilon' > 0 \) and \( \delta > 0 \), there exists \( \varepsilon > 0 \) such that for any \( \varepsilon \)-elaboration \((T, P, u)\) of \( g \), there exists an interim \( \varepsilon' \)-Bayesian Nash equilibrium \( \sigma^* = (\sigma^*_i)_{i \in I} \in \Sigma \) such that
\[
\sum_{t=(t_i)_{i \in I} \in T} P(t) \prod_{i \in I} \sigma_i^*(a_i^* | t_i) \geq 1 - \delta.
\]

By definition, if \( \sigma^* \) is robust, then it is approximately robust. Thus, the conclusion of the following proposition is a strengthening of that of Theorem 3(2).

Proposition A.1. In a binary-action supermodular game \( f \), if condition (4.3) is satisfied, then \( 1 \) is not approximately robust to incomplete information.

Proof. Take a \( \mu \in \Delta^*(\Gamma) \) that satisfies condition (4.3). Let
\[
D_i = - \sum_{S \in \mathcal{I}_i} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) f_i(S),
\]
\[
\varepsilon' = \min_{i \in I(\mu)} D_i / 2 > 0.
\]

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For each \( \eta \), define \( C_{i,\eta} = \sum_{\ell=1}^{[|I|]} (1 - \eta)^{|I| - \ell} \mu(\{ \gamma = (i_1, \ldots, i_\ell) \in \Gamma_i \mid i_\ell = i \}) \) as in Claim A.4 in Appendix A.4. Note that \( \sum_{S \in \mathcal{L}_i} (1 - \eta)^{|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) f_i(S) \to -D_i \) and \( C_{i,\eta} \to \mu(\Gamma_i) \) as \( \eta \to 0 \), and hence for any \( i \in I(\mu) \),
\[
\sum_{S \in \mathcal{L}_i} (1 - \eta)^{|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) f_i(S) / C_{i,\eta} \to -D_i
\]
as \( \eta \to 0 \).

Fix any \( \varepsilon \in (0, 1] \), and let \( \eta = 1 - (1 - \varepsilon)^{|I| - 1} \) \( \in (0, 1] \). It suffices to consider the case where \( \varepsilon \) is sufficiently small, and so is \( \eta \) so that
\[
\sum_{S \in \mathcal{L}_i} (1 - \eta)^{|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) f_i(S) / C_{i,\eta} < -\varepsilon'
\]
for any \( i \in I(\mu) \).

Given the above \( \mu \) and \( \eta \), take the type space \( (P, T) \) and the event profile \( E \) as constructed in the proof of Theorem 2 in Appendix A.4. Then, \( P(E) = 1 - \varepsilon \). Moreover, as in the proof of Claim A.5, if \( i \in I(\mu) \) and \( \tau \geq |I| \), then
\[
\mathbb{E} \left[ f_i \left( S_{E_{-i}^{\tau}}(i) \right) \mid t_i = \tau \right] = \sum_{S \in \mathcal{L}_{-i}} (1 - \eta)^{|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) f_i(S) / C_{i,\eta} < -\varepsilon',
\]
and hence for the game \( f + \varepsilon' \) defined by \( f_i(S) + \varepsilon' \) for all \( i \in I \) and \( S \in \mathcal{L}_{-i} \), we have \( P(CB^{f + \varepsilon'}(E)) = 0 \).

Finally, as in the proof of Proposition 3(2), let the payoff functions \( u \) be such that for each \( i \in I, T_i^{f_i} = E_i, \) and action 0 is a dominant action for all types \( t_i \not\in E_i \). Then in the elaboration \( (T, P, u) \), 1 is never played in any interim \( \varepsilon' \)-Bayesian Nash equilibrium. \( \blacksquare \)

Condition (4.3) in Proposition A.1 cannot be relaxed to condition (4.4). In fact, the game (4.5) in Remark 2 satisfies condition (4.4), but by the upper semi-continuity of approximately robust equilibria with respect to payoffs, both 1 and 0 are approximately robust (Haimanko and Kajii (2016)).

A.6 Robustness of Non-Extreme Action Profiles

In this section, we generalize Theorem 3 to action profiles other than 1.

**Corollary A.1.** Let a binary-action game \( f \) be supermodular. Then the following results hold.

1. If \( S^* \) is a monotone potential maximizer in \( f \), then \( 1_{S^*} \) is robust to incomplete information in \( f \).

2. For generic \( f \), if \( S^* \) is not a monotone potential maximizer in \( f \), then \( 1_{S^*} \) is not robust to incomplete information in \( f \).

Part (1) follows from Morris and Ui (2005), whereas it can also be proved by applying our Theorem 1 to the “subgames” of \( f \) introduced below.

For the proof of part (2), we utilize two “subgames” of the original game \( f = (f_i)_{i \in I} \): the **lower game** \( f^- = (f_i^+)_{i \in S^*} \) is the binary-action game among the players in \( S^* \) defined by \( f_i^-(S) = f_i(S) \) for all \( i \in S^* \) and \( S \subseteq S^* \setminus \{i\} \), where the actions of the players in \( I \setminus S^* \) are fixed at action 0; the **upper game** \( f^+ = (f_i^+)_{i \in I \setminus S^*} \) is the binary-action game among the players in \( I \setminus S^* \) defined by \( f_i^+(S) = f_i(S \cup S^*) \) for any \( i \in I \setminus S^* \) and \( S \subseteq (I \setminus S^*) \setminus \{i\} \), where the actions of the players in \( S^* \) are fixed at action 1.
Note that if \( S^* \) is a monotone potential maximizer in \( f \), then \( S^* \) is a monotone potential maximizer in the lower game \( f^− \), and \( \emptyset \) is a monotone potential maximizer in the upper game \( f^+ \). Indeed, let \( v \) be a monotone potential of \( f \) for \( S^* \) with \( \lambda = (\lambda_i)_{i \in I} \). Then \( v^− \) given by \( v^−(S) = v(S) \) for all \( S \subset S^* \) is a monotone potential of \( f^− \) for \( S^* \) with \( (\lambda_i)_{i \in S^*} \), and \( v^+ \) given by \( v^+(S) = v(S \cup S^*) \) for all \( S \subset I \setminus S^* \) is a monotone potential of \( f^+ \) for \( \emptyset \) with \( (\lambda_i)_{i \in I \setminus S^*} \). The next lemma claims that the converse also holds if \( f \) is supermodular.

**Lemma A.3.** \( S^* \) is a monotone potential maximizer in a binary-action supermodular game \( f \) if and only if \( S^* \) is a monotone potential maximizer in the lower game \( f^− \), and \( \emptyset \) is a monotone potential maximizer in the upper game \( f^+ \).

Similarly to Lemma 1 in Section 4.3, the existence of a monotone potential for \( S^* \) can be characterized in terms of the weighted sum of payoff increments. For each \( \gamma = (i_1, \ldots, i_k) \in \Gamma \) and \( \lambda = (\lambda_i)_{i \in I} \) with \( \lambda_i > 0 \), we define

\[
F(\gamma, \lambda; S^*) = \sum_{\xi = 1}^{k} \lambda_i \tilde{f}_i(S^* \triangle \{i_1, \ldots, i_\xi\}; S^*),
\]

where \( \tilde{f}_i \) is the “signed” payoff increment,

\[
\tilde{f}_i(S; S^*) = \begin{cases} f_i(S) & \text{if } i \in S^*, \\ -f_i(S \setminus \{i\}) & \text{if } i \in I \setminus S^*. \end{cases}
\]

and \( S \triangle S' = (S \setminus S') \cup (S' \setminus S) \) denotes the symmetric difference between \( S, S' \in \mathcal{I} \).

**Lemma A.4.** A binary-action game \( f \) admits a monotone potential with \( \lambda \) for \( S^* \) if and only if \( F(\gamma, \lambda; S^*) > 0 \) for all \( \gamma \in \Gamma \).

The proof, which is similar to that of Lemma 1, is omitted.

**Proof of Lemma A.3.** It suffices to show the “if” direction. Fix any \( \gamma = (i_1, \ldots, i_k) \in \Gamma \). Since \( f \) is supermodular, we have

\[
\tilde{f}_i(S^* \triangle \{i_1, \ldots, i_\xi\}; S^*) = \tilde{f}_i(S^* \triangle \{i_1, \ldots, i_\xi\}) \\
\geq \tilde{f}_i(S^* \setminus \{i_1, \ldots, i_\xi\}) = \tilde{f}_i(S^* \setminus \{i_1, \ldots, i_\xi\}; S^*)
\]

for \( i_\xi \in S^* \), and

\[
\tilde{f}_i(S^* \triangle \{i_1, \ldots, i_\xi\}; S^*) = -\tilde{f}_i(S^* \triangle \{i_1, \ldots, i_{\xi-1}\}) \\
\geq -\tilde{f}_i(S^* \cup \{i_1, \ldots, i_{\xi-1}\}) = \tilde{f}_i(S^* \cup \{i_1, \ldots, i_{\xi-1}\}; S^*)
\]

for \( i_\xi \in I \setminus S^* \). By these inequalities, we have

\[
F(\gamma, \lambda; S^*) = \sum_{\xi \in S^*} \lambda_i \tilde{f}_i(S^* \triangle \{i_1, \ldots, i_\xi\}; S^*) + \sum_{\xi \in I \setminus S^*} \lambda_i \tilde{f}_i(S^* \triangle \{i_1, \ldots, i_\xi\}; S^*) \\
\geq \sum_{\xi \in S^*} \lambda_i \tilde{f}_i(S^* \setminus \{i_1, \ldots, i_\xi\}; S^*) + \sum_{\xi \in I \setminus S^*} \lambda_i \tilde{f}_i(S^* \setminus \{i_1, \ldots, i_\xi\}; S^*) \\
= F(\gamma^−, \lambda; S^*) + F(\gamma^+, \lambda; S^*),
\]

(A.3)

where \( \gamma^− \) (resp. \( \gamma^+ \)) is the subsequence of \( \gamma \) consisting of the players in \( S^* \) (resp. \( I \setminus S^* \)). Since \( S^* \) is a monotone potential maximizer in \( f^− \), and \( \emptyset \) is a monotone potential maximizer in \( f^+ \), it follows from Lemma A.4 applied to \( f^− \) and \( f^+ \) that \( F(\gamma^−, \lambda; S^*) > 0 \) and \( F(\gamma^+, \lambda; S^*) > 0 \), and hence \( F(\gamma, \lambda; S^*) > 0 \) by inequality (A.3). Since this holds for any \( \gamma \in \Gamma \), it follows from Lemma A.4 applied to \( f \) that \( S^* \) is a monotone potential maximizer in \( f \).
Proof of Corollary A.1(2). Suppose that $S^*$ is not a monotone potential maximizer in $f$. By Lemma A.3, either $S^*$ is not a monotone potential maximizer in $f^-$, or $\emptyset$ is not a monotone potential maximizer in $f^+$. We assume the former by symmetry. By genericity, we assume that $f^-$ satisfies condition (4.3). By Proposition A.1 in Section A.5, $1_{S^*}$ is not approximately robust in $f^-$. That is, there exist $\varepsilon' > 0$ and $\delta^- > 0$ such that for any $\varepsilon > 0$, there exists an $\varepsilon'$-elaboration $(T^-, P^-, u^-)$ with $T^- = \prod_{t \in S^*} T_i$ in which for all $i \in S^*$, 0 is a dominant action for $t_i \notin T_i^{f_i}$, and every interim $\varepsilon'$-Bayesian Nash equilibrium plays $1_{S^*}$ with probability less than $1 - \delta^-$. Let $\delta = \min\{\delta^- : (2 \max_{i \in I, S \in I, i} |f_i(S)|) > 0\}$. In the following, we denote $E_i = T_i^{f_i}$ for $i \in S^*$.

Fix any $\varepsilon > 0$. Given this $\varepsilon$, we extend the above elaboration to all players in $I$ by adding a singleton type to each player in $I \setminus S^*$. Formally, construct $(T, P, u)$ with $T = \prod_{i \in I} T_i$ as follows: for $i \in S^*$, let $E_i$ be the same space as in the original elaboration, and for $i \in I \setminus S^*$, let $T_i = \{t_{i,0}\}$; let $P \in \Delta(T)$ be defined by $P(t^-, (t_{i,0})_{i \in I \setminus S^*}) = P^-(t^-)$ for all $t^- \in T^-$; and let $u$ be such that for $i \in S^*$, $T_i^{f_i} = E_i$, and 0 is a dominant action for $t_i \notin E_i$, and for $i \in I \setminus S^*$, $T_i^{f_i} = \{t_{i,0}\}$. Since $(T^-, P^-, u^-)$ is an $\varepsilon$-elaboration of $f^-$, $(T, P, u)$ is an $\varepsilon$-elaboration of $f$. If $(T, P, u)$ has a Bayesian Nash equilibrium $\sigma^* = (\sigma^*_i)_{i \in I}$ that plays $1_{S^*}$ with probability at least $1 - \delta$, then $(\sigma^*_i)_{i \in S^*}$ is an interim $\varepsilon'$-Bayesian Nash equilibrium in $(T^-, P^-, u^-)$ that plays $1_{S^*}$ with probability at least $1 - \delta \geq 1 - \delta^-$, which is a contradiction. Therefore, $1_{S^*}$ is not robust in $f$.

In fact, the above proof, with the last step appropriately modified, shows that under condition (4.3) for $f^-$ (or the corresponding condition for $f^+$), $1_{S^*}$ is not even approximately robust in $f$.

In the proof of Corollary A.1(2), we assumed condition (4.3) for $f^-$. This condition cannot be relaxed to condition (4.4) except for $S^* = I$ and symmetrically for $S^* = \emptyset$ (Remark 2 to Theorem 2). For example, consider the game

$$
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1,1,0 & 0,0,0 \\
1 & 0,0,0 & 1,1,0 \\
0 & 0,0,0 & 2,2,0 \\
1 & 0,0,0 & 0,0,0 \\
\end{array}
$$

The induced payoff increment function $f_i$ is given by

$$
f_i(S) = \begin{cases} 
2 \times 1_{\{3-i \in S\}} + 1_{\{3 \in S\}} - 1 & \text{if } i = 1, 2, \\
0 & \text{if } i = 3,
\end{cases}
$$

(A.4)

where $1_X$ is the indicator function for condition $X$. Let $S^* = \{1, 2\}$. The lower game $f^-$ between players 1 and 2, where the action of player 3 is fixed at action 0, satisfies condition (4.4), but violates condition (4.3) (note that $f^-$ is the same as the game (4.5) in Remark 2).

Proposition A.2. In the binary-action game $f$ given by (A.4), $1_{S^*} = (1, 1, 0)$ is robust to incomplete information.

Proof.\footnote{If $1_{S^*}$ is a $(p, p, 0)$-dominant equilibrium in the game (A.4) if and only if $p \geq 1/2$, and hence the proposition does not follow directly from Kajii and Morris (1997a, Proposition 5.3).} Fix any $\varepsilon > 0$ and any $\varepsilon$-elaboration $(T, P, u)$ of $f$. Recall that $T_i^{f_i}$ is the event where player $i$ knows that his payoff increment in $u$ is given by $f_i$. Denote $T^f = (T_1^{f_1}, T_2^{f_2}, T_3^{f_3})$ and $T^f = T_1^{f_1} \times T_2^{f_2} \times T_3^{f_3}$. By definition, $P(T^f) \geq 1 - \varepsilon$.\footnote{This statement is not explicitly stated in the text but follows from the definition of $T^f$.}
Let \( p_\varepsilon = (p_\varepsilon, p_\varepsilon, 0) \) with \( p_\varepsilon = 1/(2 + \sqrt{\varepsilon}) < 1/2 \). By Kajii and Morris (1997a, Proposition 4.2) (or by our Theorem 1 for the game \((f_1, f_2, f_3)\) as defined in Example 1), we have
\[
P(CB_{p_\varepsilon}(T^f)) \geq 1 - \kappa_{KM}(p_\varepsilon)(1 - P(T^f)) \geq 1 - \frac{1}{1 - 2p_\varepsilon} \times \varepsilon = 1 - (2 + \sqrt{\varepsilon})\sqrt{\varepsilon},
\]
where \( CB_{p_\varepsilon}(T^f) = CB_{1\varepsilon}(T^f) \times CB_{2\varepsilon}(T^f) \times CB_{3\varepsilon}(T^f) \).

Let \( \Sigma^* = \Sigma_1^* \times \Sigma_2^* \times \Sigma_3^* \), where for each \( i = 1, 2, \Sigma_i^* \) is the set of strategies \( \sigma_i \) such that \( \sigma_i(1|t_i) = 1 \) whenever \( t_i \in CB_{p_\varepsilon}(T^f) \), and \( \Sigma_3^* \) is the set of strategies \( \sigma_3 \) such that \( \sigma_3(1|t_3) = \sqrt{\varepsilon} \) whenever \( t_3 \in CB_{3\varepsilon}(T^f) \). For any \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \in \Sigma^* \), if \( i = 1, 2 \) and \( t_i \in CB_{1\varepsilon}(T^f) \subset T^f_i \), then
\[
E \left[ \sum_{a_3-i, a_3} f_i(\{j \neq i | a_j = 1\})\sigma_{3-1}(a_3-i)\sigma_3(a_3|) | t_i \right] = E[2\sigma_{3-i}(1|) + \sigma_3(1|) | t_i] - 1
\geq (2 + \sqrt{\varepsilon})P(CB_{p_\varepsilon}(T^f)) \times CB_{3\varepsilon}(T^f)|t_i | - 1 \geq (2 + \sqrt{\varepsilon})p_\varepsilon - 1 = 0,
\]
where the second inequality follows from the \( p_\varepsilon \)-evidence of \((CB_{p_\varepsilon}(T^f))_{j \in I}\) (or from our Proposition 2). Thus, we have \( 1 \in BR_{1}(\sigma_{-1}(t_i)) \). Also for \( i = 3 \), we have \( BR_3(\sigma_{-3}(t_3)) = \{0, 1\} \) for all \( t_3 \in CB_{3\varepsilon}(T^f) = T_3^{\beta} \), as \( f_3 \equiv 0 \). Hence, as in the proof of Proposition 3(1), we apply Kacustani’s fixed-point theorem to the best response correspondence restricted to \( \Sigma^* \), and obtain a Bayesian Nash equilibrium \( \sigma^* \in \Sigma^* \).

By the construction of \( \Sigma^* \), \( \sigma^* \) plays action profile \( 1_{\Sigma^*} \) with probability
\[
\sum_{t \in T} P(t)\sigma^*_i(1|t_1)\sigma^*_2(1|t_2)\sigma^*_3(0|t_3) \geq P(CB_{p_\varepsilon}(T^f))(1 - \sqrt{\varepsilon}) \geq (1 - (2 + \sqrt{\varepsilon})\sqrt{\varepsilon})(1 - \sqrt{\varepsilon}),
\]
which converges to 1 as \( \varepsilon \to 0 \).

A.7 Proof of Proposition 6

If \( z_i z_j > y_i y_j \) for all \( i, j \in I \) with \( i \neq j \), then as constructed in Example 6, there exists a monotone potential for \( I \). By Theorem 3(1), \( 1 \) is robust. Symmetrically, if \( z_i z_j < y_i y_j \) for all \( i, j \in I \) with \( i \neq j \), then \( 0 \) is robust.

Suppose that \( z_i z_j \leq y_i y_j \) for some \( i, j \in I \) with \( i \neq j \). Take any such \( i \) and \( j \). We show that condition (4.4) in Remark 2 holds for all parameter values in this case. For each small \( \eta > 0 \), let \( \mu \in \Delta(I) \) be such that \( \gamma = (i, *, \ldots, *, j) \) with probability \( (1 - \eta^2) y_i / (y_i + z_i) \), where \( (k, *, \ldots, *, \ell) \) denotes the sequence of length \( |I| \) such that \( k \) is listed first, \( \ell \) is listed last, and the other \( |I| - 2 \) players are listed in some fixed order (e.g., the ascending order in player indices), \( \gamma = (j, *, \ldots, *, i) \) with probability \( (1 - \eta^2) z_j / (y_j + z_j) \), and for each \( k \neq i, j, \gamma = (i, *, \ldots, *, k) \) with probability \( \eta^2 / (|I| - 2) \). For player \( i \), the left-hand side of (4.4) is
\[
(1 - \eta^2) \frac{z_i}{y_i + z_i} \times (-y_i) + (1 - \eta^{|I|-1}) \times \left( (1 - \eta^2) \frac{y_i}{y_i + z_i} + \eta^2 \right) \times z_i
= -(1 - \eta^2) \frac{y_i z_i}{y_i + z_i} \frac{(1 - (1 - \eta)^{|I|-1}) + (1 - \eta)^{|I|-1} \eta^2 z_i}{\approx (|I|-1)\eta} \times z_i < 0
\]
for small \( \eta \); for player \( j \), it is
\[
(1 - \eta^2) \frac{y_j}{y_j + z_j} \times (-y_j) + (1 - \eta^{|I|-1}) \times (1 - \eta^2) \frac{z_i}{y_i + z_i} \times z_j
\]
\[
(1 - \eta^2) \left(1 - \eta \frac{|I|-1}{y_i} \right) \frac{y_i y_j - z_i z_j}{y_i + z_i} < 0
\]
as \(y_i y_j \geq z_i z_j > 0\); and for player \(k \neq i, j\), it is \(\eta^2 / (|I| - 2) \times (-y_k) < 0\). Thus, by Remarks 2 and 3 and Proposition 3(2), for any small \(\varepsilon > 0\), there exists a dominance-solvable \(\varepsilon\)-elaboration in which \(0\) is played everywhere, which implies that no equilibrium other than \(0\) can be robust. Symmetrically, if \(z_i z_j \geq y_i y_j\) for some \(i, j \in I\) with \(i \neq j\), then no equilibrium other than \(1\) can be robust.

By combining these cases, the conclusion follows.

References


