GENERALIZED BELIEF OPERATOR AND ROBUSTNESS IN
BINARY-ACTION SUPERMODULAR GAMES1,2

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This paper studies the robustness of an equilibrium to incomplete information in
binary-action supermodular games. Using a generalized version of belief operator, we
explore the restrictions that prior beliefs impose on higher order beliefs. In particu-
lar, we obtain a non-trivial lower bound on the probability of a common belief event,
uniform over type spaces, when the underlying game has a monotone potential. Con-
versely, when the game has no monotone potential, we construct a type space with an
arbitrarily high probability event in which players never have common belief about
that event. As an implication of these results, we show for generic binary-action su-
permodular games that an action profile is robust to incomplete information if and
only if it is a monotone potential maximizer. Our study offers new methodology and
insight to the analysis of global game equilibrium selection.

Keywords: incomplete information, higher order belief, supermodular game, ro-
 bustness, contagion, duality theorem, global game.

1. INTRODUCTION

Consider a situation in which many agents make binary decisions, to be “ac-
tive” (action 1) or “inactive” (action 0), with a motive to coordinate their actions.
Examples of such situations include those of currency attacks, bank runs, liquidity
crises, and policy changes, among many others. Understanding the equilib-
rium outcomes in these games entails understanding the uncertainty that players
face, not only about payoff relevant parameters, but also about the other players’
behavior, which in turn entails understanding higher order beliefs about payoff
relevant parameters.

In the presence of both payoff uncertainty and strategic uncertainty, equi-
librium predictions may be fragile to small perturbations in information, as
demonstrated, e.g., through the “contagion” arguments by Rubinstein (1989)
and Carlsson and van Damme (1993a). Indeed, for a large class of global games
with strategic complementarity, a unique strategy profile survives the iterated
elimination of strictly dominated strategies in the limit as the noise vanishes
(Frankel et al. (2003)). In $2 \times 2$ coordination games and symmetric binary-action
supermodular games, for example, the risk-dominant equilibrium and the “Lapla-
cian” equilibrium, respectively, are selected by the global game approach inde-
pendently of the noise structure (Carlsson and van Damme (1993a), Morris and

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On the other hand, in a certain class of asymmetric games, Corsetti et al. (2004) demonstrate that different equilibria are selected depending on the noise structure.

In this paper, within the class of binary-action supermodular games, we aim at identifying when action 1 survives in all information perturbations, and conversely, when action 0 is played contagiously in some information perturbation. We emphasize two forms of generality in our analysis: we allow for general payoffs beyond two-player games or symmetric games, and examine robustness against general information perturbations, including global-game perturbations with various noise structures as particular instances. More specifically, we study the notion of robustness introduced by Kajii and Morris (1997a): a Nash equilibrium of a complete information game is robust to incomplete information if in any incomplete information game where with high probability, all players know that their payoff functions are given by those in the original complete information game, the equilibrium action profile continues to be played with high probability in some Bayesian Nash equilibrium. Our main theorems characterize the robustness and non-robustness of equilibria in this sense.

For our study, we employ a generalized belief operator (Morris and Shin (2007)), which generalizes the p-belief operator of Monderer and Samet (1989), to accommodate the class of all binary-action supermodular games. Given any binary-action game \( f = (f_i)_{i \in I} \) with a set \( I \) of players and any information structure, our \( f_i \)-belief operator \( B^f_i \) associates player \( i \)'s beliefs with his incentives in the game \( f \). In Section 3, after stating the definition and some basic properties of the operator, we establish formal connections between common \( f \)-beliefs \( CB^f \) and equilibria of incomplete information games that embed the game \( f \) (Proposition 3). It turns out that examining the robustness of action profile “all 1” amounts to examining the behavior of the probability \( P(CB^f(E)) \) of common \( f \)-belief of an event profile \( E = (E_i)_{i \in I} \) as the probability \( P(E) \) of the event \( E = \prod_{i \in I} E_i \) becomes close to one (Proposition 4).

In Section 4, we state and prove our main theorems. First, our Theorem 1 extends the Critical Path Theorem of Kajii and Morris (1997a, Proposition 4.2) from p-belief (with a vector \( p = (p_i)_{i \in I} \)) to the generalized belief. It establishes a relationship between the probability \( P(CB^f(E)) \) of common \( f \)-belief and the probability \( P(E) \) when the underlying game \( f \) has a monotone potential as introduced by Morris and Ui (2005). Formally, it shows that if “all 1” is a monotone potential maximizer in \( f \), then the probability \( P(CB^f(E)) \) of common \( f \)-belief converges to 1 as the probability \( P(E) \) converges to 1, where the convergence is uniform over all information structures and all event profiles.

Second, Theorem 2 shows a generic converse of our Critical Path Theorem. That is, for a generic game \( f \), if “all 1” is not a monotone potential maximizer in \( f \), then there exist an information structure and an arbitrarily high probability event profile such that players never have common \( f \)-belief about that event profile. For the proof of this theorem, by exploiting the duality between payoffs and
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probabilities, we construct a desired incomplete information perturbation based on a novel application of a duality theorem to the system of linear inequalities that characterizes the existence of a monotone potential. Specifically, the duality theorem gives us a distribution over sequences, or rankings, of players, which in our construction, determines the posterior beliefs about the rankings of types among the players.

Third, combining the results above, Theorem 3 and Corollary A.1 establish a complete characterization of robust equilibrium in generic binary-action supermodular games: an equilibrium is robust to incomplete information if and generically only if it is a monotone potential maximizer. Note that results by Morris and Ui (2005) prove the robustness of a monotone potential maximizer in supermodular games with many actions, but are silent about the non-existence of a robust equilibrium when there is no monotone potential.

Our study offers new methodology and insight to the analysis of global games beyond the Laplacian criterion (Morris and Shin (2003)) for the case of homogeneous players. In Section 5, as a minimal departure from symmetric games, we revisit the asymmetric currency attack game of Corsetti et al. (2004) with a large trader and small traders and characterize its robust equilibrium. In particular, we observe that the condition for the noise-dependent selection in the global game of Corsetti et al. (2004) precisely corresponds to that for the non-existence of a monotone potential in the underlying complete information game, which boils down to the solvability of a system of two linear inequalities in one variable. When no monotone potential exists, we discuss how the distribution over the rankings among players, which is obtained from the duality, corresponds to the relative precision of the signals of the large and the small traders.

1.1. Related Literature

The notion of robust equilibrium by Kajii and Morris (1997a) allows for a much richer set of payoff perturbations than the classical refinement concepts, such as Kohlberg and Mertens’ (1986) strategic stability.\(^1\) Indeed, Kajii and Morris (1997a, Example 3.1) present an example of a three-player three-action game (without supermodularity) that has a unique Nash equilibrium, which is strict, but has no robust equilibrium. Yet, several sufficient conditions for robustness have been identified. For example, Kajii and Morris (1997a) show that if the game has a \(\mathbf{p}\)-dominant equilibrium with \(\sum_{i \in I} p_i < 1\), then it is robust.\(^2\) Using the technique of potential functions, Ui (2001) shows that if the game admits a potential, then the potential maximizing action profile is robust.\(^3\) Subsequently,

\(^1\)See Kajii and Morris (1997b) for the relationship between robustness and the existing refinement concepts.

\(^2\)For \(\mathbf{p} = (p_i)_{i \in I}\), an action profile \(a^*\) is a \(\mathbf{p}\)-dominant equilibrium if for each player \(i \in I\), \(a_i\) is a best response to any belief that assigns at least probability \(p_i\) to the opponents’ playing \(a_{-i}\). This notion generalizes the notion of risk-dominance in \(2 \times 2\) coordination games.

\(^3\)Precisely, Ui (2001) uses the notion of robustness to canonical elaborations, introduced in Kajii and Morris (1997b), which is, in principle, weaker than the original notion of Kajii and
Morris and Ui (2005) extend Ui’s approach to generalized/monotone potential games. Their sufficient condition for robustness subsumes both Kajii-Morris’ and Ui’s conditions. Our paper provides an alternative proof for Morris-Ui’s result via the Critical Path Theorem and proves its (generic) converse for binary-action supermodular games.

Global games, as introduced by Carlsson and van Damme (1993a), offer a tractable modelling device that leads to equilibrium selection, thus allowing comparative statics exercises for policy implications, and have found applications where coordination of actions plays an important role. Examples include Morris and Shin (1998) for currency attacks, Morris and Shin (2004) for debt pricing, Goldstein and Pauzner (2005) for bank runs, and Bueno de Mesquita (2010), Edmond (2013), and Chen and Suen (2017) for policy changes. For the class of symmetric binary-action supermodular games, Morris and Shin (2003) show that the limit equilibrium (either under the improper uniform prior or in the limit with vanishingly small noise) is independent of the noise structure, and characterized by the “Laplacian” action, i.e., the best response to the uniform belief over the rankings of the players. This characterization has been extended to accommodate some forms of preference heterogeneity (Guimaraes and Morris (2008, Section 6.1), Sàkovic and Steiner (2012)). On the other hand, in the context of currency attacks, Corsetti et al. (2004) incorporate a large atomic player with a continuum of small players, and show that the selected equilibrium is sensitive to the relative precision of the signals between the large and the small players.

Global games also offer a tractable class of incomplete information perturbations for examining the robustness of an equilibrium. Formally, Basteck et al. (2010) and Oury and Tercieux (2007) show that for a generic supermodular game, if a robust equilibrium exists, then every global game that embeds the given game, regardless of the noise structure, must select the robust equilibrium. This result implies, in particular, that a generic supermodular game has at most one robust equilibrium, and that it has no robust equilibrium if the global game equilibrium selection depends on the noise structure. The converse of the result of Basteck et al. and Oury-Tercieux, however, does not hold for games with more than two actions (Basteck and Daniëls (2011), Oyama and Takahashi (2011)). Our Theorem 3, along with the construction in the proof of Theorem 2, in fact implies that the converse does hold for generic binary-action supermodular games, i.e., for these games, an equilibrium is robust to incomplete information if and only if it is a noise-independent global game selection (Proposition 5).

The literature has provided sporadic examples of games with no robust equilibria. For binary-action supermodular games, the two notions of robustness generally coincide by our Theorem 3 along with its proof. See Pram (2019) and Takahashi (2018) for the relationship between the two notions for general games.

4 For further developments in the literature, see, e.g., Oyama and Tercieux (2009), Nora and Uno (2014), and Haimanko and Kajii (2016).

5 Other forms of preference heterogeneity are considered in Leister et al. (2018) and Serrano-Padial (2018), where the equilibrium is characterized by a (weighted) potential maximization problem (Frankel et al. (2003)).
librium, in addition to the example by Kajii and Morris (1997a, Example 3.1) as mentioned earlier, such as those in Carlsson (1989), Frankel et al. (2003), Corsetti et al. (2004), Basteck et al. (2010), Oury (2013), and Oyama and Takahashi (2011; 2015). In demonstrating non-existence of robust equilibria, these papers rely on ad hoc constructions of incomplete information perturbations. The present paper (the proof of Theorem 2), in contrast, offers a systematic construction of perturbations for binary-action supermodular games based on a duality theorem.

Equilibrium behavior in incomplete information games can be analyzed by belief operators and the associated notions of common beliefs. For example, using the belief operator of Monderer and Samet (1989) (with a generalization to vector $p$), Kajii and Morris (1997a, Lemma 5.2) show that if $a^*$ is $p$-dominant in a complete information game, then any incomplete information game has a Bayesian Nash equilibrium in which $a^*$ is played wherever there is common $p$-belief that the players know that their payoffs are given by those in the complete information game. Kajii and Morris (1997a, Proposition 5.3) combine this lemma and their Critical Path Theorem to obtain the sufficient condition for robustness in terms of $p$-dominance. Morris and Shin (2007) generalize the $p$-belief of Monderer and Samet (1989) to accommodate general, state-dependent, supermodular payoff functions with binary actions. Using their generalized belief operator, Morris and Shin (2007) give a common belief foundation for global game selection by identifying conditions on rank beliefs under which the game has a unique rationalizable strategy for the case of separable-symmetric payoffs.

We follow Kajii and Morris (1997a) to adopt an ex ante perspective on robustness. Weinstein and Yildiz (2004; 2007) instead consider the robustness to perturbations in interim beliefs. They show that under a richness assumption, for any type $t_i$ in the universal type space and any rationalizable action $a_i$ for type $t_i$, there exists a type arbitrarily close to $t_i$ (in the product topology on the universal type space) for which $a_i$ is uniquely rationalizable. Thus, according to Weinstein and Yildiz (2004), an action profile is interim robust if and only if it is uniquely rationalizable. Note that Weinstein-Yildiz' analysis does not impose any restriction on the ex ante probability of the event that the payoffs are given by those of the underlying game, and that due to Lipman (2003), their result holds irrespective of whether the common prior assumption holds or not. In our ex ante approach, in contrast, the Critical Path Theorem (of Kajii and Morris (1997a) or ours) quantifies non-trivial implications that the common prior assumption imposes on the ex ante probabilities of higher order belief events.

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6 The first five papers in this list demonstrate noise-dependent global game selection, which implies non-existence of a robust equilibrium by Basteck et al. (2010) and Oury and Tercieux (2007); see Basteck et al. (2013, Table 1) for a list of games with noise-dependent global game selection.

7 Morris et al. (2016) instead restrict attention to two-player binary-action supermodular games.

8 See Oyama and Tercieux (2010; 2012) for robustness and contagion under non-common
1.2. Organization of the Paper

Section 2 introduces our setup and defines robust equilibria. Section 3 defines the generalized belief operator and states its basic properties and connections to incomplete information elaborations. Section 4 defines monotone potentials and presents our main theorems (Theorems 1–3). Section 5 studies the asymmetric incomplete information elaborations. Section 6 concludes.

2. PRELIMINARIES

2.1. Complete Information Games

Let \( I = \{1, \ldots, |I|\} \) be the set of players with \(|I| \geq 2\). We write \( \mathcal{I} = 2^I \) for the collection of all subsets of \( I \) (including the empty set), and for each \( i \in I \), \( \mathcal{I}_- = 2^I \setminus \{i\} \) for the collection of all subsets of \( I \setminus \{i\} \).

The finite set of actions available to each player \( i \in I \) is denoted by \( A_i \), where we write \( A = \prod_{i \in I} A_i \) and \( A_{-i} = \prod_{j \neq i} A_j \) as usual. A complete information game is then represented by a profile \( g = (g_i)_{i \in I} \) of payoff functions \( g_i : A \to \mathbb{R} \), \( i \in I \).

2.2. Type Spaces and Incomplete Information Elaborations

A type space \( (T, P) \) consists of a countable set \( T_i \) of each player \( i \)'s types and a common prior distribution \( P \in \Delta(T) \), where \( T = \prod_{i \in I} T_i \).\(^9\) We assume \( P(\{t_i\} \times T_{-i}) > 0 \) for each \( i \in I \) and \( t_i \in T_i \), where \( T_{-i} = \prod_{j \neq i} T_j \). For any \( i \in I \) and \( t_i \in T_i \), the posterior of type \( t_i \) is given by

\[
P(E_{-i} | t_i) = \frac{P(\{t_i\} \times E_{-i})}{P(\{t_i\} \times T_{-i})}
\]

for \( E_{-i} \subset T_{-i} \).

With \( I \) and \( (A_i)_{i \in I} \) fixed, an incomplete information elaboration consists of a type space \( (T, P) \) and a profile \( u = (u_i)_{i \in I} \) of payoff functions \( u_i : A \times T \to \mathbb{R} \), \( i \in I \). A (behavioral) strategy for player \( i \in I \) is a function \( \sigma_i : T_i \to \Delta(A_i) \).

Denote by \( \Sigma_i \) the set of all strategies for player \( i \), and write \( \Sigma = \prod_{i \in I} \Sigma_i \) and \( \Sigma_{-i} = \prod_{j \neq i} \Sigma_j \). The expected payoff to player \( i \) of type \( t_i \in T_i \) from playing \( a_i \in A_i \) against opponents' strategy profile \( \sigma_{-i} = (\sigma_j)_{j \neq i} \in \Sigma_{-i} \) is

\[
\mathbb{E}[u_i((a_i, \sigma_{-i}(.)), (t_i, .))]_{t_i} = \sum_{t_{-i} \in T_{-i}} P(t_{-i} | t_i) u_i((a_i, \sigma_{-i}(t_{-i})), (t_i, t_{-i}))
\]

where \( u_i((a_i, .), t) \) is extended to \( \prod_{j \neq i} \Delta(A_j) \) in the usual manner. Let the correspondence \( BR_i : \Sigma_{-i} \times T_i \to A_i \) for each \( i \) be defined by

\[
BR_i(\sigma_{-i})(t_i) = \arg \max_{a_i \in A_i} \mathbb{E}[u_i((a_i, \sigma_{-i}(.)), (t_i, .))]_{t_i}.
\]

\(^9\)For a finite or countably infinite set \( X \), we write \( \Delta(X) \) for the set of all probability distributions on \( X \).
A strategy profile \( \sigma^* = (\sigma^*_i)_{i \in I} \in \Sigma \) is a Bayesian Nash equilibrium of \((T, P, u)\) if for all \( i \in I \), all \( t_i \in T_i \), and all \( a_i \in A_i \),

\[
\sigma^*_i(a_i|t_i) > 0 \Rightarrow a_i \in BR_i(\sigma^*_i)(t_i).
\]

2.3. Robust Equilibria

Given a complete information game \( g \) and an incomplete information elaboration \((T, P, u)\), we denote

\[
T^g_i = \{ t_i \in T_i \mid u_i(a, (t_i, t_{-i})) = g_i(a) \text{ for all } a \in A \text{ and for all } t_{-i} \in T_{-i} \text{ with } P(t_i, t_{-i}) > 0 \}.
\]

For \( \varepsilon \in [0, 1] \), an incomplete information elaboration \((T, P, u)\) is an \( \varepsilon \)-elaboration of \( g \) if \( P(\prod_{i \in I} T^g_i) \geq 1 - \varepsilon \). The notion of information robustness of equilibrium is due to Kajii and Morris (1997a).

**Definition 1** A Nash equilibrium \( a^* = (a^*_i)_{i \in I} \in A \) of a complete information game \( g \) is robust (to incomplete information) in \( g \) if for any \( \varepsilon > 0 \) such that for any \( \varepsilon \)-elaboration \((T, P, u)\) of \( g \), there exists a Bayesian Nash equilibrium \( \sigma^* = (\sigma^*_i)_{i \in I} \in \Sigma \) such that

\[
\sum_{t = (t_i)_{i \in I} \in T} P(t) \prod_{i \in I} \sigma^*_i(a^*_i|t_i) \geq 1 - \delta.
\]

Not all (strict) Nash equilibria are robust to incomplete information (Rubinstein (1989)), and even a unique equilibrium, which is strict, may not be robust (Kajii and Morris (1997a)). For \( p = (p_i)_{i \in I} \in [0, 1]^I \), an action profile \( a^* \in A \) is a \( p \)-dominant equilibrium of \( g \) if for all \( i \in I \) and all \( \nu_i \in \Delta(A_{-i}) \),

\[
\nu_i(a^*_{-i}) \geq p_i \Rightarrow a^*_i \in \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} g_i(a_i, a_{-i})\nu_i(a_{-i}).
\]

Kajii and Morris (1997a) show that a \( p \)-dominant equilibrium with \( \sum_{i \in I} p_i < 1 \) is robust to incomplete information.

**Remark 1** In Kajii and Morris (1997a), robustness is defined for action distributions (elements of \( \Delta(A) \)) whereas it is defined for pure-action profiles (elements of \( A \)) here. This is without loss of generality in generic supermodular games, where a robust equilibrium, if any, is known to be a pure-action profile (and unique) by Basteck et al. (2010; 2013); in fact, the proof of Theorem 11 in Basteck et al. (2010) shows that the support of any robust action distribution lies between the smallest and the largest pure-action profiles selected by global games, and these action profiles coincide for generic payoffs by Basteck et al. (2013, Theorem 2). In nongeneric supermodular games, nondegenerate action distributions may become robust; for example, any mixed-action profile is robust in games with constant payoff functions.
3. GENERALIZED BELIEF OPERATORS

3.1. Binary-Action Supermodular Games

In what follows, we consider binary-action games, where \( A_1 = \cdots = A_{|I|} = \{0,1\} \). For \( S \in \mathcal{I} \), we denote by \( 1_S \) the action profile such that all players in \( S \) play action 1, and the others play action 0; by convention, we write \( 1 \) for \( 1_I \). Given a binary-action complete information game \( g \), for each \( i \in I \), we define the “payoff increment” function \( f_i : \mathcal{I}_i \to \mathbb{R} \) by

\[
f_i(S) = g_i(1_{S \cup \{i\}}) - g_i(1_S)
\]

for \( S \in \mathcal{I}_i \). That is, \( f_i(S) \) is the payoff increment for player \( i \) from playing action 1 over playing action 0 when the set of opponent players playing action 1 is \( S \).

Without loss of generality, we identify the binary-action complete information game with the profile \( f = (f_i)_{i \in I} \) of payoff increment functions. Throughout our analysis, we focus on supermodular games: we assume that every player has weakly increasing payoff increments, i.e., for all \( i \in I \) and all \( S, S' \in \mathcal{I}_i \),

\[
f_i(S) \leq f_i(S') \text{ whenever } S \subseteq S'.
\]

3.2. Generalized Beliefs

Let a binary-action supermodular game \( f = (f_i)_{i \in I} \) and a type space \((\mathcal{T}, P)\) be given as above. We denote \( \mathcal{T}_i = 2^{\mathcal{T}_i} \), \( \mathcal{T} = \prod_{i \in I} \mathcal{T}_i \), and \( \mathcal{T}_i = \prod_{j \neq i} \mathcal{T}_j \) for \( i \in I \).

We define \( f_i \)-belief for each player \( i \) as follows. This notion has been introduced in Morris and Shin (2007) and generalizes that of \( p \)-belief by Monderer and Samet (1989). The reader should bear in mind that an event \( E_i \in \mathcal{T}_i \) can be thought as player \( i \)'s (pure) strategy, in a binary-action incomplete information elaboration, that plays action 1 on, and only on, \( E_i \), while \( E_{-i} = (E_j)_{j \neq i} \in \mathcal{T}_{-i} \) as a profile of opponent players’ strategies, and that the \( f_i \)-belief operator is closely related to the best response function for player \( i \)'s types whose payoffs are given by \( f_i \), as discussed formally in Section 3.4.

For \( i \in I \) and \( E_{-i} \in \mathcal{T}_{-i} \), define the function \( S_{E_{-i}} : \mathcal{T}_{-i} \to \mathcal{T}_i \) by

\[
S_{E_{-i}}(t_{-i}) = \{ j \neq i \mid t_j \in E_j \}.
\]

Then the expectation of the random variable \( t_{-i} \mapsto f_i(S_{E_{-i}}(t_{-i})) \) conditional on \( t_i \in \mathcal{T}_i \) is

\[
\mathbb{E}[f_i(S_{E_{-i}}(\cdot))|t_i] = \sum_{t_{-i} \in \mathcal{T}_{-i}} P(t_{-i}|t_i) f_i(\{ j \neq i \mid t_j \in E_j \})
\]

\[
= \sum_{S \in \mathcal{I}_{-i}} P \left( \prod_{j \in S} E_j \times \prod_{j \notin S \cup \{i\}} (T_j \setminus E_j) \bigg| t_i \right) f_i(S).
\]
Definition 2. For $i \in I$, $t_i \in T_i$, and $E = (E_i)_{i \in I} \in \mathcal{T}$, a type $t_i$ of player $i$ is said to have $f_i$-belief about $E$ if $t_i \in E_i$ and $\mathbb{E}[f_i(S_{E_{-i}}(\cdot))|t_i] \geq 0$. Player $i$’s $f_i$-belief operator $B_i^{f_i} : \mathcal{T} \to T_i$ is defined by

$$B_i^{f_i}(E) = \{t_i \in E_i \mid \mathbb{E}[f_i(S_{E_{-i}}(\cdot))|t_i] \geq 0\}$$

for $E = (E_i)_{i \in I} \in \mathcal{T}$, i.e., $B_i^{f_i}(E)$ is the set of player $i$’s types that have $f_i$-belief about $E$.

As we explain below, the notion of $f_i$-belief generalizes the notion of $p_i$-belief and accommodates some other interesting cases.

Example 1 ($p_i$-Belief) The notion of $f_i$-belief generalizes that of $p_i$-belief. For $p_i \in [0, 1]$, player $i$’s $p_i$-belief operator $B_i^{p_i} : \mathcal{T} \to T_i$ is defined by

$$B_i^{p_i}(E) = \{t_i \in E_i \mid P(E_{-i}|t_i) \geq p_i\},$$

where $E_{-i} = \prod_{j \neq i} E_j$. To see that the notion of $p_i$-belief is a special case of that of $f_i$-belief, given $p_i$, define the payoff increment function $f_i^{p_i}$ by

$$f_i^{p_i}(S) = \begin{cases} 1 - p_i & \text{if } S = I \setminus \{i\}, \\ -p_i & \text{otherwise} \end{cases}$$

for $S \in \mathcal{I}_{-i}$. Then one can verify that for any $E \in \mathcal{T}$,

$$\mathbb{E}[f_i^{p_i}(S_{E_{-i}}(\cdot))|t_i] = (1 - p_i)P(E_{-i}|t_i) - p_i(1 - P(E_{-i}|t_i)) = P(E_{-i}|t_i) - p_i,$$

and hence $B_i^{f_i^{p_i}}(E) = B_i^{p_i}(E)$.

The payoff increment function $f_i^{p_i}$ is constant for all $S \subseteq I \setminus \{i\}$, and hence $B_i^{p_i}(E)$ is determined only by $E_i$ and the posterior probability that all opponents belong to $E_{-i}$. In contrast, if $|I| \geq 3$, then the generalized belief operator may depend on other statistics such as the number of opponents belonging to $E_j$.

Example 2 (Aggregation) For a weakly increasing function $h : \{0, \ldots, |I| - 1\} \to \mathbb{R}$ and a threshold value $c_i \in \mathbb{R}$, define the payoff increment function $f_i^{h,c_i}$ by

$$f_i^{h,c_i}(S) = h(|S|) - c_i$$

for $S \in \mathcal{I}_{-i}$. Then a type has $f_i^{h,c_i}$-belief about $E$ if and only if the type lies in $E_i$ and the expected value of the number of players $j \neq i$ whose types belong to $E_j$, transformed by the $h$ function, is at least $c_i$, i.e.,

$$\sum_{k=0}^{|I|-1} P(\{t_{-i} \in T_{-i} \mid ||\{j \neq i \mid t_j \in E_j\}| = k\}|t_i) h(k) \geq c_i.$$
This class of belief operators subsumes the \( p_i \)-belief operator in Example 1 with
\[
h(k) = \begin{cases} 
1 & \text{if } k = n - 1, \\
0 & \text{otherwise}
\end{cases}
\]
and \( c_i = p_i \) for all \( i \). Coordinated attack models with symmetric players, often studied in the global game literature (e.g., Morris and Shin (1998)), also belong to this class, where \( h \) is of the form
\[
h(k) = \begin{cases} 
1 & \text{if } k = \bar{k}, \\
0 & \text{otherwise}
\end{cases}
\]
and a common threshold value, i.e., \( c_i = c \) for all \( i \). More generally, all symmetric binary-action supermodular games can be written as \( f_i^{h,c} \) with a common threshold value \( c \).

Allowing for arbitrary \( f \), our analysis is not restricted to the class covered in Example 2. In particular, in Section 5, we will discuss the currency attack game with asymmetric traders studied by Corsetti et al. (2004), where the outcome depends on the identities of the attackers, especially whether the large trader joins the attack or not.

**Example 3 (Unanimity)** For \( y_i, z_i > 0 \), define the payoff increment function \( f_i^{y_i, z_i} \) by
\[
f_i^{y_i, z_i}(S) = \begin{cases} 
y_i & \text{if } S = \emptyset, \\
z_i & \text{if } S = I \setminus \{i\}, \\
0 & \text{otherwise}
\end{cases}
\]
for \( S \in \mathcal{I} \). Then a type has \( f_i^{y_i, z_i} \)-belief about \( E \) if and only if the type lies in \( E_i \) and the ratio between the posterior probabilities of \( \prod_{j \neq i} E_j \) and of \( \prod_{j \neq i} (T_j \setminus E_j) \) is at least \( y_i/z_i \).

Similarly to the \( p \)-belief operator by Monderer and Samet (1989), the \( f_i \)-belief operator satisfies the following properties. For \( E = (E_i)_{i \in I}, E' = (E'_i)_{i \in I} \in \mathcal{T} \), we write \( E \subset E' \) if \( E_i \subset E'_i \) for all \( i \in I \); for a sequence \( (E^n)_{n=0}^{\infty} = ((E^n_i)_{i \in I})_{n=0}^{\infty} \) in \( \mathcal{T} \), we write \( \bigcap_{n=0}^{\infty} E^n \) for \( ((\bigcap_{n=0}^{\infty} E^n_i)_{i \in I})_{n=0}^{\infty} \).

**Proposition 1** Let a payoff increment function \( f_i \) be weakly increasing.
1. \( B_i^{f_i}(E) \subset E_i \).
2. If \( E \subset E' \), then \( B_i^{f_i}(E) \subset B_i^{f_i}(E') \).
3. If \( (E^n)_{n=0}^{\infty} \) is a weakly decreasing sequence, then \( B_i^{f_i}(\bigcap_{n=0}^{\infty} E^n) = \bigcap_{n=0}^{\infty} B_i^{f_i}(E^n) \).

\(^{10}\)For this class of games, Carlson and van Damme (1993b) and Kim (1996) investigate various approaches to equilibrium selection, including the global game approach.
Property (1) holds by definition, while property (2) by the monotonicity of \( f_i \).
In property (3), the inclusion “\( \supset \)" follows from the continuity of the probability measure for monotone sequences, while the reverse inclusion “\( \subset \)" from property (2).

3.3. Common Beliefs

Following Monderer and Samet (1989) and Morris and Shin (2007), we use the \( f_i \)-belief operators to define common \( f \)-belief.

First, a profile \( F = (F_j)_{j \in I} \) is \( f \)-evident if \( F_i \subset B^{f_i}_i(F) \) for all \( i \in I \). By property (1) in Proposition 1, the condition is equivalent to that \( F_i = B^{f_i}_i(F) \) for all \( i \in I \), i.e., that \( F \) is a fixed point of \( (B^{f_i}_i)_{i \in I} \).

We next define common \( f \)-belief by iteration of the \( f_i \)-belief operators. Let

\[
B^{f_i,0}_i(E) = E_i, \\
B^{f_i,n+1}_i(E) = B^{f_i}_i((B^{f_{j\neq i}}_j(E))_{j \in I}).
\]

The sequence \( (B^{f_i,n}_i(E))_{n=0}^{\infty} \) is weakly decreasing by properties (1) and (2). Now define

\[
CB^{f_i}_i(E) = \bigcap_{n=0}^{\infty} B^{f_i,n}_i(E).
\]

We say that \( t_i \) has common \( f \)-belief about \( E \) if \( t_i \in CB^{f_i}_i(E) \).

By property (3), \( (CB^{f_i}_i(E))_{i \in I} \) is \( f \)-evident. By property (2), if \( F \subset E \), and \( F \) is \( f \)-evident, then \( F_i \subset CB^{f_i}_i(E) \) for all \( i \in I \). Thus we have:

**Proposition 2** Let a binary-action game \( f \) be supermodular. For \( E \in \mathcal{T} \), \( (CB^{f_i}_i(E))_{i \in I} \) is the largest \( f \)-evident event profile contained in \( E \).

This is a straightforward generalization of the corresponding result for common \( p \)-belief (Monderer and Samet (1989), Kajii and Morris (1997a)), where for \( p = (p_i)_{i \in I} \in [0,1]^I \), \( CB^p_i \) and \( p \)-evidence are defined similarly to the above with \( (B^p_i)_{i \in I} \) (Example 1) in place of \( (B^{f_i}_i)_{i \in I} \).

3.4. Connections to Incomplete Information Elaborations

Given a binary-action supermodular game \( f \), consider an incomplete information elaboration \( (T, P, u) \) of \( f \). Denote

\[
T^{f_i}_i = \{ t_i \in T_i \mid u_i(1_{S \cup \{t_i\}}, (t_i, t_{-i})) - u_i(1_S, (t_i, t_{-i})) = f_i(S) \text{ for all } S \in I_{-i} \text{ and for all } t_{-i} \in T_{-i} \text{ with } P(t_i, t_{-i}) > 0 \},
\]

and \( T^f = (T^{f_i}_i)_{i \in I} \in \mathcal{T} \).
We identify an event $E_i \in \mathcal{T}_i$ with player $i$'s (pure) strategy $\sigma_i$ such that $\sigma_i(1|t_i) = 1$ if and only if $t_i \in E_i$. Through this identification, type $t_i \in T_i^{f_i}$ has $f_i$-belief about $(\{t_i\}, E_{-i})$ if and only if action 1 is a best response to (the strategy profile identified with) $E_{-i}$ for type $t_i$.

Common $f$-belief in type space $(T, P)$ is closely related to the iterated elimination procedure of strictly dominated strategies and Bayesian Nash equilibria in elaboration $(T, P, u)$. To see this, take a “canonical” elaboration where 0 is a dominant action for all types outside $T_i^{f_i}$. In this elaboration, $B_{f_i}^{I_i}(T_i^{f_i}, E_{-i})$ corresponds to the largest best response to $E_{-i}$, i.e., $t_i \in B_{f_i}^{I_i}(T_i^{f_i}, E_{-i})$ if and only if $1 \in BR_i(\sigma_{-i})(t_i)$, where $\sigma_{-i}$ is the strategy profile such that for all $j \neq i$, $\sigma_j(1|t_j) = 1$ if and only if $t_j \in E_j$. The $n$-times iteration $B_{f_i}^{I_i:n}(T^f)$ corresponds to the largest strategy that survives the $n$-times iterated elimination of strictly dominated strategies, and the limit $CB_{f_i}^{I_i}(T^f)$ corresponds to the largest strategy that survives the iterated elimination of strictly dominated strategies. In fact, the profile $(CB_{f_i}^{I_i}(T^f))_{i \in I}$ is the largest Bayesian Nash equilibrium in $(T, P, u)$.

In a general elaboration, where the payoff functions of the types outside $T_i^{f_i}$ are arbitrary, the largest best response to $E_{-i}$ may be strictly larger than $B_{f_i}^{I_i}(T_i^{f_i}, E_{-i})$. As shown in the proposition below, there exists a Bayesian Nash equilibrium at least as large as the largest $f$-evident event profile $(CB_{f_i}^{I_i}(T^f))_{i \in I}$, i.e., a Bayesian Nash equilibrium $\sigma^*$ such that for each $i \in I$, $\sigma^*_i(1|t_i) = 1$ whenever $t_i \in CB_{f_i}^{I_i}(T^f)$.

The next proposition formally states these relations.

**Proposition 3** Let a binary-action game $f$ be supermodular.

1. For any elaboration $(P, T, u)$ of $f$, there exists a Bayesian Nash equilibrium $\sigma^*$ such that for each $i \in I$, $\sigma^*_i(1|t_i) = 1$ whenever $t_i \in CB_{f_i}^{I_i}(T^f)$.

2. For any type space $(T, P)$ and any event profile $E \in T$, there exists a profile $u$ of payoff functions such that in elaboration $(T, P, u)$, $T^f = E$ and for each $i \in I$, the largest strategy $\sigma_i$ that survives the iterated elimination of strictly dominated strategies is such that $\sigma_i(1|t_i) = 1$ if and only if $t_i \in CB_{f_i}^{I_i}(T^f)$.

**Proof:** For part (1), let $\Sigma^*_i \subset \Sigma_i$ be the set of all strategies $\sigma_i$ such that $\sigma_i(1|t_i) = 1$ whenever $t_i \in CB_{f_i}^{I_i}(T^f)$. Write $\Sigma^* = \prod_{i \in I} \Sigma^*_i$ and $\Sigma_{-i}^* = \prod_{j \neq i} \Sigma^*_j$. Note that $\Sigma^*$ is a nonempty, convex, and compact subset of a Banach space. We define the correspondence $\beta: \Sigma^* \to \Sigma^*$ by

$$\beta(\sigma) = \{\sigma' \in \Sigma^* \mid \text{for all } i \in I: \sigma'_i(a_i|t_i) > 0 \Rightarrow a_i \in BR_i(\sigma_{-i})(t_i)\}.$$ 

By Proposition 2, $(CB_{f_i}^{I_i}(T^f))_{i \in I}$ is $f$-evident and contained in $T^f$. Thus, if $t_i \in CB_{f_i}^{I_i}(T^f)$ and $\sigma_{-i} \in \Sigma_{-i}^*$, then $1 \in BR_i(\sigma_{-i})(t_i)$, implying the nonempty-valuedness of $\beta$. We can also verify that $\beta$ is convex- and compact-valued and upper semi-continuous. Hence, it follows from Kakutani’s fixed-point theorem that $\beta$ has a fixed point $\sigma^*$ in $\Sigma^*$, which is a Bayesian Nash equilibrium of the elaboration $(P, T, u)$. 

For part (2), let the payoff functions $u$ be such that for each $i \in I$, $T_i^f = E_i$, and action 0 is a dominant action for all types $t_i \notin E_i$. The conclusion then follows as in the discussion above.

Q.E.D.

By Proposition 3, we obtain the following characterization of the robustness of $1 = (1, \ldots, 1) \in A$ in terms of common $f$-belief: the “if” and the “only if” parts follow from parts (1) and (2) of Proposition 3, respectively.

**Proposition 4**  Let a binary-action game $f$ be supermodular. Then $1$ is robust to incomplete information in $f$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any type space $(T, P)$ and any event profile $E = (E_i)_{i \in I} \in T$, we have

$$P(E) \geq 1 - \varepsilon \Rightarrow P\left(CB^f(E)\right) \geq 1 - \delta,$$

where $E = \prod_{i \in I} E_i$ and $CB^f(E) = \prod_{i \in I} CB^f_i(E)$.

Our next task is to explore the relationship between $P(E)$ and $P(CB^f(E))$. For each $n$, if $P(E)$ is close to 1, then $P(\prod B_i^{f,n}(E))$ is necessarily close to 1 (as long as $f_i(I \setminus \{i\}) > 0$ for all $i$), but, in general, this implication does not hold for $P(CB^f(E))$. Our main theorems characterize a (generic) necessary and sufficient condition for $f$ under which this implication holds, uniformly over all type spaces and event profiles thereof.

### 4. Robust Equilibria in Binary-Action Supermodular Games

#### 4.1. Potentials

Monderer and Shapley (1996) introduce the notion of potential for general normal form games. In our notation for binary-action games, the definition is written as follows:

**Definition 3**  A function $v: I \rightarrow \mathbb{R}$ is a potential of a binary-action game $f = (f_i)_{i \in I}$ if

$$f_i(S) = v(S \cup \{i\}) - v(S)$$

for all $i \in I$ and $S \in I_{-i}$.

$S^* \in I$ is a potential maximizer in $f$ if $f$ admits a potential $v$ that is strictly maximized at $S^*$, i.e., $v(S^*) > v(S)$ for all $S \in I$ with $S \neq S^*$.

By definition, a potential maximizer is a strict Nash equilibrium.

Not every game admits a potential. It is easy to see that a binary-action game $f$ admits a potential if and only if

$$f_i(S \cup \{j\}) - f_i(S) = f_j(S \cup \{i\}) - f_j(S)$$
for any $i \neq j$ and $S \subset I \setminus \{i, j\}$. If $f$ admits a potential, then the potential is determined uniquely up to constants:

$$v(S) = v(\emptyset) + \sum_{t=1}^{k} f_{i_t}([i_1, \ldots, i_{t-1}])$$

for $S = \{i_1, \ldots, i_k\} \in I$, where the summation is independent of the order of players in $S$.

**Example 4 (p$_i$-Belief)** Suppose that each player $i \in I$ has the payoff increment function $f^p_i$ as in Example 1. Then the binary-action game $(f^p_i)_{i \in I}$ admits the following potential:

$$v(S) = \begin{cases} 1 - \sum_{i \in I} p_i & \text{if } S = I, \\ -\sum_{i \in S} p_i & \text{otherwise.} \end{cases}$$

Note that this function $v$ is strictly maximized at $I$ if and only if $\sum_{i \in I} p_i < 1$.

**Example 5 (Aggregation)** Suppose that each player $i \in I$ has the payoff increment function $f_{h,c_i}^i$ as in Example 2, where we assume that $c_1 \leq c_2 \leq \cdots \leq c_{|I|}$ without loss of generality. Then the binary-action game $(f_{h,c_i}^i)_{i \in I}$ admits the following potential:

$$v(S) = \sum_{n=0}^{|S|-1} h(n) - \sum_{i \in S} c_i,$$

which is strictly maximized at $I$ if and only if $\sum_{n=k}^{|I|-1} h(n) > \sum_{i=k+1}^{|I|} c_i$ for any $0 \leq k \leq |I| - 1$.

Our main tool is the following generalization of potential, which is in parallel to characteristic/monotone potentials for general normal form games introduced by Morris and Ui (2005).

**Definition 4** $S^* \in I$ is a **monotone potential maximizer** in a binary-action game $f = (f_i)_{i \in I}$ if there exist a function $v: I \to \mathbb{R}$ and $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i > 0$ such that

$$\lambda_i f_i(S) \geq v(S \cup \{i\}) - v(S)$$

for all $i \in S^*$ and $S \in \mathcal{I}_{-i}$,

$$\lambda_i f_i(S) \leq v(S \cup \{i\}) - v(S)$$

for all $i \in I \setminus S^*$ and $S \in \mathcal{I}_{-i}$, and $v(S^*) > v(S)$ for all $S \in \mathcal{I}$ with $S \neq S^*$.

Such a function $v$ is called a monotone potential of $f$ for $S^*$. 
Clearly, if $S^*$ is a potential maximizer in $f$, then it is a monotone potential maximizer in $f$. In the definition of a monotone potential, the equality in the definition of a potential is replaced with inequalities, while the multiplier $\lambda_i > 0$ is for normalization.\footnote{Our definition is in fact equivalent to strict monotone potential introduced by Oyama et al. (2008), and slightly stronger than monotone potential in the sense of Morris and Ui (2005), which requires the existence of $\mathcal{X} = (\lambda'_i)_{i \in I}$ with $\lambda'_i \geq 0$ such that $f_i(S) \geq (\text{resp.} \leq) \lambda'_i(v(S \cup \{i\}) - v(S))$ for all $i \in S^*$ (resp. $i \in I \setminus S^*$) and $S \in \mathcal{I}_v$.}

**Remark 2** As a corollary of their main results, Oyama et al. (2008, p. 176) show that a supermodular game has at most one monotone potential maximizer. That is, if $f$ is supermodular, and $v$ is a monotone potential of $f$ for $S^*$, then there is no monotone potential of $f$ for any $S \neq S^*$. On the other hand, there is a nonempty open set of supermodular games that have no monotone potential maximizer. See, e.g., Morris and Ui (2005, Section 7.2) and Example 7 below.

In what follows, we focus on the case of $S^* = I$ for expositional ease. In particular, our main theorems (Theorems 1–3) will be stated in terms of the (non-)existence of a monotone potential for $I$. Nevertheless, we demonstrate in Appendix A.4 that Theorem 3 extends to general $S$ (Corollary A.1).

Before stating the main theorems in the subsequent subsections, we give a characterization for the existence of a monotone potential with a fixed $\gamma$. Let $\Gamma$ be the set of all finite sequences $\gamma = (i_1, \ldots, i_k)$ of distinct players in $I$, where $1 \leq k \leq |I|$. For each $\gamma = (i_1, \ldots, i_k) \in \Gamma$ and $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i > 0$, we define

$$F(\gamma, \lambda) = \sum_{\ell=1}^k \lambda_i f_i(I \setminus \{i_1, \ldots, i_\ell\}).$$

The following is a special case of Okada and Tercieux (2012, Proposition 1).

**Lemma 1** For a binary-action game $f = (f_i)_{i \in I}$ and $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i > 0$, $f$ admits a monotone potential with $\lambda$ for $I$ if and only if $F(\gamma, \lambda) > 0$ for all $\gamma \in \Gamma$.

**Proof:** Suppose that $v : \mathcal{I} \to \mathbb{R}$ is a monotone potential of $f$ with $\lambda$ for $I$. Fix any $\gamma = (i_1, \ldots, i_k) \in \Gamma$. Then we have

$$\lambda_i f_i(I \setminus \{i_1, \ldots, i_{\ell-1}\}) \geq v(I \setminus \{i_1, \ldots, i_{\ell-1}\}) - v(I \setminus \{i_1, \ldots, i_{\ell}\})$$

for any $\ell = 1, \ldots, k$. Summing up over all $\ell$, we have $F(\gamma, \lambda) \geq v(I) - v(I \setminus \{i_1, \ldots, i_k\}) > 0$.

Conversely, if $F(\gamma, \lambda) > 0$ for every $\gamma \in \Gamma$, then it is easy to check that

$$v(S) = \begin{cases} 0 & \text{if } S = I, \\ \min_{\gamma \in (i_1, \ldots, i_k)} F(\gamma, \lambda) & \text{otherwise} \end{cases}$$

for all $S \in \mathcal{I}_v$. 

\footnote{Our definition is in fact equivalent to strict monotone potential introduced by Oyama et al. (2008), and slightly stronger than monotone potential in the sense of Morris and Ui (2005), which requires the existence of $\mathcal{X} = (\lambda'_i)_{i \in I}$ with $\lambda'_i \geq 0$ such that $f_i(S) \geq (\text{resp.} \leq) \lambda'_i(v(S \cup \{i\}) - v(S))$ for all $i \in S^*$ (resp. $i \in I \setminus S^*$) and $S \in \mathcal{I}_v$.}
is a monotone potential of \( f \) with \( \lambda \) for \( I \).

Q.E.D.

Example 6 (Unanimity) Suppose that each player \( i \in I \) has the payoff increment function \( f_{i}^{y_{i}} \), as in Example 3. By Lemma 1, the binary-action game \((f_{i}^{y_{i}})_{i \in I}\) admits a monotone potential with \( \lambda = (\lambda_{i})_{i \in I} \) for \( I \) if and only if \( \lambda_{i}z_{i} > \lambda_{j}y_{j} \) for all \( i, j \in I \) with \( i \neq j \). In fact, if \((f_{i}^{y_{i}})_{i \in I}\) admits a monotone potential with some \( \lambda \) for \( I \), then it is necessary to have \((\lambda_{i}z_{i})(\lambda_{j}z_{j}) > (\lambda_{j}y_{j})(\lambda_{i}y_{i})\), and hence \( z_{i}z_{j} > y_{i}y_{j} \) for all \( i, j \in I \) with \( i \neq j \). Conversely, if \( z_{i}z_{j} > y_{i}y_{j} \) for all \( i, j \in I \) with \( i \neq j \), then \((f_{i}^{y_{i}})_{i \in I}\) admits a monotone potential with \( \lambda_{i} = 1/\sqrt{y_{i}z_{i}} \) for each \( i \in I \), as we have \( \lambda_{i}z_{i} = \sqrt{z_{i}/y_{i}} > \sqrt{y_{j}/z_{j}} = \lambda_{j}y_{j} \) for all \( i, j \in I \) with \( i \neq j \).

4.2. The Critical Path Theorem

Using the \( p \)-belief operator, Kajii and Morris (1997a, Proposition 4.2) show the following “Critical Path Theorem”: For \( p = (p_{i})_{i \in I} \in [0, 1]^{I} \), if \( \sum_{i \in I} p_{i} < 1 \), then for any type space \((T, P)\) and any event profile \( E \in T \), we have

\[
P\left(CB^{p}(E)\right) \geq 1 - \kappa^{KM}(p) \left(1 - P(E)\right),
\]

with

\[
\kappa^{KM}(p) = \frac{1 - \min_{i \in I} p_{i}}{1 - \sum_{i \in I} p_{i}}
\]

where \( CB^{p}(E) = \prod_{i \in I} CB_{i}^{p}(E) \) and \( E = \prod_{i \in I} E_{i} \). Note that the coefficient \( \kappa^{KM}(p) \) depends only on \( p \) and is independent of the type space \((T, P)\) or the event profile \( E \). Thus, the inequality implies that, with a fixed \( p \) satisfying \( \sum_{i \in I} p_{i} < 1 \), we have \( P\left(CB^{p}(E)\right) \to 1 \) as \( P(E) \to 1 \). Combining this theorem with an argument similar to our Propositions 3 and 4, Kajii and Morris (1997a, Proposition 5.3) establish the robustness of a \( p \)-dominant equilibrium with \( \sum_{i \in I} p_{i} < 1 \).

We extend the Critical Path Theorem to the \( f \)-belief operator, where the assumption “\( \sum_{i \in I} p_{i} < 1 \)” is replaced with the existence of a monotone potential of \( f \) for \( I \). Let \( v: I \to \mathbb{R} \) be a function that is strictly maximized at \( I \). Define

\[
\kappa(v) = 1 + \frac{M}{v(I) - v'},
\]

where

\[
v' = \max_{S \subseteq I} v(S),
\]

\[
M = \max_{S \subseteq S' \subseteq I} (v(S) - v(S')).
\]

Similarly to \( \kappa^{KM}(p) \), \( \kappa(v) \) depends only on the function \( v \).

\footnote{See also Morris and Ui (2005, Section 7.2).}
Generalized Belief Operator

Theorem 1 Let a binary-action game \( f \) be supermodular. Suppose that \( I \) is a monotone potential maximizer in \( f \) with a monotone potential \( v \). Then for any type space \( (T, P) \) and any event profile \( E = (E_i)_{i \in T} \in T \), we have

\[
P \left( CB^f(E) \right) \geq 1 - \kappa(v) (1 - P(E)),
\]

where \( CB^f(E) = \prod_{i \in T} CB^f_i(E) \) and \( E = \prod_{i \in T} E_i \).

Theorem 1 reduces to Kajii and Morris (1997a, Proposition 4.2) in the case of common \( p \)-belief with \( \sum_{i \in T} p_i < 1 \). To verify this, given \( p \) let the binary-game \( (f^p_i)_{i \in T} \) be as given in Example 1,

\[
f^p_i(S) = \begin{cases} 1 - p_i & \text{if } S = I \setminus \{i\}, \\ -p_i & \text{otherwise}, \end{cases}
\]

and the potential \( v \) be as given in Example 4,

\[
v(S) = \begin{cases} 1 - \sum_{i \in T} p_i & \text{if } S = I, \\ -\sum_{i \not\in S} p_i & \text{otherwise}, \end{cases}
\]

which is strictly maximized at \( I \) if and only if \( \sum_{i \in T} p_i < 1 \). Then, we have \( v' = 0 \) and \( M = \max_{i \in T} \sum_{j \neq i} p_j \), and thus

\[
\kappa(v) = 1 + \frac{\max_{i \in T} \sum_{j \neq i} p_j}{1 - \sum_{i \in T} p_i} = 1 - \min_{i \in T} p_i = \kappa^{KM}(p).
\]

Our proof, given in Appendix A.1, proceeds along the same line as the higher order beliefs approach of Kajii and Morris (1997a) in that we apply the \( f^p_i \)-belief operators to \( E \) iteratively, and evaluate the probability of the event that survives in each step.\(^{13}\) Our proof is, however, significantly simpler and more transparent, in our opinion, than the proof by Kajii and Morris (1997a), even in the case of common \( p \)-belief, i.e., when \( f = (f^p_i)_{i \in T} \). Technically, the iteration steps give us inequalities involving the prior probabilities of relevant events weighted with the values of \( f^p_i \)’s. We sum these inequalities across different players and different steps, utilize the common function \( v \) to “aggregate” the player-specific weights, and thus obtain the estimate of \( P \left( CB^f(E) \right) \).

4.3. Contagion

Theorem 1 implies that if \( I \) is a monotone potential maximizer in \( f \), then for any type space and any high-probability event, players have common \( f \)-belief.

\(^{13}\)In our working paper (Oyama and Takahashi (2019, Appendix A.2)), we provide an alternative proof that adopts the potential maximization approach of Ui (2001) and Morris and Ui (2005).
about that event with high probability. In this section, we prove the generic converse (Theorem 2): if $I$ is not a monotone potential maximizer in $f$, then there generically exist a type space and a high-probability event such that players never have common $f$-belief about that event.

Our proof exploits the duality between payoffs and probabilities. Recall that we have in Lemma 1 a characterization for the existence of a monotone potential with a fixed $\lambda$. Notice that the condition $F(\gamma, \lambda) > 0$ for all $\gamma \in \Gamma$ constitutes a system of linear inequalities in $\lambda$. By a duality theorem, we eliminate the variable $\lambda$ there to obtain a dual condition in terms of expected payoffs, where the dual variable is a probability distribution, denoted $\mu$, over the set of finite sequences of distinct players (Lemma 2). We then construct a type space à la global games (Carlsson and van Damme (1993a)) with the desired property, where the “noise” structure is determined by $\mu$ obtained above (as sketched after the statement of Theorem 2).

For $i \in I$, let $\Gamma_i$ be the set of sequences in $\Gamma$ that contain $i$, and for $\gamma \in \Gamma_i$, let $S(i, \gamma) \in \mathcal{L}_i$ be the set of player $i$’s opponents who are not listed in $\gamma$ earlier than $i$, i.e., the set of players who are either listed in $\gamma$ later than $i$ or not listed in $\gamma$. For $\mu \in \Delta(\Gamma)$, let $I(\mu) = \{i \in I \mid \mu(\Gamma_i) > 0\}$, i.e., the set of players who are listed in some $\gamma$ such that $\mu(\gamma) > 0$. Let $\Delta^*(\Gamma) = \{\mu \in \Delta(\Gamma) \mid \mu(\Gamma_i) = 1 \text{ for all } i \in I(\mu)\}$. Note that $\mu \in \Delta^*(\Gamma)$ assigns positive probability only to permutations of $I(\mu)$.

Lemma 2. For a binary-action game $f = (f_i)_{i \in I}$, (1) either $I$ is a monotone potential maximizer in $f$, or there exists $\mu \in \Delta(\Gamma)$ such that
\[ \sum_{S \in \mathcal{L}_i} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})f_i(S) \leq 0 \] (4.1)
for all $i \in I$, but not both; and (2) in the latter case, if $f$ is supermodular, then there exists $\mu \in \Delta^*(\Gamma)$ such that (4.1) holds for all $i \in I$.

Proof: By Lemma 1, the existence of a monotone potential for $I$ is equivalent to the existence of $\lambda$ with $\lambda_i > 0$ such that $F(\gamma, \lambda) > 0$ for any $\gamma \in \Gamma$. By a variant of Farkas’ lemma, either this condition holds, or there exists $\mu \in \Delta(\Gamma)$ such that
\[ \sum_{\gamma \in \Gamma_i} \mu(\gamma)f_i(S(i, \gamma)) \leq 0 \]
for all $i \in I$, but not both. The desired expression (4.1) in part (1) follows by summing up $\mu(\gamma)$ over all $\gamma$ with the same $S(i, \gamma)$. See Oyama and Takahashi (2019, Section A.3) for the proof of part (2).

For generic supermodular $f$, the condition (4.1) can be strengthened to its strict version: there exists $\mu \in \Delta^*(\Gamma)$ such that
\[ \sum_{S \in \mathcal{L}_i} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})f_i(S) < 0 \] (4.2)
for all \( i \in I(\mu) \). More precisely, the set of all binary-action supermodular games that satisfy (4.2) for some \( \mu \in \Delta^*(\Gamma) \) is open and dense in the set of those that satisfy (4.1) for some \( \mu \in \Delta^*(\Gamma) \). The former set is open because for each \( \mu \), (4.2) imposes finitely many strict inequalities; it is dense in the latter set because, for all \( i \in I(\mu) \), perturbing \( f_i \) by \( f'_i(S) = f_i(S) - \varepsilon \) for all \( S \in \mathcal{I}_i \) would make the corresponding inequality strict.

The following theorem shows that the existence of a monotone potential for \( I \) is not only sufficient, but also necessary for the Critical Path Theorem for a generic choice of \( f \).

**Theorem 2** Let a binary-action game \( f \) be supermodular. For generic \( f \), if \( I \) is not a monotone potential maximizer in \( f \), then for any \( \varepsilon \in (0, 1] \), there exist a type space \((T, P)\) and an event profile \( E = (E_i)_{i \in I} \in T \) such that \( P(E) = 1 - \varepsilon \) and \( P(CB^f(E)) = 0 \), where \( E = \prod_{i \in I} E_i \) and \( CB^f(E) = \prod_{i \in I} CB_i^f(E) \).

**Remark 3** Theorem 2 in fact holds under the following condition: for any sufficiently small \( \eta > 0 \), there exists \( \mu \in \Delta^*(\Gamma) \) such that

\[
\sum_{S \in \mathcal{I}_i} (1 - \eta)^{|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) f_i(S) < 0 \tag{4.3}
\]

for all \( i \in I(\mu) \). Clearly, condition (4.3) is satisfied if \( f \) is generic in the sense of condition (4.2). The converse does not hold. For example, consider the two-player game

\[
\begin{array}{ccc}
0 & \frac{1}{1} & 0 \frac{0}{0} \\
1 & 0 \frac{1}{0} & \frac{1}{1} \\
\end{array}
\]

The induced profile of payoff increment functions,

\[
f_i(\emptyset) = -1, \ f_i(\{3 - i\}) = 1, \tag{4.4}
\]

satisfies condition (4.3), but violates condition (4.2).

**Remark 4** If (4.2) or (4.3) holds with \( \mu \in \Delta^*(\Gamma) \), then, as can be seen in the proof of Theorem 2, the conclusion of the theorem holds with \( CB^f_i(E) = \emptyset \) for all \( i \in I(\mu) \) in place of \( P(CB^f(E)) = 0 \).

The proof of Theorem 2, provided in Appendix A.2, is by construction: given (a family of) \( \mu \in \Delta^*(\Gamma) \) satisfying condition (4.3), we construct a desired type space \((T, P)\) along with event profile \( E \). Here, we describe how it is generated by a signal structure à la global games.

---

\footnote{We need to impose some genericity condition to exclude the trivial game \( f \equiv 0 \), where \( I \) is not a monotone potential maximizer, and yet the Critical Path Theorem holds with \( \kappa = 1 \).}
Let $\mu \in \Delta^*(\Gamma)$ be given. For simplicity, we assume that $I(\mu) = I$. The “state of the world” $m$ is drawn according to the geometric distribution on nonnegative integers with parameter $\eta > 0$. Independently of $m$, a permutation $\gamma = (i_1, \ldots, i_{|I|})$ of $I$ is drawn according to $\mu$, and the “noise” profile $(\xi_i)_{i \in I}$ is determined by $\xi_i = \ell$ if and only if $i = i_\ell$. Given the realization of $(m, \gamma)$, each player $i$ observes the “signal”

$$t_i = m + \xi_i.$$ 

Thus, $T_i = \{1, 2, \ldots\}$, and $P \in \Delta(T)$ is the law of the random variable $t = (t_i)_{i \in I}$. Then let $E_i = \{|I|, |I| + 1, \ldots\}$ for all $i \in I$. Note that $P(E)$ is close to 1 for small $\eta > 0$.

The key property of this type space is that the posterior beliefs about the rankings of signals among the players are given by $\mu$. Note that if $m$ followed the improper uniform distribution on the nonnegative integers, then for player $i$ with signal $t_i$, the posterior beliefs about opponent players who receive signals no smaller than $t_i$ would be given by

$$P(\{j \neq i \mid t_j \geq t_i\} = S \mid t_i) = \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})$$

for each $S \in \mathcal{L}_i$. As $\eta > 0$, the actual posterior beliefs about $t_j$’s are slightly skewed toward smaller values, so that $P(\{j \neq i \mid t_j \geq t_i\} = S \mid t_i)$ is proportional to $(1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})$. Therefore, if $\mu$ satisfies condition (4.3), then the condition for $t_i = \tau \notin B^t_i(E^i)$, where $E^i = \{\tau, \tau + 1, \ldots\}$, is precisely (4.3). By iteration, we have $CB^t_i(E) = \emptyset$ for all $i$.

Rubinstein’s (1989) email game is perhaps the earliest example of Theorem 2 for symmetric $2 \times 2$ coordination games. In our terminology, he constructs a two-player type space $(T, P)$ and an event profile $E$ such that $P(E)$ is bounded away from 0, and $CB^t_1(E) = CB^t_2(E) = \emptyset$ with $p = (1/2, 1/2)$. For many-player games, in the special case where $f$ is given by

$$f_i(S) = -f^p_i(I \setminus (S \cup \{i\})) = \begin{cases} p_i - 1 & \text{if } S = \emptyset, \\ p_i & \text{otherwise} \end{cases}$$

for some $p = (p_i)_{i \in I}$ such that $\sum_{i \in I} p_i \leq 1$ (in which 0 is a $p$-dominant equilibrium), Kajii and Morris (1997a, Lemma 5.5) present a similar construction of type space in which the conclusion of Theorem 2 holds. Also, if there exists a monotone potential of $f$ for some $S^* \neq I$, then $1_{S^*}$ is selected by global games regardless of the noise structure (Frankel et al. (2003, Theorem 4)), and thus an appropriate discretization of the global-game perturbations can be used to obtain the conclusion of Theorem 2.

Note that we prove Theorem 2 under the non-existence of a monotone potential for $I$. Recall that any supermodular game has at most one monotone potential maximizer, and a nonempty open set of games has no monotone potential maximizer (Remark 2 to Definition 4). Therefore, if some $S^* \neq I$ is a
monotone potential maximizer in \( f \), then \( I \) is not a monotone potential maximizer in \( f \), while the converse does not hold in general. That is, our assumption is strictly weaker than the existence of a monotone potential for some \( S^* \neq I \). In particular, when \( f \) has no monotone potential maximizer, our proof uses the fine details of the payoff structure of \( f \) through condition (4.3) to specify the noise structure in the construction there.

4.4. A Generic Characterization of Robust Equilibria

Combining Theorems 1 and 2 with Proposition 4, we provide a generic characterization of robust equilibria in binary-action supermodular games. Here, it is stated in terms of (non-)robustness of the action profile \( 1 = (1, \ldots, 1) \); it applies also to \( 0 = (0, \ldots, 0) \) by reversing the action labels 0 and 1.

Theorem 3 Let a binary-action game \( f \) be supermodular. Then the following results hold.
(1) If \( I \) is a monotone potential maximizer in \( f \), then \( 1 \) is robust to incomplete information in \( f \).
(2) For generic \( f \), if \( I \) is not a monotone potential maximizer in \( f \), then \( 1 \) is not robust to incomplete information in \( f \).

Part (1) of Theorem 3 is a special case of Proposition 2 of Morris and Ui (2005), which applies to many-action games; our contribution is to provide a new proof, as discussed in Section 4.2. The generic converse, part (2), as well as its proof is new in the literature; see Section 4.3.

The qualification “for generic \( f \)” in Theorem 3(2) is inherited from Theorem 2. It can be dispensed with at least in the subclass of binary-action supermodular games, unanimity games, to be discussed in Example 7, where we verify condition (4.3) in Remark 3 whenever \( I \) is not a monotone potential maximizer.\(^{15}\)

Remark 5 Haimanko and Kajii (2016) introduce the notion of approximate robustness, a weakening of the Kajii-Morris robustness. Under the stronger genericity condition (4.2), we can strengthen the conclusion of Theorem 3(2) with “\( 1 \) is not approximately robust” in place of “\( 1 \) is not approximately robust”; see Appendix A.3.

Remark 6 The characterization extends to action profiles other than \( 1 \) and \( 0 \); see Appendix A.4.

Our results also have an implication about global game selection. In a global game, a state of the world \( \theta \) is drawn from the real line and determines the payoffs \( u_i(a, \theta) \) of the players. Each player observes a noisy signal \( \theta + \nu \varepsilon_i \), where

\(^{15}\)For another such example, games with cyclic symmetry, see our working paper (Oyama and Takahashi (2019, Example 8)).
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$(\varepsilon_1, \ldots, \varepsilon_{|I|})$ is a noise profile that is independent of the state $\theta$, and $\nu > 0$ is a scale parameter. Under supermodularity and state-monotonicity in payoffs and the existence of dominance regions, Frankel et al. (2003) show, for many-player many-action games, that an essentially unique equilibrium survives iterative deletion of dominated strategies as $\nu \to 0$ (limit uniqueness), while the limit equilibrium may depend on the joint distribution of noise terms (noise dependence). An action profile $a^*$ is a noise-independent global game selection in $u(\cdot, \theta)$ if the limit equilibrium plays $a^*$ at $\theta$ independent of the noise distribution.

By appropriately modifying the discrete type spaces in the proof of Theorem 2 to the corresponding global games with continuous states and noises, we can show that if $I$ is not a monotone potential maximizer, then $I$ is not a noise-independent global game selection. Together with Theorem 3 (and Corollary A.1 in Appendix A.4), this leads to the following characterization.

**Proposition 5** Let a binary-action game $f$ be supermodular. For generic $f$, $a^*$ is a noise-independent global game selection in $f$ if and only if $a^*$ is robust to incomplete information in $f$.

The “if” direction holds for many-player many-action supermodular games (Basteck et al. (2010), Oury and Tercieux (2007)). The “only if” direction is new in the literature, and its proof is given in Appendix B.1 in the Supplementary Material; it does not extend to more than two actions (Basteck and Daniél (2011), Oyama and Takahashi (2011)).

**Example 7** (Unanimity) Recall the class of binary-action unanimity games discussed in Examples 3 and 6. In this class of games, the existence of a monotone potential and its implications have been studied in Morris and Ui (2005, Section 7.2), Oyama et al. (2008, Section 4.3.4), Oyama et al. (2011, Section 4), and Okada and Tercieux (2012, Example 1). The following proposition establishes a full characterization of robust equilibria, where the condition is given by the pairwise Nash products. The proof is provided in Appendix A.5.

**Proposition 6** In the binary-action unanimity game $f = (f_{i,y,z})_{i \in I}$,

1. If $z_i z_j > y_i y_j$ for all $i, j \in I$ with $i \neq j$, then $I$ is a unique robust equilibrium;
2. If $z_i z_j < y_i y_j$ for all $i, j \in I$ with $i \neq j$, then $0$ is a unique robust equilibrium; and
3. Otherwise, there is no robust equilibrium.

---

16 We follow the original formulation of Carlsson and van Damme (1993a) to allow correlation in noise terms among players. The results of Frankel et al. (2003) continue to hold even without their assumption of independence among noise terms.

17 This condition differs from the comparison of the $|I|$-player Nash products, $\prod_{i \in I} z_i \geq \prod_{i \in I} y_i$. Namely, the condition in (1) (resp. (2)) in Proposition 6 implies $\prod_{i \in I} z_i > (resp. <) \prod_{i \in I} y_i$, but not vice versa.
As a parametric illustration, consider a three-player example with payoffs \( y_1 = 6 + c, \ y_2 = y_3 = 1, \) and \( z_1 = z_2 = z_3 = 2, \) where \( c > -6. \) In this case, by Proposition 6, if \( c < -2, \) then 1 is a monotone potential maximizer, hence a robust equilibrium, and if \( c \geq -2, \) then the game has no monotone potential maximizer, hence no robust equilibrium.\(^{18}\)

Carlsson (1989) has shown that the global game selection may depend on the noise structure for the class of three-player binary-action unanimity games. Namely, he constructed a noise structure for which 1 (resp. 0) is selected if \( z_1 z_2 (z_3)^2 > (\text{resp. } <) y_1 y_2 (y_3)^2 \) and another noise structure for which 1 (resp. 0) is selected if \( (z_1)^2 z_2 z_3 > (\text{resp. } <) (y_1)^2 y_2 y_3, \) and therefore, the selected equilibrium depends on the noise structure if these inequalities hold with opposite directions. Note that this last condition is stronger than that for Proposition 6(3). With the above parameterization, for example, Carlsson’s sufficient condition for noise dependence is that \(-2 < c < 10,\) while there is noise dependence also for \(c \geq 10\) according to our (generic) characterization by Propositions 5 and 6.

5. APPLICATION: ASYMMETRIC CURRENCY ATTACK GAME

In this section, we consider a model of speculative currency attacks with possibly asymmetric traders, as studied in the global games of Morris and Shin (1998) (symmetric traders) and Corsetti et al. (2004) (one large trader and symmetric small traders). We are interested in characterizing when this game has a robust equilibrium, which, in light of Theorem 3, amounts to identifying the condition for the existence of a monotone potential. As we studied in Section 4.3, this in turn reduces to examining the solvability of a finite system of linear inequalities. In the case of one large trader and symmetric small traders, we explicitly express the solvability condition in terms of the exogenous parameters, which coincides with the limit thresholds as the noise vanishes in the analysis of Corsetti et al. (2004). With this result in hand, we discuss how the noise-(in)dependent selection results by Corsetti et al. (2004) can be understood from our perspective, as well as compare the methodology between theirs and ours.

There are finitely many heterogeneous traders \( i \in I.\) These traders simultaneously choose whether to attack the currency of a certain country by short-selling it for dollars \((a_i = 1)\) or not \((a_i = 0),\) where each trader \(i\) has a limit of the amount \(L_i > 0\) of credit available to take a short position. Then the central bank abandons the currency peg if and only if the total amount of the currency short-sold is larger than or equal to \(\Theta,\) where \(\Theta\) represents the strength of fundamentals. We denote by \(S\) the collection of sets \(S \in \mathcal{I}\) of traders such that \(\sum_{i \in S} L_i \geq \Theta.\) Thus an action profile \(a = (a_i)_{i \in I}\) leads to a successful attack if and only if \(\{i \in I \mid a_i = 1\} \in S.\) Note that \(S\) is monotone, i.e., if \(S \in S\) and \(S \subset S',\) then \(S' \in S.\) Short-selling costs \(c_i, 0 < c_i < 1,\) for each trader \(i.\) If the

\(^{18}\)Morris and Ui (2005, Section 7.2) have used these payoff values with \(c = 0\) as an example where no monotone potential maximizer exists, but they did not discuss its implication to the non-existence of robust equilibria.
attack is successful, each trader who joins the currency attack receives a payoff normalized to 1. Thus trader \( i \)'s net payoff from attack \((a_i = 1)\) is given by

\[
f_i(S) = \begin{cases} 
1 - c_i & \text{if } S \cup \{i\} \in S, \\
-c_i & \text{if } S \cup \{i\} \notin S,
\end{cases}
\]

where \( S = \{j \in I \setminus \{i\} \mid a_j = 1\} \), while that from not attack \((a_i = 0)\) is normalized to 0. By the monotonicity of \( S \), the currency attack game \( f = (f_i)_{i \in I} \) is supermodular. Note that if \( \max_{i \in I} L_i < \Theta \leq \sum_{i \in I} L_i \), then this game has two pure-action Nash equilibria: “all attack” \((S = I)\) and “no one attacks” \((S = \emptyset)\).

Let \( F \) be the collection of sets \( F \in I \) such that \( F \notin S \) and \( F \cup \{i\} \in S \) for some \( i \in I \setminus F \). That is, \( F \) is the collection of sets of traders whose attacks fail by “narrow margins.”

The following characterizes the condition under which the game \( f \) has a monotone potential.\(^{19}\)

**Lemma 3** Suppose that \( \max_{i \in I} L_i < \Theta \leq \sum_{i \in I} L_i \). The currency attack game admits a monotone potential with \( \lambda = (\lambda_i)_{i \in I} \) for “all attack” (resp. “no one attacks”) if and only if

\[
\sum_{i \in S} \lambda_i > (\text{resp. } <) \sum_{i \in I} c_i \lambda_i \tag{5.1}
\]

for all \( S \in I \) such that \( I \setminus S \in F \).

**Proof:** Follows immediately from Lemma 1. \( Q.E.D. \)

In the rest of this section, as in Corsetti et al. (2004), we focus on the case where there is a large trader (player 1), and the others (players \( 2, \ldots, |I| \)) are homogeneous small traders. We set \( L_1 = L \geq 1, L_2 = \cdots = L_{|I|} = 1 \), and \( c_1 = c_2 = \cdots = c_{|I|} = c \). In this case, “all attack” and “no one attacks” are Nash equilibria when \( L < \Theta \leq L + |I| - 1 \). To ease our computation, we assume that \( L \) and \( \Theta \) are positive integers.

The condition (5.1) in Lemma 3 defines a system of linear inequalities in \( \lambda \). In the next lemma, by examining the solvability of this system, we identify the conditions for the existence of a monotone potential in terms of the parameter

\(^{19}\)If \( c_i = c \) for all \( i \in I \), then the inequalities (5.1) in Lemma 3 can be interpreted as \( \lambda \) belonging to the strict core of a coalitional game: “all attack” is a monotone potential maximizer if and only if the strict core is nonempty in the coalitional game \( w^A : I \to \mathbb{R} \) given by \( w^A(I) = 1, w^A(S) = c \) for all \( S \in I \) such that \( I \setminus S \in F \), and \( w^A(S) = 0 \) otherwise; “no one attacks” is a monotone potential maximizer if and only if the strict core is nonempty in the coalitional game \( w^N : I \to \mathbb{R} \) given by \( w^N(I) = 1, w^N(S) = 1 - c \) for all \( S \in F \), and \( w^N(S) = 0 \) otherwise.
GENERALIZED BELIEF OPERATOR

By the symmetry of small traders, with the normalization \( \lambda_1 = \lambda \) and \( \lambda_2 = \cdots = \lambda_{|I|} = 1 \), the system in fact reduces to the following simpler system with one variable \( \lambda \):

\[
\lambda + |I| - 1 - k \geq c(\lambda + |I| - 1) \quad (k = \Theta - L, \ldots, \min(|I|, \Theta) - 1), \tag{5.2}
\]

\[
L + |I| - \Theta \geq c(\lambda + |I| - 1). \tag{5.3}
\]

Define \( \Theta^A \) and \( \Theta^N \) by the following:

\[
\Theta^A = \begin{cases} 
L + (|I| - 1) \left(1 - \frac{c}{1-c}\right) & \text{if } \frac{L}{L+|I|-1} > c, \\
(L + |I| - 1)(1-c) & \text{if } \frac{L}{L+|I|-1} \leq c
\end{cases} \tag{5.4}
\]

if \( c \leq 1/2 \), and

\[
\Theta^A = \begin{cases} 
L & \text{if } \frac{L}{L+|I|-1} > 1-c, \\
(L + |I| - 1)(1-c) & \text{if } \frac{L}{L+|I|-1} \leq 1-c
\end{cases} \tag{5.5}
\]

if \( c > 1/2 \), and

\[
\Theta^N = L + |I|(1-c) + 1. \tag{5.6}
\]

One can verify that \( L \leq \left\lfloor \Theta^A \right\rfloor < |\Theta^N| \leq L + |I|. \tag{20}

**Lemma 4** In the currency attack game with one large trader and \( |I| - 1 \) small traders, “all attack” (resp. “no one attacks”) is a monotone potential maximizer if and only if \( \Theta \leq \left\lfloor \Theta^A \right\rfloor \) (resp. \( \Theta \geq \lfloor \Theta^N \rfloor \)).

**Proof:** See Oyama and Takahashi (2019, Section 5). \( Q.E.D. \)

We thus have the following.

**Proposition 7** In the currency attack game with one large trader and \( |I| - 1 \) small traders,

1. if \( \Theta \leq \left\lfloor \Theta^A \right\rfloor \), then “all attack” is a unique robust equilibrium;
2. if \( \Theta \geq \lfloor \Theta^N \rfloor \), then “no one attacks” is a unique robust equilibrium; and
3. if \( \left\lfloor \Theta^A \right\rfloor < \Theta < \lfloor \Theta^N \rfloor \), then there is no robust equilibrium.

Given Lemma 4, it follows from Theorem 3(1) that if \( \Theta \leq \left\lfloor \Theta^A \right\rfloor \), then “all attack” (\( a = 1 \)) is robust while if \( \Theta \geq \lfloor \Theta^N \rfloor \), then “no one attacks” (\( a = 0 \)) is robust. On the other hand, if \( \Theta < \left\lfloor \Theta^N \right\rfloor \), then one can verify that condition (4.3) in Remark 3 is satisfied, and hence by Theorem 2 and Proposition 3(2) together with Remark 4, “all attack” is contagious, i.e., for any \( \varepsilon > 0 \), there exists a dominance-solvable \( \varepsilon \)-elaboration in which “attack” is played by every

---

\( \text{For } x \in \mathbb{R}, \lfloor x \rfloor \) denotes the largest integer that is smaller than or equal to \( x \); \( \lceil x \rceil \) denotes the smallest integer that is larger than or equal to \( x \).
21Corsetti et al. (2004) use the notations $\lambda$ and $t$ for the size of the large trader and the
observe $\theta + \tau \eta$ and $\theta + \sigma \varepsilon_i$, respectively, where $\theta$ is distributed according to the (improper) uniform prior on $\mathbb{R}$, and $\eta$ and $\varepsilon_i$ are random variables with smooth symmetric densities, independent across all traders and identical among small traders. They show that the game has a unique equilibrium for each pair $(\sigma, \tau)$ of precision levels, which consists of threshold strategies of the large and the small traders, characterize the common limit $\bar{\theta}(r)$ of the thresholds as $\sigma, \tau \to 0$ with $\sigma/\tau \to r$, and show that $\bar{\theta}(r)$ is increasing in $r$. Therefore, for sufficiently small $\sigma$ and $\tau$, (1) if $\theta < \bar{\theta}(0) := \lim_{r \to 0} \bar{\theta}(r)$ (resp. $\theta > \bar{\theta}(\infty) := \lim_{r \to \infty} \bar{\theta}(r)$), then all traders attack (resp. no trader attacks) independently of the noise structure within the class of global game perturbations; and (3) if $\bar{\theta}(0) < \theta < \bar{\theta}(\infty)$, then whether all traders attack or no trader attacks depends on the noise structure through the relative precision $r$.

Note that our setting is an appropriate discretization of the continuous setting of Corsetti et al. (2004) up to normalization. In particular, our thresholds $\Theta^A$ and $\Theta^N$ correspond to $\bar{\theta}(0)$ and $\bar{\theta}(\infty)$, respectively. Indeed, given $\Theta^A$ and $\Theta^N$ as defined by (5.4)--(5.5) and (5.6), one can check that as $L/|I| \to \infty$ with $L/(L+|I|-1) \to \ell$, the normalized values $\Theta^A/(L+|I|-1)$ and $\Theta^N/(L+|I|-1)$ converge to $\bar{\theta}(0)$ and $\bar{\theta}(\infty)$, respectively. Figure 2 depicts $\Theta^A/(L+|I|-1)$ and $\Theta^N/(L+|I|-1)$ as functions of the relative size $L/(L+|I|-1)$ of the large trader, where $L$ and $|I|$ are varied with the total size $L+|I|-1$ held fixed to 1000, for $c = 0.4$ (panel (a)) and $c = 0.6$ (panel (b)). Observe that panel (a) well approximates Figure 3 in Corsetti et al. (2004).

Our non-robustness result, namely part (3) of Proposition 7, corresponds to Corsetti et al.’s (2004) noise dependence result, and in particular, our approach offers new insight to understand the role played by the relative precision $\sigma/\tau$ in Corsetti et al. (2004). Specifically, in the case (3) where $[\Theta^A] < \Theta < [\Theta^N]$, neither of “all attack” and “no one attacks” is a monotone potential maximizer, and thus, there are dominance-solvable elaborations, one in which “attack” is played everywhere and another in which “not attack” is played everywhere. Indeed, following the proof of Theorem 2, we have explicitly constructed such elaborations as in (N1)--(N2) and (A1)--(A2), respectively. While there are apparent differences between our and Corsetti et al.’s perturbations (such as discrete or

22There is an error in the expression of $\bar{\theta}(0)$ in Corsetti et al. (2004). On page 100, they claim (in our notation) that regardless of $c \leq 1/2$ or $c > 1/2$,

$$
\bar{\theta}(0) = \begin{cases} 
\ell + (1 - \ell \left(1 - \frac{1}{c}\right) & \text{if } \ell > c,
1 - c & \text{if } \ell \leq c.
\end{cases}
$$

This would imply $\bar{\theta}(0) < \ell$ if $c > 1/2$ and $\ell > 1 - c$, contradicting $\bar{\theta}(r) \geq \ell$ for all $r$ (which follows from their equation (4.2)). The correct expression of $\bar{\theta}(0)$ is

$$
\bar{\theta}(0) = \begin{cases} 
\ell & \text{if } \ell > 1 - c,
1 - c & \text{if } \ell \leq 1 - c.
\end{cases}
$$

for $c > 1/2$. Compare our equation (5.5).
Figure 2.— Thresholds (normalized)

For the class of binary-action supermodular games, the present paper has studied the robustness of an equilibrium to incomplete information in the sense of Kajii and Morris (1997a). Using the generalized belief operator (Morris and Shin (2007), Morris et al. (2016)), we proved a generalized version of the Critical Path Theorem of Kajii and Morris (1997a), which provides a non-trivial lower bound on the prior probability of a common belief event, uniform over type spaces, when the underlying game has a monotone potential (Morris and Ui (2005)). Conversely, when the game has no monotone potential, we con-
constructed a type space with an arbitrarily high probability event in which players never have common belief about that event. Our construction is based on a novel application of a duality theorem applied to the system of linear inequalities that characterizes the existence of a monotone potential. Combining these results, we established a generic equivalence between robustness and monotone potential maximization for binary-action supermodular games. Finally, we discussed how the methodology developed in this paper enriches our understanding of global game equilibrium selection. In particular, for the asymmetric global game of Corsetti et al. (2004), we demonstrated that the solvability condition for a simple system of linear inequalities characterizes the noise-(in)dependent selection.

The generalized belief operator and the Critical Path Theorem allow us to analyze the properties of equilibria across all common prior type spaces at once. In this study, we used these tools to address the particular question of robustness to incomplete information. Nonetheless, our theory has broader applicability. An example is information design with adversarial equilibrium selection (see Bergemann and Morris (2019, Section 7) and references therein). Specifically, consider the information design problem where the information designer chooses an $\varepsilon$-elaboration of $f$ (as $\varepsilon \to 0$) to minimize the probability of action profile $1$ to be played in the worst-case equilibrium thereof, i.e.,

$$W^* = \lim_{\varepsilon \to 0} \inf_{(T, P, u)} \max_{\sigma^*} \sum_{t \in T} P(t) \prod_{i \in I} \sigma^*_i(1|t_i),$$

where the “inf” is taken over all $\varepsilon$-elaborations $(T, P, u)$ of $f$ and the “max” over all Bayesian Nash equilibria $\sigma^*$ of $(T, P, u)$. By the definition of robustness, $W^* = 1$ if and only if 1 is robust in $f$. Our results imply that for generic $f$, $W^*$ equals either 1 or 0, and more specifically, we have $W^* = 1$ (resp. $W^* = 0$) if $I$ is (resp. is not) a monotone potential maximizer in $f$.

Extending this information design problem with state dependent payoffs and general objective functions is a potential topic for future research.

APPENDIX A

A.1. Proof of Theorem 1

Fix a binary-action supermodular game $f$ that admits a monotone potential $v$. Fix a type space $(T, P)$ and an event profile $E = (E_i)_{i \in I} \in T$ with $E = \prod_{i \in I} E_i$. We consider a sequence of events obtained by applying the $B^j_i$ operators to $E$.
This identity obtains from the fact that the function $v$ is not indexed by $i$ and, combined with Claim A.1, plays an important role in proving the following Claim.
Claim A.2 \( x \leq \frac{M}{v(I) - v'} \).

Proof: We have

\[
0 \geq \sum_{i \in I} \sum_{n:1 \leq n_i < \infty} \pi(n)(v(S(n_i, n) \cup \{i\}) - v(S(n_i, n)))
= \sum_{n:min(n) < \infty} \sum_{i:1 \leq n_i < \infty} \pi(n)(v(S(n_i, n) \cup \{i\}) - v(S(n_i, n)))
= \sum_{n:min(n) < \infty} \pi(n)(v(S(0, n)) - v(S(\infty, n)))
= \sum_{n:1 \leq min(n) < \infty} \pi(n)((v(I) - v(S(\infty, n)))
+ \sum_{n:min(n) = 0} \pi(n)(v(S(0, n)) - v(S(\infty, n)))
\geq x(v(I) - v') - \varepsilon M,
\]

where the inequality in the first line follows from Claim A.1, the equality in the second line is obtained by sorting all terms according to \( n \), the equality in the third line follows from the identity (A.1), and the inequality in the last line follows from the definitions of \( v' \), \( M \), \( x \), and \( \varepsilon \). Then the claim follows because \( v(I) - v' > 0 \).

Q.E.D.

By Claim A.2, we have

\[
1 - P\left(CB^f(E)\right) = \sum_{n:min(n) < \infty} \pi(n) = \varepsilon + x \leq \left(1 + \frac{M}{v(I) - v'}\right) \varepsilon = \kappa(v) \varepsilon,
\]

as desired.

A.2. Proof of Theorem 2

Fix any \( \varepsilon \in (0, 1] \), and let \( \eta = 1 - (1 - \varepsilon)^{1/(|I| - 1)} \in (0, 1] \). It suffices to consider the case where \( \varepsilon \) is sufficiently small, and so is \( \eta \).\footnote{Once we construct a pair \((T, P)\) and \( E \) such that \( P(CB^f(E)) = 0 \), it is easy to construct another pair \((T', P')\) and \( E' \) such that \( P'(E') \) takes an arbitrary value in \([0, P(E)]\) while \( P'(CB^f(E')) = 0 \).} Take a \( \mu \in \Delta^*(\Gamma) \) that satisfies condition (4.3) in Remark 3.

We construct the type space \((T, P)\) as follows. For each \( i \in I \), let

\[
T_i = \begin{cases} 
\{1, 2, \ldots\} & \text{if } i \in I(\mu), \\
\{\infty\} & \text{otherwise.}
\end{cases}
\]
Let $P \in \Delta(T)$ be given by

\[ P(t) = \begin{cases} 
\eta(1 - \eta)^m \mu(\gamma) & \text{if there exist } m \in \mathbb{N} \text{ and } \gamma = (i_1, \ldots, i_k) \in \Gamma \\
0 & \text{otherwise}
\end{cases} \]

for each $t = (t_i)_{i \in I} \in T$, where

\[ \ell(i, \gamma) = \begin{cases} 
\ell & \text{if there exists } \ell \in \{1, \ldots, k\} \text{ such that } i_\ell = i, \\
\infty & \text{otherwise}
\end{cases} \]

for each $i \in I$ and $\gamma = (i_1, \ldots, i_k) \in \Gamma$.

For $\tau \geq 1$, we write $E^\tau_i = \{\tau, \tau + 1, \ldots\}$ for $i \in I(\mu)$ and $E^\tau_i = \{\infty\}$ otherwise, and write $E^\tau = (E^\tau_i)_{i \in I}$ and $E^\tau_{-i} = (E^\tau_j)_{j \neq i}$. Let $E = E^{|I|}$. Then

\[ P(E) = \sum_{m=|I|-1}^{\infty} \eta(1 - \eta)^m = (1 - \eta)^{|I|-1} = 1 - \varepsilon. \]

**Claim A.3** For any $i \in I(\mu)$ and any $\tau \geq |I|$,

\[ P(S_{E^\tau_{-i}}(t_{-i}) = S|t_i = \tau) = (1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})/C_{i, \eta} \]

for all $S \in \mathcal{L}_{-i}$, where $C_{i, \eta} = \sum_{l=1}^{|I|} (1 - \eta)^{|I|-l} \mu(\{\gamma = (i_1, \ldots, i_k) \in \Gamma_i \mid i_\ell = i\}) > 0$.

**Proof:** For each $S \in \mathcal{L}_{-i}$,

\[ P(S_{E^\tau_{-i}}(t_{-i}) = S|t_i = \tau) = P(t_i = \tau, S_{E^\tau_{-i}}(t_{-i}) = S)/P(t_i = \tau) \]

\[ = \eta(1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})/P(t_i = \tau) \]

\[ = (1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})/C_{i, \eta}, \]

as claimed. \(Q.E.D.\)

**Claim A.4** For any $i \in I(\mu)$ and any $\tau \geq |I|$, $B^\tau_i(E^\tau) \subset E^\tau_{i+1}$.

**Proof:** Consider type $t_i = \tau$. Then

\[ \mathbb{E}\left[f_i \left(S_{E^\tau_{-i}}(t) \right) \mid t_i = \tau\right] = \sum_{S \in \mathcal{L}_{-i}} (1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i \mid S(i, \gamma) = S\})f_i(S)/C_{i, \eta} < 0, \]

where the equality follows from Claim A.3 and the inequality from condition (4.3). Thus, $\tau \notin B^\tau_i(E^\tau)$. \(Q.E.D.\)

By applying Claim A.4 and Proposition 1 inductively, we have $B^\tau_{i,n}(E) \subset E^{|I|+n}_{i+1}$ for all $i \in I(\mu)$ and all $n \geq 0$, and hence $CB^\tau_i(E) = \emptyset$ for all $i \in I(\mu)$. Thus, $P(CB^\tau_i(E)) = 0$. 

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A.3. Approximate Robustness

In this section, we prove a version of Theorem 3(2) with a stronger conclusion in terms of approximate robustness (Haimanko and Kajii (2016)) by assuming the stronger genericity condition (4.2) (see Remark 5).

Given an elaboration \((T, P, u)\) and \(\varepsilon > 0\), we say that a strategy profile \(\sigma^* = (\sigma^*_i)_{i \in I} \in \Sigma\) is an interim \(\varepsilon\)-Bayesian Nash equilibrium of \((T, P, u)\) if for all \(i \in I\), all \(t_i \in T_i\), and all \(a_i, a'_i \in A_i\),

\[
\sigma^*_i(a_i|t_i) > 0 \Rightarrow \mathbb{E}[u_i((a_i, \sigma^*_{-i}(.)), (t_i, \cdot))|t_i] \geq \mathbb{E}[u_i((a'_i, \sigma^*_{-i}(.)), (t_i, \cdot))|t_i] - \varepsilon.
\]

**Definition A.1** A Nash equilibrium \(a^* = (a^*_i)_{i \in I} \in A\) of a complete information game \(g\) is approximately robust (to incomplete information) in \(g\) if for any \(\varepsilon' > 0\) and \(\delta > 0\), there exists \(\varepsilon > 0\) such that for any \(\varepsilon\)-elaboration \((T, P, u)\) of \(g\), there exists an interim \(\varepsilon'\)-Bayesian Nash equilibrium \(\sigma^* = (\sigma^*_i)_{i \in I} \in \Sigma\) such that

\[
\sum_{t=(t_i)_{i \in I} \in T} P(t) \prod_{i \in I} \sigma^*_i(a^*_i|t_i) \geq 1 - \delta.
\]

By definition, if \(a^*\) is robust, then it is approximately robust. Thus, the conclusion of the following proposition is a strengthening of that of Theorem 3(2).

**Proposition A.1** In a binary-action supermodular game \(f\), if condition (4.2) is satisfied, then \(1\) is not approximately robust to incomplete information.

**Proof:** Take a \(\mu \in \Delta^*(\Gamma)\) that satisfies condition (4.2). Let

\[
D_i = - \sum_{S \in \mathcal{L}_i} \mu(\{\gamma \in \Gamma_i | S(i, \gamma) = S\})f_i(S),
\]

\[
\varepsilon' = \min_{i \in I(\mu)} D_i/2 > 0.
\]

For each \(\eta\), define \(C_{i,\eta} = \sum_{i \in \Gamma} (1 - \eta)^{|i| - t} \mu(\{\gamma = (i_1, \ldots, i_k) \in \Gamma_i | i_t = i\})\) as in Claim A.3 in Appendix A.2. Note that \(\sum_{S \in \mathcal{L}_i} (1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i | S(i, \gamma) = S\})f_i(S) \to -D_i\) and \(C_{i,\eta} \to \mu(\Gamma_i) (= 1)\) as \(\eta \to 0\), and hence for any \(i \in I(\mu),\)

\[
\sum_{S \in \mathcal{L}_i} (1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i | S(i, \gamma) = S\})f_i(S)/C_{i,\eta} \to -D_i
\]

as \(\eta \to 0\).

Fix any \(\varepsilon \in (0, 1]\), and let \(\eta = 1 - (1 - \varepsilon)^{1/(|I|-1)} \in (0, 1]\). It suffices to consider the case where \(\varepsilon\) is sufficiently small, and so is \(\eta\) so that

\[
\sum_{S \in \mathcal{L}_i} (1 - \eta)^{|S|} \mu(\{\gamma \in \Gamma_i | S(i, \gamma) = S\})f_i(S)/C_{i,\eta} < -\varepsilon'
\]
for any $i \in I(\mu)$.

Given the above $\mu$ and $\eta$, take the type space $(P,T)$ and the event profile $E$ as constructed in the proof of Theorem 2 in Appendix A.2. Then, $P(E) = 1 - \varepsilon$.

Moreover, as in the proof of Claim A.4, if $i \in I(\mu)$ and $\tau \geq |I|$, then

$$
E \left[ f_i \left( S_{E_i}^i \right) \mid t_i = \tau \right] = \sum_{S \in \mathcal{I}_i} (1 - \eta)^{|S|} \mu(\{ \gamma \in \Gamma_i \mid S(i, \gamma) = S \}) f_i(S)/C_i, \eta < -\varepsilon',
$$

and hence for the game $f + \varepsilon'$ defined by $f_i(S) + \varepsilon'$ for all $i \in I$ and $S \in \mathcal{I}_i$, we have $P(CB^{f + \varepsilon'}(E)) = 0$.

Finally, as in the proof of Proposition 3(2), let the payoff functions $u$ be such that for each $i \in I$, $T_i f_i = E_i$, and action 0 is a dominant action for all types $t_i \notin E_i$. Then in the elaboration $(T,P,u)$, 1 is never played in any interim $\varepsilon'$-Bayesian Nash equilibrium. Q.E.D.

Condition (4.2) in Proposition A.1 cannot be relaxed to condition (4.3). In fact, the game (4.4) in Remark 3 satisfies condition (4.3), but by the upper semi-continuity of approximately robust equilibria with respect to payoffs, both 1 and 0 are approximately robust (Haimanko and Kajii (2016)).

**A.4. Robustness of Non-Extreme Action Profiles**

In this section, we generalize Theorem 3 to action profiles other than 1.

**Corollary A.1** Let a binary-action game $f$ be supermodular. Then the following results hold.

1. If $S^*$ is a monotone potential maximizer in $f$, then $1_{S^*}$ is robust to incomplete information in $f$.
2. For generic $f$, if $S^*$ is not a monotone potential maximizer in $f$, then $1_{S^*}$ is not robust to incomplete information in $f$.

Part (1) follows from Morris and Ui (2005), whereas it can also be proved by applying our Theorem 1 (to the “subgames” of $f$ introduced below).

For the proof of part (2), we utilize two “subgames” of the original game $f = (f_i)_{i \in I}$: the lower game $f^- = (f^-_i)_{i \in S^*}$ is the binary-action game among the players in $S^*$ defined by $f^-_i(S) = f_i(S)$ for all $i \in S^*$ and $S \subset S^* \setminus \{i\}$, where the actions of the players in $I \setminus S^*$ are fixed at action 0; the upper game $f^+ = (f^+_i)_{i \in I \setminus S^*}$ is the binary-action game among the players in $I \setminus S^*$ defined by $f^+_i(S) = f_i(S \cup S^*)$ for any $i \in I \setminus S^*$ and $S \subset (I \setminus S^*) \setminus \{i\}$, where the actions of the players in $S^*$ are fixed at action 1.

Note that if $S^*$ is a monotone potential maximizer in $f$, then $S^*$ is a monotone potential maximizer in the lower game $f^-$, and $\emptyset$ is a monotone potential maximizer in the upper game $f^+$. Indeed, let $v$ be a monotone potential of $f$ for $S^*$.
with $\lambda = (\lambda_i)_{i \in I}$. Then $v^-$ given by $v^-(S) = v(S)$ for all $S \subset S^*$ is a monotone potential of $f^-$ for $S^*$ with $(\lambda_i)_{i \in S^*}$, and $v^+$ given by $v^+(S) = v(S \cup S^*)$ for all $S \subset I \setminus S^*$ is a monotone potential of $f^+$ for $\emptyset$ with $(\lambda_i)_{i \in I \setminus S^*}$. The next lemma claims that the converse also holds if $f$ is supermodular.

**Lemma A.1** $S^*$ is a monotone potential maximizer in a binary-action supermodular game $f$ if and only if $S^*$ is a monotone potential maximizer in the lower game $f^-$, and $\emptyset$ is a monotone potential maximizer in the upper game $f^+$.

**Proof:** See Oyama and Takahashi (2019, Section A.6). Q.E.D.

Similarly to Lemma 1 in Section 4.3, the existence of a monotone potential for $S^*$ can be characterized in terms of the weighted sum of payoff increments. For each $\gamma = (i_1, \ldots, i_k) \in \Gamma$ and $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i > 0$, we define

$$F(\gamma, \lambda; S^*) = \sum_{\ell=1}^k \lambda_{i_\ell} \tilde{f}_{i_\ell}(S^* \triangle \{i_1, \ldots, i_k\}; S^*),$$

where $\tilde{f}_i$ is the “signed” payoff increment,

$$\tilde{f}_i(S; S^*) = \begin{cases} f_i(S) & \text{if } i \in S^*, \\ -f_i(S \setminus \{i\}) & \text{if } i \in I \setminus S^*, \end{cases}$$

and $S \triangle S' = (S \setminus S') \cup (S' \setminus S)$ denotes the symmetric difference between $S, S' \in \mathcal{I}$.

**Lemma A.2** A binary-action game $f$ admits a monotone potential with $\lambda$ for $S^*$ if and only if $F(\gamma, \lambda; S^*) > 0$ for all $\gamma \in \Gamma$.

The proof, which is similar to that of Lemma 1, is omitted.

**Proof of Corollary A.1(2):** Suppose that $S^*$ is not a monotone potential maximizer in $f$. By Lemma A.1, either $S^*$ is not a monotone potential maximizer in $f^-$, or $\emptyset$ is not a monotone potential maximizer in $f^+$. We assume the former by symmetry. By genericity, we assume that $f^-$ satisfies condition (4.2). By Proposition A.1 in Section A.3, $1_{S^*}$ is not approximately robust in $f^-$. That is, there exist $\varepsilon' > 0$ and $\delta^- > 0$ such that for any $\varepsilon > 0$, there exists an $\varepsilon$-elaboration $(T^-, P^-, u^-)$ with $T^- = \prod_{i \in S^*} T_i$ in which for all $i \in S^*$, 0 is a dominant action for $t_i \notin T'^-_i$, and every interim $\varepsilon'$-Bayesian Nash equilibrium plays $1_{S^*}$ with probability less than $1 - \delta^-$. Let $\delta = \min\{\delta^-, \varepsilon'/(2 \max_{i \in I, S \in \mathcal{X}, |f_i(S)|)}\} > 0$.

In the following, we denote $E_i = T'^-_i$ for $i \in S^*$.

Fix any $\varepsilon > 0$. Given this $\varepsilon$, we extend the above elaboration to all players in $I$ by adding a singleton type to each player in $I \setminus S^*$. Formally, construct $(T, P, u)$ with $T = \prod_{i \in I} T_i$ as follows: for $i \in S^*$, let $T_i$ be the same space as in the original elaboration, and for $i \in I \setminus S^*$, let $T_i = \{t_{i,0}\}$; let $P \in \Delta(T)$
be defined by \( P(t^-, (t_i, 0)_{i \in I \setminus S^*}) = P^-(t^-) \) for all \( t^- \in T^- \); and let \( u \) be such that for \( i \in S^* \), \( T_i^{f_i} = E_i \), and 0 is a dominant action for \( t_i \notin E_i \), and for \( i \in I \setminus S^* \), \( T_i^{f_i} = \{ t_i, 0 \} \). Since \( (T^-, P^-, u^-) \) is an \( \varepsilon \)-elaboration of \( f^- \), \((T, P, u)\) is an \( \varepsilon \)-elaboration of \( f \). If \((T, P, u)\) has a Bayesian Nash equilibrium \( \sigma^* = (\sigma^*_i)_{i \in I} \) that plays \( 1_{S^*} \) with probability at least \( 1 - \delta \), then \((\sigma^*_i)_{i \in S^*}\) is an interim \( \varepsilon' \)-Bayesian Nash equilibrium in \((T^-, P^-, u^-)\) that plays \( 1_{S^*} \) with probability at least \( 1 - \delta \geq 1 - \delta^- \), which is a contradiction. Therefore, \( 1_{S^*} \) is not robust in \( f \). \( \text{Q.E.D.} \)

In fact, the above proof, with the last step appropriately modified, shows that under condition (4.2) for \( f^- \) (or the corresponding condition for \( f^+ \)), \( 1_{S^*} \) is not even approximately robust in \( f \).

In the proof of Corollary A.1(2), we assumed condition (4.2) for \( f^- \). This condition cannot be relaxed to condition (4.3) except for \( S^* = \emptyset \) (Remark 3 to Theorem 2). See our working paper (Oyama and Takahashi (2019, Appendix A.6)) for a counter-example.

A.5. Proof of Proposition 6

If \( z_i z_j > y_i y_j \) for all \( i, j \in I \) with \( i \neq j \), then as shown in Example 6, \( I \) is a monotone potential maximizer. By Theorem 3(1), \( 1 \) is robust. Symmetrically, if \( z_i z_j < y_i y_j \) for all \( i, j \in I \) with \( i \neq j \), then \( 0 \) is robust.

Suppose that \( z_i z_j \leq y_i y_j \) for some \( i, j \in I \) with \( i \neq j \). Take any such \( i \) and \( j \). We show that condition (4.3) in Remark 3 holds for all parameter values in this case. For each small \( \eta > 0 \), let \( \mu \in \Delta(\Gamma) \) be such that \( \gamma = (i, *, \ldots, *, j) \) with probability \( (1 - \eta^2) y_i / (y_i + z_i) \), where \((k, *, \ldots, *, \ell)\) denotes the sequence of length \(|I|\) such that \( k \) is listed first, \( \ell \) is listed last, and the other \(|I| - 2\) players are listed in some fixed order (e.g., the ascending order in player indices), \( \gamma = (j, *, \ldots, *, i) \) with probability \((1 - \eta^2) z_i / (y_i + z_i) \), and for each \( k \neq i, j \), \( \gamma = (i, *, \ldots, *, k) \) with probability \( \eta^2 / (|I| - 2) \). For player \( i \), the left-hand side of (4.3) is

\[
(1 - \eta^2) \frac{z_i}{y_i + z_i} \times (-y_i) + (1 - \eta)^{|I| - 1} \times \left( 1 - \eta^2 \right) \frac{y_i}{y_i + z_i} \times \eta^2 \times z_i
\]

\[
= -(1 - \eta^2) \frac{y_i z_i}{y_i + z_i} \left( 1 - (1 - \eta)^{|I| - 1} \right) + (1 - \eta)^{|I| - 1} \eta^2 z_i < 0
\]

for small \( \eta \); for player \( j \), it is

\[
(1 - \eta^2) \frac{y_j}{y_j + z_i} \times (-y_j) + (1 - \eta)^{|I| - 1} \times (1 - \eta^2) \frac{z_j}{y_j + z_i} \times z_j
\]

\[
= (1 - \eta^2) \left( 1 - \eta \right)^{|I| - 1} z_i z_j - y_i y_j / y_i + z_i < 0
\]
as $y_i, y_j \geq z_i, z_j > 0$; and for player $k \neq i, j$, it is $\eta^2/(|I| - 2) \times (-y_k) < 0$. Thus, by Remarks 3 and 4 and Proposition 3(2), for any small $\varepsilon > 0$, there exists a dominance-solvable $\varepsilon$-elaboration in which $0$ is played everywhere, which implies that no equilibrium other than $0$ can be robust. Symmetrically, if $z_i, z_j \geq y_i, y_j$ for some $i, j \in I$ with $i \neq j$, then no equilibrium other than $1$ can be robust.

By combining these cases, the conclusion follows.

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