Iterated Potential and Robustness of Equilibria*

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Abstract

For any given set-valued solution concept, it is possible to consider iterative elimination of actions outside the solution set. This paper applies such a procedure to define the concept of *iterated monotone potential maximizer* (*iterated MP-maximizer*). It is shown that under some monotonicity conditions, an iterated MP-maximizer is robust to incomplete information (Kajii and Morris, *Econometrica* **65** (1997)) and absorbing and globally accessible under perfect foresight dynamics for a small friction (Matsui and Matsuyama, *Journal of Economic Theory* **65** (1995)). Several simple sufficient conditions under which a game has an iterated MP-maximizer are also provided. *Journal of Economic Literature* Classification Numbers: C72, C73, D82.

KEYWORDS: equilibrium selection; robustness; incomplete information; perfect foresight dynamics; iteration; monotone potential; **p**-dominance.

Contents

1 Introduction

Economic modeling, by its nature, is based on simplified assumptions that schematize a given economic phenomenon. One way of assessing the role of the assumptions postulated is to compare the model with its "perturbed variants" based on slightly weakened assumptions. It is now well known in game theory that outcomes of a game may dramatically change when we allow for small departures from a given assumption (one may think of departure from the common knowledge assumption as demonstrated, among others, by Rubinstein (1989) or Carlsson and van Damme (1993)).[1] Let us say that an equilibrium is robust with respect to a given assumption if it is still an equilibrium when this assumption is slightly weakened.

The lack of robustness of some Nash equilibria has led game theorists to consider criteria that guarantees robustness. In bold strokes, two types of methods have proved to be powerful in identifying equilibria that are robust in various aspects: namely, the potential method (due to Monderer and Shapley (1996); see also Blume (1993), Hofbauer and Sorger (1999, 2002), Ui (2001)) and the risk-dominance method (due to Harsanyi and Selten (1988); see also Kandori, Mailath, and Rob (1993), Young (1993), Matsui and Matsuyama (1995), Morris, Rob, and Shin (1995), and Kajii and Morris (1997)). These criteria, however, are very demanding and such concepts fail to exist in many games. In this paper, we introduce a notion of *iterative construction* that enables us to enlarge the class of games where these approaches apply and hence to extend the existing sufficient conditions for equilibria to be robust.

In this paper, we consider two robustness tests. The first one is the so-called "robustness to incomplete information" test as originally defined by Kajii and Morris (1997). To motivate this approach, consider an analyst who plans to model some strategic situation by a particular complete information game. This analyst should be aware that his prediction might be (in some games) highly dependent on the assumption of complete information. Hence, if it is guaranteed that the analyst's prediction based on the complete information game is not qualitatively different from some equilibrium of the real incomplete information game being played, then he is justified in choosing the simplified assumption of complete information. To be more precise, robustness to incomplete information is defined as follows. A (pure) Nash equilibrium a^* of a complete information game $\mathbf{g}$ is *robust to incom-*

[1] Sensitivity to simplified assumptions has also been discussed in many economic applications. For instance, Morris and Shin (1998) and Goldstein and Pauzner (2005) respectively consider how the predictions of standard models on currency crises and on bank runs which rely on the existence of multiple self-fulfilling beliefs are modified when allowing for slight departure from the complete information assumption. In a series of papers, Matsuyama (1991, 1992a, 1992b) departs from the perfect reversibility assumption on action revisions such as career choice decisions and underlines its consequences in models of sectoral adjustment and economic development.

plete information if every "nearby" incomplete information elaboration of **g** has a Bayesian Nash equilibrium that generates an (ex-ante) distribution over actions assigning a weight close to one to a^*. "Nearby" incomplete information elaborations are incomplete information games such that the sets of players and actions are the same as in the complete information game **g**, and with high probability, each player knows that his payoffs are the same as in **g**. Thus, payoffs of the incomplete information elaboration are allowed to be very different of **g** with very low probability.

The second robustness test we consider is the one introduced by Matsui and Matsuyama (1995), namely, the perfect foresight dynamics approach. To motivate this approach, assume that an analyst considers a one-shot complete information game to predict the long-run outcome of a given repeated interaction. Consider a Nash equilibrium of this game and embed the game in a dynamic game with a large society of agents. If there is no link between time periods, then, regardless of the initial action distribution of the society, the Nash equilibrium is the limit of some equilibrium path in this dynamic game. But what if we slightly depart from such a simplified assumption and assume that there exists a small amount of irreversibility or friction in action revisions? If in this modified dynamic game, the Nash equilibrium is always the limit of an equilibrium path regardless of the initial action distribution, then the analyst can ignore the subtle complications induced by intertemporal effects through irreversibility. To be more precise, we consider a large society with continua of agents (one for each player position of **g**), in which a one-shot game **g** is played repeatedly in a random matching fashion. There is friction in action revisions: each agent cannot change his action at every point in time. Action revision opportunities follow independent Poisson processes. Agents, when given a revision opportunity, take actions that maximize their expected discounted payoffs. The degree of friction is then measured by the discounted average duration of a commitment. A *perfect foresight path* is a feasible path of action distribution along which each revising agent takes a best response to the future course of play. A Nash equilibrium a^* is *globally accessible* if for any initial action distribution, there exists a perfect foresight path that converges to a^*; a^* is *linearly absorbing* if the feasible path converging linearly to a^* is the unique perfect foresight path from each initial action distribution in a neighborhood of a^*. If a Nash equilibrium that is globally accessible is also absorbing, then it is the unique globally accessible equilibrium.

It has been known that even a strict Nash equilibrium may fail to be robust in each sense above. In 2×2 coordination games, for instance, while the risk-dominant equilibrium is robust in the above senses, the risk-dominated equilibrium is not: the risk-dominated equilibrium is never played in any Bayesian Nash equilibrium under some incomplete information structures

(Rubinstein (1989), Morris, Rob, and Shin (1995))[2] and it is never played along any equilibrium path for some initial action distributions (Matsui and Matsuyama (1995)). That is, even strict Nash equilibria which are often considered as being immune against most perturbations (see Kohlberg and Mertens (1986)) can be very sensitive to slight departure from some simplified assumptions.

In finding sufficient conditions for an equilibrium to be robust in each sense above, the two concepts of potential maximizer and **p**-dominance (the latter is a generalization of risk-dominance) have proved to be powerful. Kajii and Morris (1997) show that if the complete information game has a **p**-dominant equilibrium with low **p**, then it is robust to incomplete information,[3] while Ui (2001) shows that in potential games, the potential maximizer is robust to incomplete information. For perfect foresight dynamics, Hofbauer and Sorger (1999, 2002) show that a potential maximizer is stable for any small degree of friction, while the **p**-dominance condition is studied by Oyama (2002) (in a single population setting).[4] Furthermore, Morris and Ui (2005) introduce a generalization of potential and establishes the robustness of generalized potential maximizer to incomplete information. Oyama, Takahashi, and Hofbauer (2003, OTH henceforth) consider the stability of monotone potential maximizer (a special case of generalized potential maximizer) under the perfect foresight dynamics. The class of games with a monotone potential maximizer contains games with a **p**-dominant equilibrium with a low **p**, and therefore the results on generalized/monotone potential maximizer unify the potential maximizer and the **p**-dominance conditions.

This paper applies an *iterative construction* to potential and **p**-dominance methods to generate new sufficient conditions that are obtained by iterating the existing conditions above. Considering monotone potential, which unifies the two methods, we introduce *iterated monotone potential maximizer* (*iterated MP-maximizer*). Roughly speaking, our iterative procedure to build this concept can be described as follows. An action profile a^* is said to be an iterated MP-maximizer if there exists a sequence of subsets of action profiles $S^0 \supset S^1 \supset \cdots \supset S^m = \{a^*\}$ such that for all $k = 1, \cdots, m$, S^k is an MP-maximizer set in the game restricted to S^{k-1}, where S^0 is the set of all action profiles. We show that under certain monotonicity conditions, an iterated MP-maximizer is robust to incomplete information and globally accessible and linearly absorbing for a small friction. This is proved by ex-

[2] Kajii and Morris (1997) also provide a three-player three-action game where a unique Nash equilibrium, which is strict, is not robust to incomplete information.

[3] Tercieux (2006) proves a set-valued extension of this result.

[4] Kojima (2006) considers another generalization of risk-dominance and establishes the stability result in a multiple population setting. Kim (1996) reports a similar result for binary games with many identical players. Tercieux (2004) considers a set-valued extension of the **p**-dominance condition.

ploiting the similarity between the mathematical structures of incomplete information elaborations and perfect foresight dynamics, which may be of independent interest.[5]

Tercieux (2004) considers iteration of **p**-dominance and defines *iterated* **p**-*dominant equilibrium*. We prove that if a game has an iterated **p**-dominant equilibrium with low **p**, then this equilibrium is actually an iterated MP-maximizer and the relevant monotonicity conditions for our robustness results to hold are satisfied. While finding iterated MP-maximizers or even simply MP-maximizers can sometimes be a difficult task, iterated **p**-dominance indeed provides a simpler procedure to find robust Nash equilibria. Restricting our attention to specific classes of games, we also give several other tools which are much easier to manipulate. In particular, for supermodular games, we introduce the concept of *iterated pairwise* **p**-*dominance* and, as a special case, that of *iterated risk-dominance* for two-player supermodular coordination games, which are based on (a generalization of) the pairwise risk-dominance concept considered by Kandori and Rob (1998) and thus rely only on local properties of the payoff structure. We also consider a 3×3 game example (due to Young (1993)) in which iterated **p**-dominance is shown to be a strictly stronger concept than **p**-dominance.

The paper is organized as follows. Section 2 introduces the concept of iterated MP-maximizer. Section 3 considers the informational robustness of iterated MP-maximizer, while Section 4 considers the stability of iterated MP-maximizer under the perfect foresight dynamics. Section 5 discusses some extensions.

2 Iterated Monotone Potential Maximizer

2.1 Underlying Game

Throughout our analysis, we fix the set of players, $I = \{1, 2, \cdots, N\}$, and the linearly ordered set of actions, $A_i = \{0, 1, \ldots, n_i\}$, for each player $i \in I$. We denote $\prod_{i \in I} A_i$ by A and $\prod_{j \neq i} A_j$ by A_{-i}. A one-shot complete information game is specified by, and identified with, a profile of payoff functions, $\mathbf{g} = (g_i)_{i \in I}$, where $g_i \colon A \to \mathbb{R}$ is the payoff function for player i. For $S = S_1 \times \cdots \times S_N$ where $S_i \subset A_i$, $g_i|_S$ denotes the restriction of g_i to S. We identify $\mathbf{g}|_S = (g_i|_S)_{i \in I}$ with the restricted game with the sets of actions S_i.

For any nonempty, at most countable set S, we denote by $\Delta(S)$ the set of all probability distributions on S. We sometimes identify each action in A_i with the element of $\Delta(A_i)$ that assigns one to the corresponding coordinate.

[5] Takahashi (2005) reports a formal correspondence between perfect foresight dynamics and global games (with a certain class of noise structures) for games with linear payoff functions.

For $x_i, y_i \in \Delta(A_i)$, we write $x_i \precsim y_i$ if

$$\sum_{k=h}^{n_i} x_{ik} \leq \sum_{k=h}^{n_i} y_{ik}$$

for all $h \in A_i$. We write $x \precsim y$ for $x, y \in \prod_i \Delta(A_i)$ if $x_i \precsim y_i$ for all $i \in I$, and $x_{-i} \precsim y_{-i}$ for $x_{-i}, y_{-i} \in \prod_{j \neq i} \Delta(A_j)$ if $x_j \precsim y_j$ for all $j \neq i$. For $\pi_i, \pi_i' \in \Delta(A_{-i})$, we write $\pi_i \precsim \pi_i'$ if

$$\sum_{a_{-i} \in S_{-i}} \pi_i(a_{-i}) \leq \sum_{a_{-i} \in S_{-i}} \pi_i'(a_{-i})$$

for any increasing subset $S_{-i} \subset A_{-i}$.[6] The game $\mathbf{g}$ is said to be *supermodular* if whenever $h < k$, the difference $g_i(k, a_{-i}) - g_i(h, a_{-i})$ is nondecreasing in $a_{-i} \in A_{-i}$, i.e., if $a_{-i} \leq b_{-i}$, then

$$g_i(k, a_{-i}) - g_i(h, a_{-i}) \leq g_i(k, b_{-i}) - g_i(h, b_{-i}).$$

It is well known that this property extends to $\Delta(A_{-i})$: if $h < k$ and $\pi_i \precsim \pi_i'$, then

$$g_i(k, \pi_i) - g_i(h, \pi_i) \leq g_i(k, \pi_i') - g_i(h, \pi_i').$$

We endow $\prod_{i \in I} \Delta(A_i)$, $\Delta(A)$, and $\Delta(A_{-i})$, $i \in I$, with the sup (or max) norm: $|x| = \max_{i \in I} \max_{h \in A_i} x_{ih}$ for $x \in \prod_{i \in I} \Delta(A_i)$, $|\pi| = \max_{a \in A} \pi(a)$ for $\pi \in \Delta(A)$, and $|\pi_i| = \max_{a_{-i} \in A_{-i}} \pi_i(a_{-i})$ for $\pi_i \in \Delta(A_{-i})$. For $\varepsilon > 0$, denote $B_\varepsilon(x) = \{x' \in \prod_i \Delta(A_i) \mid |x' - x| < \varepsilon\}$ for $x \in \prod_{i \in I} \Delta(A_i)$, $B_\varepsilon(\pi) = \{\pi' \in \Delta(A) \mid |\pi' - \pi| < \varepsilon\}$ for $\pi \in \Delta(A)$, and $B_\varepsilon(\pi_i) = \{\pi_i' \in \Delta(A_{-i}) \mid |\pi_i' - \pi_i| < \varepsilon\}$ for $\pi_i \in \Delta(A_{-i})$. Write $B_\varepsilon(F) = \bigcup_{\pi \in F} B_\varepsilon(\pi)$ for $F \subset \Delta(A)$ and $B_\varepsilon(F_{-i}) = \bigcup_{\pi_i \in F_{-i}} B_\varepsilon(\pi_i)$ for $F_{-i} \subset \Delta(A_{-i})$.

Let f be a function from A to $\mathbb{R}$. With abuse of notion, $f(a_i, \cdot)$ are extended to $\prod_{j \neq i} \Delta(A_j)$ and $\Delta(A_{-i})$, and $f(\cdot)$ to $\prod_{j \in I} \Delta(A_j)$ and $\Delta(A)$ in the usual way. For $S_i \subset A_i$, let

$$br_f^i(x_{-i}|S_i) = \arg\max\{f(h, x_{-i}) \mid h \in S_i\}$$

for $x_{-i} \in \prod_{j \neq i} \Delta(A_j)$, and

$$br_f^i(\pi_i|S_i) = \arg\max\{f(h, \pi_i) \mid h \in S_i\}$$

for $\pi_i \in \Delta(A_{-i})$. We also denote $br_f^i(x_{-i}) = br_f^i(x_{-i}|A_i)$ and $br_f^i(\pi_i) = br_f^i(\pi_i|A_i)$.

Let S_i^* be a nonempty subset of A_i for each $i \in I$, and $S^* = \prod_{i \in I} S_i^*$. We say that S^* is a *best response set* of $\mathbf{g}$ if for all $i \in I$, $br_{g_i}^i(\pi_i) \cap S_i^* \neq \emptyset$ for all $\pi_i \in \Delta(S_{-i})$ and that S^* is a *strict best response set* of $\mathbf{g}$ if for all $i \in I$, $br_{g_i}^i(\pi_i) \subset S_i^*$ for all $\pi_i \in \Delta(S_{-i})$. An action profile $a^* \in A$ is a (strict) Nash equilibrium of $\mathbf{g}$ if $\{a^*\}$ is a (strict) best response set of $\mathbf{g}$.

[6] $S_{-i} \subset A_{-i}$ is said to be increasing if $a_{-i} \in S_{-i}$ and $a_{-i} \leq b_{-i}$ imply $b_{-i} \in S_{-i}$.

2.2 Iterated MP-Maximizer

In this subsection, we define our main concept of iterated monotone potential maximizer (iterated MP-maximizer, in short). In the sequel, we denote $[\underline{a}_i, \overline{a}_i] = \{h \in A_i \mid \underline{a}_i \leq h \leq \overline{a}_i\}$, and for $\underline{a} = (\underline{a}_i)_{i \in I}$ and $\overline{a} = (\overline{a}_i)_{i \in I}$, $[\underline{a}, \overline{a}] = \prod_{i \in I} [\underline{a}_i, \overline{a}_i]$ and $[\underline{a}_{-i}, \overline{a}_{-i}] = \prod_{j \neq i} [\underline{a}_j, \overline{a}_j]$. We say that $S \subset A$ is an order interval, or simply an interval, if $S = [\underline{a}, \overline{a}]$ for some $\underline{a}, \overline{a} \in A$ such that $\underline{a}_i \leq \overline{a}_i$ for all $i \in I$, and denote $S_i = [\underline{a}_i, \overline{a}_i]$ and $S_{-i} = [\underline{a}_{-i}, \overline{a}_{-i}]$.

We employ a refinement of the MP-maximizer concept due to Morris and Ui (2005).[7]

Definition 2.1. An interval $S^* \subset A$ is a *strict MP-maximizer set* of $\mathbf{g}$ if there exists a function $v \colon A \to \mathbb{R}$ such that $S^* = \arg\max_{a \in A} v(a)$, and for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min br^i_v(\pi_i | [\min A_i, \min S^*_i]) \leq \min br^i_{g_i}(\pi_i | [\min A_i, \max S^*_i]), \tag{2.1}$$

and

$$\max br^i_v(\pi_i | [\max S^*_i, \max A_i]) \geq \max br^i_{g_i}(\pi_i | [\min S^*_i, \max A_i]). \tag{2.2}$$

Such a function v is called a *strict monotone potential function.*

Now our concept of iterated strict MP-maximizer is obtained by iteration of strict MP-maximizer.

Definition 2.2. An interval $S^* \subset A$ is an *iterated strict MP-maximizer set* of $\mathbf{g}$ if there exists a sequence of intervals $S^0, S^1, \ldots, S^m$ with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ such that S^k is a strict MP-maximizer set of $\mathbf{g}|_{S^{k-1}}$ for each $k = 1, \ldots, m$.

An action profile $a^* \in A$ is an *iterated strict MP-maximizer* of $\mathbf{g}$ if $\{a^*\}$ is an iterated strict MP-maximizer set of $\mathbf{g}$.

For supermodular games, an iterated strict MP-maximizer is unique if it exists, due to Theorems 4.1 and 4.7 given in Section 4.

We also introduce a weaker, but more complicated, version of iterated MP-maximizer, which is sufficient to obtain the robustness to incomplete information and the stability under perfect foresight dynamics.

Definition 2.3. Let S^* and S be intervals such that $S^* \subset S \subset A$. S^* is an *MP-maximizer set* of $\mathbf{g}$ *relative to* S if there exist a function $v \colon A \to \mathbb{R}$ and a real number $\eta > 0$ such that $S^* = \arg\max_{a \in A} v(a)$, and for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S_{-i}))$,

$$\min br^i_v(\pi_i | [\min S_i, \min S^*_i]) \leq \max br^i_{g_i}(\pi_i | [\min S_i, \max S^*_i]), \tag{2.3}$$

[7] This refinement has been introduced by OTH (2003, Definition 4.2) for action profiles (singleton sets).

and

$$\max br^i_v(\pi_i|[\max S^*_i, \max S_i]) \geq \min br^i_{g_i}(\pi_i|[\min S^*_i, \max S_i]). \tag{2.4}$$

Such a function v is called a *monotone potential function relative to* $B_\eta(\Delta(S))$.

Notice the 'max' and the 'min' in the right hand sides of (2.3) and (2.4), respectively (cf. those of (2.1) and (2.2)). Notice also that v is defined on the whole set A and that (2.3) and (2.4) must be satisfied also for beliefs π_i that assign small probability (less than η) to actions outside S_{-i}, which is an indispensable requirement for the informational robustness and the stability; see Example 2.7.

Definition 2.4. An interval $S^* \subset A$ is an *iterated MP-maximizer set* of $\mathbf{g}$ if there exists a sequence of intervals $S^0, S^1, \ldots, S^m$ with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ such that S^k is an MP-maximizer set relative to S^{k-1} for each $k = 1, \ldots, m$.

An action profile $a^* \in A$ is an *iterated MP-maximizer* of $\mathbf{g}$ if $\{a^*\}$ is an iterated MP-maximizer set of $\mathbf{g}$.

For an iterated (strict) MP-maximizer set S^*, the sequence $S^0, S^1, \ldots, S^m$ in the definition will be called *associated intervals* of S^*.

Remark 2.1. In Definition 2.3, let $\mathcal{P}_i = \{S^*_i\} \cup \{\{a_i\} \mid a_i \notin S^*_i\}$ and $\mathcal{P} = \{\prod_{i \in I} X_i \mid X_i \in \mathcal{P}_i \text{ for } i \in I\}$. If v is $\mathcal{P}$-measurable, then "$[\min S_i, \min S^*_i]$" in the left hand side of (2.1) and (2.3) and "$[\max S^*_i, \max S_i]$" in the left hand side of (2.2) and (2.4) can be replaced with "$[\min S_i, \max S^*_i]$" and "$[\min S^*_i, \max S_i]$", respectively. If S^* is an MP-maximizer set relative to A with v being $\mathcal{P}$-measurable, then it is an MP-maximizer (with respect to $\mathcal{P}$) in the sense of Morris and Ui (2005, Definition 8).

Here we show that iterated strict MP-maximizer is actually a refinement of iterated MP-maximizer.

Proposition 2.1. *An iterated strict MP-maximizer set is an iterated MP-maximizer set.*

It is sufficient to show the following.

Lemma 2.2. *Let S^* and S be intervals such that $S^* \subset S \subset A$. If S^* is a strict MP-maximizer set of $\mathbf{g}|_S$ with a strict monotone potential function $v\colon S \to \mathbb{R}$, then there exist a function $\tilde{v}\colon A \to \mathbb{R}$ and a real number $\eta > 0$ such that $S^* = \arg\max_{a \in A} \tilde{v}(a)$, and* (2.1) *and* (2.2) *with $A = S$ hold for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S_{-i}))$.*

Moreover, if $v|_S$ is supermodular, then $\tilde{v}$ can be taken so that $\tilde{v}|_A$ is supermodular.

We call such a function $\tilde{v}$ a *strict monotone potential function relative to* $B_\eta(\Delta(S))$.

Proof. See Appendix. ▮

Finally, we report a useful fact for reference.

Lemma 2.3. *Suppose that* $\mathbf{g}$ *has an iterated MP-maximizer* S^* *with* $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ *and* $(v^k)_{k=1}^m$. *Then, there exists* $\eta > 0$ *such that for all* $k = 1, \ldots, m$ *and for all* $i \in I$ *and all* $\pi_i \in B_\eta(\Delta(S_{-i}^k))$,

$$br_{g_i}^i(\pi_i) \cap S_i^k \neq \emptyset.$$

Proof. Note first that for all $\ell = 1, \ldots, k$, $S^\ell = \arg\max_{a \in S^{\ell-1}} v^\ell(a)$, and therefore we can take $\varepsilon^\ell > 0$ such that for all $i \in I$ and all $\pi_i \in B_{\varepsilon^\ell}(\Delta(S_{-i}^\ell))$,

$$br_{v^\ell}^i(\pi_i | [\min S_i^{\ell-1}, \min S_i^\ell]) = \min S_i^\ell,$$
$$br_{v^\ell}^i(\pi_i | [\max S_i^\ell, \max S_i^{\ell-1}]) = \max S_i^\ell$$

due to the continuity of $v^\ell(h, \pi_i)$ in π_i. By definition, for all $\ell = 1, \ldots, k$, there exists $\eta^\ell > 0$ such that for all $i \in I$ and all $\pi_i \in B_{\eta^\ell}(\Delta(S_{-i}^\ell))$,

$$\max br_{g_i}^i(\pi_i | [\min S_i^{\ell-1}, \max S_i^\ell]) \geq \min br_{v^\ell}^i(\pi_i | [\min S_i^{\ell-1}, \min S_i^\ell]),$$
$$\min br_{g_i}^i(\pi_i | [\min S_i^\ell, \max S_i^{\ell-1}]) \leq \max br_{v^\ell}^i(\pi_i | [\max S_i^\ell, \max S_i^{\ell-1}]).$$

Setting $\eta = \min_\ell \varepsilon^\ell \wedge \min_\ell \eta^\ell$, we have that for all $\ell = 1, \ldots, k$ and for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S_{-i}^k))$ $(\subset B_\eta(\Delta(S_{-i}^\ell)))$,

$$\max br_{g_i}^i(\pi_i | [\min S_i^{\ell-1}, \max S_i^{\ell-1}]) \geq \min S_i^\ell,$$
$$\min br_{g_i}^i(\pi_i | [\min S_i^{\ell-1}, \max S_i^{\ell-1}]) \leq \max S_i^\ell,$$

and therefore,

$$br_{g_i}^i(\pi_i | S_i^{\ell-1}) \cap S_i^\ell \neq \emptyset.$$

An induction argument thus proves that

$$br_{g_i}^i(\pi_i) \cap S_i^k \neq \emptyset$$

for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S_{-i}^k))$, as claimed. ▮

2.3 Iterated p-Dominance

This subsection provides simple ways to find iterated monotone potentials using iteration of $\mathbf{p}$-dominance as considered in Tercieux (2004).

Let $\mathbf{p} = (p_i)_{i \in I} \in [0, 1)^N$. Let us first review the definition of strict $\mathbf{p}$-dominant equilibrium due to Kajii and Morris (1997).

Definition 2.5. An action profile $a^* \in A$ is a *strict* $\mathbf{p}$*-dominant equilibrium* of $\mathbf{g}$ if for all $i \in I$,

$$\{a_i^*\} = br_{g_i}^i(\pi_i)$$

holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) > p_i$.

Next we define strict **p**-best response set. This concept is a set-valued extension of the strict **p**-dominance concept (see Tercieux (2004, 2006)). The set $S = \prod_{i \in I} S_i$ ($S_i \subset A_i$, $i \in I$) is a strict **p**-best response set if, whenever any player i believes with probability strictly greater than p_i that the other players will play actions in S_{-i}, all of his best responses are contained in S_i.

Definition 2.6. Let S_i^* be a nonempty subset of A_i for each $i \in I$, and $S^* = \prod_{i \in I} S_i^*$. The set S^* is a *strict* **p**-*best response set* of **g** if for all $i \in I$,

$$br^i_{g_i}(\pi_i) \subset S_i^*$$

holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(S^*_{-i}) > p_i$.

Now with the two steps procedure that we used to define an iterated MP-maximizer, we define iterated (strict) **p**-dominant equilibrium. Formally, this can be stated as follows.

Definition 2.7. Let S_i^* be a nonempty subset of A_i for each $i \in I$, and $S^* = \prod_{i \in I} S_i^*$. The set S^* is an *iterated strict* **p**-*best response set* of **g** if there exists a sequence $S^0, S^1, \ldots, S^m$ with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ such that S^k is a strict **p**-best response set in $\mathbf{g}|_{S^{k-1}}$ for each $k = 1, \ldots, m$.

An action profile $a^* \in A$ is an *iterated strict* **p**-*dominant equilibrium* of **g** if $\{a^*\}$ is an iterated strict **p**-best response set of **g**.

For an iterated strict **p**-best response set S^*, the sequence $S^0, S^1, \ldots, S^m$ in the definition will be called *associated subsets* of S^*.

We now prove a link between iterated **p**-dominant equilibrium and iterated MP-maximizer.

Proposition 2.4. *Let a^* be an iterated strict* **p**-*dominant equilibrium of* **g** *with $\sum_{i \in I} p_i < 1$, and $A = S^0 \supset S^1 \supset \cdots \supset S^m = \{a^*\}$ associated subsets. Then, there exists an order $<$ on A such that S^k's are intervals and a^* is an iterated strict MP-maximizer with monotone potential functions v^k ($k = 1, \ldots, m$) that are supermodular and of the form:*

$$v^k(a) = \begin{cases} 1 - \sum_{i \in I} p_i & \text{if } a \in S^k, \\ -\sum_{i \in C^k(a)} p_i & \text{otherwise,} \end{cases} \tag{2.5}$$

where $C^k(a) = \{i \in I \mid a_i \in S_i^k\}$.

To have v^k's be supermodular, re-order the actions so that for all $i \in I$, for all $k = 1, \ldots, m$, and for all $a_i \in S_i^k$, $a_i' \in S_i^{k-1} \setminus S_i^k$, $a_i' < a_i$. Note that this implies that $a^* = \max A = \max S^1 = \cdots \max S^m$. One can verify that for all k, v^k is supermodular with respect to the new order.

Now Proposition 2.4 follows from the following lemma.

Lemma 2.5. *Let $(S^k)_{k=0}^m$ be intervals such that $A = S^0 \supset S^1 \supset \cdots \supset S^m$ and $\max S^k = \max A$ for all $k = 1, \dots, m$. If for each $k = 1, \dots, m$, S^k is a strict $\mathbf{p}^k$-best response set in $\mathbf{g}|_{S^{k-1}}$ with $\sum_{i \in I} p_i^k < 1$, then S^m is an iterated strict MP-maximizer set of* $\mathbf{g}$.

Proof. For each $k = 1, \dots, m$, let v^k be given as in (2.5) with $p_i = p_i^k$. Consider any $k = 1, \dots, m$ and any $i \in I$. It is now sufficient to show that v^k is a strict monotone potential functions for S^k in $\mathbf{g}|_{S^{k-1}}$. Denote $\underline{a}_j^\ell = \min S_j^\ell$ for each $j \in I$ and $\ell = k-1, k$. We want to show that for all $\pi_i \in \Delta(S_{-i}^{k-1})$,

$$\min br_{v^k}^i(\pi_i | S_i^{k-1}) \leq \min br_{g_i}^i(\pi_i | S_i^{k-1})$$

(note that $br_{v^k}^i(\pi_i | S_i^{k-1}) = br_{v^k}^i(\pi_i | [\underline{a}_i^{k-1}, \underline{a}_i^k])$ by construction).

Fix any $\pi_i \in \Delta(S_{-i}^{k-1})$. Observe that

$$v^k(h, \pi_i) = \sum_{a_{-i} \in S_{-i}^{k-1}} \pi_i(a_{-i}) v^k(h, a_{-i})$$

takes only two different values: one for $h < \underline{a}_i^k$ and another for $h \geq \underline{a}_i^k$. Hence,

$$\min br_{v^k}^i(\pi_i | S_i^{k-1}) \in \{\underline{a}_i^{k-1}, \underline{a}_i^k\}.$$

It is sufficient to consider the case where $\min br_{v^k}^i(\pi_i | S_i^{k-1}) = \underline{a}_i^k$. For such $\pi_i \in (S_{-i}^{k-1})$, we have

$$\begin{aligned} 0 < v^k(\underline{a}_i^k, \pi_i) - v^k(\underline{a}_i^{k-1}, \pi_i) &= \sum_{a_{-i} \in S_{-i}^k} \pi_i(a_{-i})(1 - p_i^k) - \sum_{a_{-i} \notin S_{-i}^k} \pi_i(a_{-i}) p_i^k \\ &= \sum_{a_{-i} \in S_{-i}^k} \pi_i(a_{-i}) - p_i^k, \end{aligned}$$

and thus $\pi_i(S_{-i}^k) > p_i^k$. Since S^k is a strict $\mathbf{p}^k$-best response set in $\mathbf{g}|_{S^{k-1}}$, $br_{g_i}^i(\pi_i | S_i^{k-1}) \subset S_i^k$. Therefore, we have $\min br_{g_i}^i(\pi_i | S_i^{k-1}) \geq \underline{a}_i^k = \min br_{v^k}^i(\pi_i | S_i^{k-1})$, completing the proof. ▮

In the case where $\mathbf{g}$ is supermodular, we have a simple characterization of iterated $\mathbf{p}$-dominant equilibrium by means of the notion of iterated pairwise $\mathbf{p}$-dominance.

Definition 2.8. An action profile $a^* \in A$ is an *iterated pairwise strict* $\mathbf{p}$-*dominant equilibrium* of $\mathbf{g}$ if there exists a sequence $0 = \underline{a}_i^0 \leq \underline{a}_i^1 \leq \cdots \leq \underline{a}_i^m = a_i^* = \overline{a}_i^m \leq \cdots \leq \overline{a}_i^1 \leq \overline{a}_i^0 = n_i$ for each $i \in I$ such that for all $k = 1, \dots, m$, $\underline{a}^k$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[\underline{a}^{k-1}, \underline{a}^k]}$ and $\overline{a}^k$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[\overline{a}^k, \overline{a}^{k-1}]}$.

Proposition 2.6. *Suppose that* $\mathbf{g}$ *is supermodular. If* a^* *is an iterated pairwise strict* $\mathbf{p}$*-dominant equilibrium of* $\mathbf{g}$*, then* a^* *is an iterated strict* $\mathbf{p}$*-dominant equilibrium of* $\mathbf{g}$*.*

Hence, by Proposition 2.4, if a^* is an iterated pairwise strict $\mathbf{p}$-dominant equilibrium of a supermodular game $\mathbf{g}$ with $\sum_{i\in I} p_i < 1$, then a^* is an iterated strict MP-maximizer of $\mathbf{g}$.

The proof utilizes the following fact.

Lemma 2.7. *Suppose that* $\mathbf{g}$ *is supermodular. Let* S *be an interval such that* $\max S = \max A$*. If* $\min S$ *is a strict* $\mathbf{p}$*-dominant equilibrium in* $\mathbf{g}|_{[0,\min S]}$*, then* S *is a strict* $\mathbf{p}$*-best response set of* $\mathbf{g}$*.*

Proof. Given S as above, denote $\underline{a}_i = \min S_i$ for each $i \in I$. Take any $i \in I$ and any $\pi_i \in \Delta(A_{-i})$ such that $\pi_i(S_{-i}) > p_i$. We want to show that $br^i_{g_i}(\pi_i) \subset S_i$. Define $\pi'_i \in \Delta(A_{-i})$ by

$$\pi'_i(a_{-i}) = \begin{cases} \pi_i(S_{-i}) & \text{if } a_{-i} = \underline{a}_{-i}, \\ 1 - \pi_i(S_{-i}) & \text{if } a_{-i} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\pi'_i(\underline{a}_i) > p_i$, we have $br^i_{g_i}(\pi'_i|[0, \underline{a}_i]) = \{\underline{a}_i\}$ by the assumption that $\underline{a}$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[0,\underline{a}]}$, so that $\min br^i_{g_i}(\pi'_i) \geq \underline{a}_i$. On the other hand, since $\pi'_i \precsim \pi_i$, we have $\min br^i_{g_i}(\pi'_i) \leq \min br^i_{g_i}(\pi_i)$ due to the supermodularity of $\mathbf{g}$. It thus follows that $\min br^i_{g_i}(\pi_i) \geq \underline{a}_i$, which implies that $br^i_{g_i}(\pi_i) \subset S_i$. ∎

Proof of Proposition 2.6. Suppose that a^* is an iterated pairwise $\mathbf{p}$-dominant equilibrium. It is sufficient to show that (a) for each $k = 1, \dots, m$, $[\underline{a}^k, \overline{a}^0]$ is a strict $\mathbf{p}$-best response set in $\mathbf{g}|_{[\underline{a}^{k-1},\overline{a}^0]}$, and (b) for each $k = 1, \dots, m$, $[a^*, \overline{a}^k]$ is a strict $\mathbf{p}$-best response set in $\mathbf{g}|_{[a^*,\overline{a}^{k-1}]}$. But, since $\underline{a}^k$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[\underline{a}^{k-1},\underline{a}^k]}$, (a) follows from Lemma 2.7 with $A = [\underline{a}^{k-1}, \overline{a}^0]$ and $S = [\underline{a}^k, \overline{a}^0]$. One can similarly prove (b) by Lemma 2.7 (by reversing the order on actions). ∎

Remark 2.2. For supermodular games, it is simple to check whether $\underline{a}^k$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[\underline{a}^{k-1},\underline{a}^k]}$ for some $\mathbf{p}$ with $\sum_{i\in I} p_i < 1$. Indeed, it is necessary and sufficient to check that for each $i \in I$, $br^i_{g_i}(\pi_i|[\underline{a}^{k-1}_i, \underline{a}^k_i]) = \{\underline{a}^k_i\}$ for $\pi_i \in \Delta([\underline{a}^{k-1}_{-i}, \underline{a}^k_{-i}])$ such that $\pi_i(\underline{a}^{k-1}_{-i}) = 1 - p_i$ and $\pi_i(\underline{a}^k_{-i}) = p_i$.

2.4 Examples

2.4.1 Iterated Risk-Dominance

We consider the class of two-player coordination games, where there are two players with the same action set $A_i = \{0, 1, \dots, n\}$ for each $i = 1, 2$, and all the action profiles on the diagonal are strict Nash equilibria, i.e., (h,k) is a strict Nash equilibrium if and only if $h = k$.

We provide a simpler way to find iterated strict MP-maximizers in two-player supermodular coordination games. Let us first generalize the notion of pairwise risk-dominance by Kandori and Rob (1998) to asymmetric two-player games and then define our notion of iterated risk-dominance.

Definition 2.9. Let $\mathbf{g}$ be a two-player coordination game. We say that (h,h) *pairwise risk dominates* (k,k) in $\mathbf{g}$ if

$$\begin{aligned}&\big(g_1(h,h) - g_1(k,h)\big) \times \big(g_2(h,h) - g_2(k,h)\big)\\ &\qquad > \big(g_1(k,k) - g_1(h,k)\big) \times \big(g_2(k,k) - g_2(h,k)\big),\end{aligned} \tag{2.6}$$

and write (h,h) PRD (k,k).

Definition 2.10. Let $\mathbf{g}$ be a two-player coordination game. (h^*,h^*) is an *iterated risk-dominant equilibrium* of $\mathbf{g}$ if

1. (h,h) PRD $(h-1,h-1)$ for each $h = 1,\dots,h^*$, and
2. (h,h) PRD $(h+1,h+1)$ for each $h = h^*,\dots,n-1$.

Proposition 2.8. *Suppose that* $\mathbf{g}$ *is a two-player supermodular coordination game. If* (h^*,h^*) *is an iterated risk-dominant equilibrium of* $\mathbf{g}$*, then it is an iterated strict MP-maximizer of* $\mathbf{g}$.

Proof. Suppose that (h^*,h^*) is an iterated risk-dominant equilibrium. In light of Lemma 2.5, it is sufficient to show that (a) for each $h = 1,\dots,h^*$, $[h,n]\times[h,n]$ is a strict $\mathbf{p}^h$-best response set in $\mathbf{g}|_{[h-1,n]\times[h-1,n]}$ for some $\mathbf{p}^h$ such that $p_1^h + p_2^h < 1$, and (b) for each $k = h^*,\dots,n-1$, $[h^*,k]\times[h^*,k]$ is a strict $\mathbf{p}^k$-best response set in $\mathbf{g}|_{[h^*,k+1]\times[h^*,k+1]}$ for some $\mathbf{p}^k$ such that $p_1^k + p_2^k < 1$. We only show (a).

Consider any $h = 1,\dots,h^*$, and let

$$p_i^h = \frac{g_i(h-1,h-1) - g_i(h,h-1)}{g_i(h,h) - g_i(h-1,h) + g_i(h-1,h-1) - g_i(h,h-1)} > 0$$

and $\mathbf{p}^h = (p_1^h, p_2^h)$. Verify that $p_1^h + p_2^h < 1$ due to the condition (2.6) and that (h,h) is a strict $\mathbf{p}^h$-dominant equilibrium in $\mathbf{g}|_{[h-1,h]\times[h-1,h]}$. It therefore follows from Lemma 2.7 that $[h,n]\times[h,n]$ is a strict $\mathbf{p}^h$-best response set in $\mathbf{g}|_{[h-1,n]\times[h-1,n]}$. ∎

Example 2.1. Consider the following asymmetric supermodular game:

	0	1	2
0	$3,1$	$0,0$	$-2,-2$
1	$0,0$	$2,2$	$0,0$
2	$-2,-2$	$0,0$	$1,3$

In this game, $(1, 1)$ is an iterated risk-dominant equilibrium and hence an iterated strict MP-maximizer. Note that this game has no iterated **p**-dominant equilibrium for $p_1 + p_2 < 1$.

If we consider *symmetric* games (i.e., $g_2(k, h) = g_1(h, k)$ for all $h \in A_1$ and $k \in A_2$), the proof of Proposition 2.8 in fact shows also the following link between iterated **p**-dominance and iterated risk-dominance.

Proposition 2.9. *Suppose that* $\mathbf{g}$ *is a symmetric two-player supermodular coordination game. If* (h^*, h^*) *is an iterated risk-dominant equilibrium of* $\mathbf{g}$*, then it is an iterated strict* (p, p)*-dominant equilibrium of* $\mathbf{g}$ *for some* $p < 1/2$*.*

Example 2.2. Consider the following symmetric supermodular game:

	0	1	2
0	$1, 1$	$0, 0$	$-3, -6$
1	$0, 0$	$2, 2$	$0, 0$
2	$-6, -3$	$0, 0$	$3, 3$

In this game, $(2, 2)$ is an iterated risk-dominant equilibrium and indeed an iterated strict $(2/5, 2/5)$-dominant equilibrium. Observe that this game has no (p, p)-dominant equilibrium for any $p < 1/2$.[8]

2.4.2 Morris' Example

Example 2.3. Consider the following symmetric 4×4 supermodular game due to Morris (1999):

	0	1	2	3
0	$50, 50$	$46, 41$	$32, 23$	$8, 3$
1	$41, 46$	$50, 50$	$42, 47$	$27, 29$
2	$23, 32$	$47, 42$	$50, 50$	$41, 52$
3	$3, 8$	$29, 27$	$52, 41$	$50, 50$

Morris (1999) shows that this game has no robust equilibrium to incomplete information. Therefore, this game has no iterated MP-maximizer due to our Theorem 3.1.

Example 2.4. Consider the following slight modification of the above game:

[8] Note also that this game has no globally risk-dominant equilibrium as defined by Kandori and Rob (1998); see Remark 2.4 below.

	0	1	2	3
0	50, 50	40, 41	32, 23	8, 3
1	41, 40	50, 50	42, 47	27, 29
2	23, 32	47, 42	50, 50	41, 52
3	3, 8	29, 27	52, 41	50, 50

Notice that $g_1(0,1) = g_2(1,0) = 40$. In this game, $(3,3)$ is an iterated pairwise strict (p,p)-dominant equilibrium for some $p < 1/2$ and hence an iterated strict MP-maximizer, with a sequence $S^1 = \{1,2,3\} \times \{1,2,3\}$, $S^2 = \{2,3\} \times \{2,3\}$, and $S^3 = \{(3,3)\}$.

2.4.3 Young's Example

Example 2.5. Consider the following symmetric 3×3 game due to Young (1993):

	0	1	2
0	6, 6	0, 5	0, 0
1	5, 0	7, 7	5, 5
2	0, 0	5, 5	8, 8

This game does not have any (p,p)-dominant equilibrium for $p < 3/5$, but $(2,2)$ is an iterated strict $(2/5, 2/5)$-dominant equilibrium with a sequence $S^1 = \{1,2\} \times \{1,2\}$ and $S^2 = \{(2,2)\}$.

Remark 2.3. For the above game, OTH (2003) have reported that $(2,2)$ is a strict MP-maximizer with a strict monotone potential function that is supermodular. Therefore, the results by Morris and Ui (2005) and OTH (2003) show that $(2,2)$ is robust to incomplete information and globally accessible and linearly accessible under perfect foresight dynamics with small friction. On the other hand, our Proposition 3.8 shows that $(2,2)$ is the *unique* robust equilibrium to incomplete information.

Remark 2.4. In the above game, $(2,2)$ is *globally pairwise risk-dominant* (Kandori and Rob (1998)), i.e., $(2,2)$ PRD (h,h) for all $h \neq 2$. In general, however, an iterated strict (p,p)-dominant equilibrium with $p < 1/2$ need not be globally pairwise risk-dominant (see Example 2.2).

2.4.4 A Binary Game with Three Players

Example 2.6. Consider the following $2 \times 2 \times 2$ supermodular game:

	0	1
0	$1,1,x$	$0,0,x$
1	$0,0,x$	$2,2,1$

0

	0	1
0	$1,1,0$	$0,0,0$
1	$0,0,0$	$2,2,2$

1

where $x > 0$ is arbitrarily large. In this game, $(1,1,1)$ is an iterated strict (1/3,1/3,0)-dominant equilibrium and hence an iterated strict MP-maximizer with a sequence $S^1 = \{1\} \times \{1\} \times \{0,1\}$ and $S^2 = \{(1,1,1)\}$.

2.4.5 A Degenerate Game with No Robust Equilibrium

Example 2.7. Consider the following 2×3 supermodular game:

	0	1	2
0	$1,0$	$1,1$	$0,0$
1	$0,0$	$1,1$	$1,0$

In this game, both $(0,1)$ and $(1,1)$ were iterated MP-maximizers if η in Definition 2.3 were allowed to be set to zero. But one can verify that none of them are robust to incomplete information or globally accessible under perfect foresight dynamics. Hence, this example shows that the requirement (in the definition of iterated MP-maximizer) that the conditions be satisfied for all $\pi_i \in B_\eta(\Delta(S^{k-1}))$ (where $\eta > 0$) is indispensable for robustness to incomplete information and stability under perfect foresight dynamics.

3 Robustness to Incomplete Information

3.1 ε-Elaborations and Robust Equilibria

Given the game $\mathbf{g}$, we consider the following class of incomplete information games. Each player $i \in I$ has a countable set of types, denoted by T_i. We write $T = \prod_{i \in I} T_i$ and $T_{-i} = \prod_{j \neq i} T_i$. The prior probability distribution on T is given by P. We assume that P satisfies that $\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) > 0$ for all $i \in I$ and $t_i \in T_i$. Let $\Delta_0(T)$ be the set of such probability distributions on T. Under this assumption, the conditional probability of t_{-i} given t_i, $P(t_{-i}|t_i)$, is well-defined by $P(t_{-i}|t_i) = P(t_i, t_{-i}) / \sum_{t'_{-i} \in T_{-i}} P(t_i, t'_{-i})$. An event $T' \subset T$ is said to be a *simple event* if it is a product of sets of types of each player, i.e., $T' = \prod_{i \in I} T'_i$ where each $T'_i \subset T_i$. Given a simple event T', we write $T'_{-i} = T'_1 \times \cdots \times T'_{i-1} \times T'_{i+1} \times \cdots \times T'_N$ and $P(T'_{-i}|t_i) = \sum_{t_{-i} \in T'_{-i}} P(t_{-i}|t_i)$. The payoff function for player $i \in I$ is a bounded function $u_i \colon A \times T \to \mathbb{R}$. Denote $\mathbf{u} = (u_i)_{i \in I}$. Fixing type space T, we represent an incomplete information game by $(\mathbf{u}, P)$.

A (behavioral) strategy for player i is a function $\sigma_i\colon T_i \to \Delta(A_i)$, where $\Delta(A_i)$ is the set of probability distributions over A_i. Denote by Σ_i the set of strategies for player i, and let $\Sigma = \prod_{i\in I}\Sigma_i$, $\sigma = (\sigma_1,\dots,\sigma_n) \in \Sigma$, $\Sigma_{-i} = \prod_{j\neq i}\Sigma_j$, and $\sigma_{-i} = (\sigma_1,\dots,\sigma_{i-1},\sigma_{i+1},\dots,\sigma_n) \in \Sigma_{-i}$. For a strategy σ_i, we denote by $\sigma_i(a_i|t_i)$ the probability that $a_i \in A_i$ is chosen at $t_i \in T_i$. We write $\sigma(a|t) = \prod_{i\in I}\sigma_i(a_i|t_i)$ and $\sigma_{-i}(a_{-i}|t_{-i}) = \prod_{j\neq i}\sigma_j(a_j|t_j)$. We also write $\sigma_P(a) = \sum_{t\in T}P(t)\sigma(a|t)$. We endow Σ with the topology of uniform convergence on finite subsets of T.[9] The set Σ is convex, and compact with respect to this topology.

We define $\sigma_i \precsim \sigma'_i$ for $\sigma_i,\sigma'_i \in \Sigma_i$ by $\sigma_i(t_i) \precsim \sigma'_i(t_i)$ for all $t_i \in T_i$; $\sigma \precsim \sigma'$ for $\sigma,\sigma' \in \Sigma$ by $\sigma_i \precsim \sigma'_i$ for all $i \in I$; and $\sigma_{-i} \precsim \sigma'_{-i}$ for $\sigma_{-i},\sigma'_{-i} \in \Sigma_{-i}$ by $\sigma_j \precsim \sigma'_j$ for all $j \neq i$.

The expected payoff to player i with type $t_i \in T_i$ playing $h \in A_i$ against strategy profile σ_{-i} is given by

$$U_i(h,\sigma_{-i})(t_i) = \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\, u_i\big((h,\sigma_{-i}(t_{-i})),(t_i,t_{-i})\big),$$

where $u_i((h,\sigma_{-i}(t_{-i})),t) = \sum_{a_{-i}\in A_{-i}}\sigma_{-i}(a_{-i}|t_{-i})u_i((h,a_{-i}),t)$. Let $BR^i\colon \Sigma_{-i}\times T_i \to A_i$ be defined for each i by

$$BR^i(\sigma_{-i})(t_i) = \arg\max\{U_i(h,\sigma_{-i})(t_i) \mid h \in A_i\}.$$

Note that for each $i \in I$, the correspondence BR^i is upper semi-continuous since U_i is continuous.

Definition 3.1. A strategy profile $\sigma \in \Sigma$ is a *Bayesian Nash equilibrium* of $(\mathbf{u},P)$ if for all $i \in I$, all $h \in A_i$, and all $t_i \in T_i$,

$$\sigma_i(h|t_i) > 0 \Rightarrow h \in BR^i(\sigma_{-i})(t_i).$$

Let $\beta^i\colon \Sigma_{-i} \to \Sigma_i$ be player i's best response correspondence in $(\mathbf{u},P)$, defined by

$$\begin{aligned}\beta^i(\sigma_{-i}) = \big\{\xi_i \in \Sigma_i \,\big|\, \forall\, h \in A_i,\ \forall\, t_i \in T_i :\\ \big[\xi_i(h|t_i) > 0 \Rightarrow h \in BR^i(\sigma_{-i})(t_i)\big]\big\},\end{aligned} \tag{3.1}$$

and $\beta\colon \Sigma \to \Sigma$ be given by $\beta(\sigma) = \prod_{i\in I}\beta^i(\sigma_{-i})$. A Bayesian Nash equilibrium of $(\mathbf{u},P)$, $\sigma \in \Sigma$, is a fixed point of β, i.e., $\sigma \in \beta(\sigma)$. Since β is nonempty-, convex-, and compact-valued and upper semi-continuous, the

[9]This topology is metrizable by the metric d_μ defined by

$$d_\mu(\sigma,\sigma') = \sup_{t\in T}\mu(t)\big|\sigma(t) - \sigma'(t)\big|$$

for $\mu \in \Delta(T)$ such that $\mathrm{supp}(\mu) = T$.

existence of Bayesian Nash equilibria then follows from Kakutani's fixed point theorem.

Given $\mathbf{g}$, let $T_i^{g_i}$ be the set of types t_i such that payoffs of player i of type t_i is given by g_i and he knows his payoffs:

$$T_i^{g_i} = \{t_i \in T_i \mid u_i(a,(t_i,t_{-i})) = g_i(a) \\ \text{for all } a \in A \text{ and all } t_{-i} \in T_{-i} \text{ with } P(t_i,t_{-i}) > 0\}.$$

Denote $T^{\mathbf{g}} = \prod_i T_i^{g_i}$.

Definition 3.2. Let $\varepsilon \in [0,1]$. An incomplete information game $(\mathbf{u}, P)$ is an *ε-elaboration* of $\mathbf{g}$ if $P(T^{\mathbf{g}}) = 1 - \varepsilon$.

Following Kajii and Morris (1997), we say that a^* is robust if, for small $\varepsilon > 0$, every ε-elaboration of $\mathbf{g}$ has a Bayesian Nash equilibrium σ with $\sigma_P(a^*)$ close to 1.

Definition 3.3. Action profile $a^* \in A$ is *robust to all elaborations* in $\mathbf{g}$ if for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \le \bar{\varepsilon}$, any ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$ has a Bayesian Nash equilibrium σ such that $\sigma_P(a^*) \ge 1 - \delta$.

Given $P \in \Delta_0(T)$, we write for any function $f\colon A \to \mathbb{R}$

$$BR_f^i(\sigma_{-i}|S_i)(t_i) = \arg\max_{h \in S_i} \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) f(h, \sigma_{-i}(t_{-i})),$$

where $S_i \subset A_i$, $\sigma_{-i} \in \Sigma_{-i}$, and $t_i \in T_i$. Note that this can be written as

$$BR_f^i(\sigma_{-i}|S_i)(t_i) = br_f^i(\pi_i^{t_i}(\sigma_{-i})|S_i)$$

where $\pi_i^{t_i}(\sigma_{-i}) \in \Delta(A_{-i})$ is given by

$$\pi_i^{t_i}(\sigma_{-i})(a_{-i}) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \sigma_{-i}(a_{-i}|t_{-i}).$$

Thus, if $f|_{S_i \times A_{-i}}$ is supermodular, then whenever $\sigma_{-i} \precsim \sigma'_{-i}$, we have

$$\min BR_f^i(\sigma_{-i}|S_i)(t_i) \le \min BR_f^i(\sigma'_{-i}|S_i)(t_i),$$
$$\max BR_f^i(\sigma_{-i}|S_i)(t_i) \le \max BR_f^i(\sigma'_{-i}|S_i)(t_i).$$

3.2 Informational Robustness of Iterated MP-Maximizer

In this subsection, we state and prove our first main result, which shows that under certain monotonicity conditions, an iterated MP-maximizer is robust to incomplete information.

Theorem 3.1. *Suppose that* $\mathbf{g}$ *has an iterated MP-maximizer* a^* *with associated intervals* $(S^k)_{k=0}^m$ *and monotone potential functions* $(v^k)_{k=1}^m$*. If for each* $k = 1, \ldots, m$*,* $g_i|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$ *or* $v^k|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$*, then* a^* *is robust to all elaborations in* $\mathbf{g}$.

Due to Lemma 2.2, we immediately have the following.

Corollary 3.2. *Suppose that* $\mathbf{g}$ *has an iterated strict MP-maximizer* a^* *with associated intervals* $(S^k)_{k=0}^m$ *and strict monotone potential functions* $(v^k)_{k=1}^m$*. If for each* $k = 1, \ldots, m$*,* $g_i|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$ *or* $v^k|_{S^{k-1}}$ *is supermodular, then* a^* *is robust to all elaborations in* $\mathbf{g}$.

Suppose that a^* is an iterated MP-maximizer of $\mathbf{g}$ with monotone potential functions $(v^k)_{k=1}^m$ that are relative to $B_{2\eta}(S^{k-1})$ respectively for $k = 1, \ldots, m$, where $\eta > 0$ is sufficiently small so that for all $i \in I$ and all $k = 1, \ldots, m$,

$$br_{g_i}^i(\pi_i) \cap S_i^k \neq \emptyset,$$

and therefore,

$$br_{g_i}^i(\pi_i | S_i^k) \subset br_{g_i}^i(\pi_i)$$

hold for $\pi_i \in B_{2\eta}(S_{-i}^k)$ (see Lemma 2.3). For each $k = 0, 1, \ldots, m$ and $i \in I$, write $S_i^k = [\underline{a}_i^k, \overline{a}_i^k]$, where $0 = \underline{a}_i^0 \leq \underline{a}_i^1 \leq \cdots \leq \underline{a}_i^m = a_i^* = \overline{a}_i^m \leq \cdots \leq \overline{a}_i^1 \leq \overline{a}_i^0 = n_i$. We assume without loss of generality that for all $k = 1, \ldots, m$, $S^k \neq S^{k-1}$, i.e., for some $i \in I$, $\underline{a}_i^k \neq \underline{a}_i^{k-1}$ or $\overline{a}_i^k \neq \overline{a}_i^{k-1}$.

Now, given $P \in \Delta_0(T)$, define $J_P^k \colon \Sigma \to \mathbb{R}$ for each $k = 1, \ldots, m$ to be

$$J_P^k(\sigma) = \sum_{t \in T} P(t) v^k(\sigma(t)),$$

and for any $\xi, \zeta \in \Sigma$ such that $\xi(t) \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^{k-1}])$ and $\zeta(t) \in \prod_i \Delta([\overline{a}_i^{k-1}, \overline{a}_i^0])$ for all $t \in T$, and any simple event $T' \subset T$, let

$$\begin{aligned}
\Sigma_{\xi,T'}^{k,-} = \{\sigma \in \Sigma \,|\, \forall\, i \in I : \sigma_i(t_i) &= \xi_i(t_i) \ \forall\, t_i \in T_i \setminus T_i', \\
\sigma_i(t_i) &\in \Delta([\underline{a}_i^{k-1}, \underline{a}_i^k]) \ \forall\, t_i \in T_i'\}, \\
\Sigma_{\zeta,T'}^{k,+} = \{\sigma \in \Sigma \,|\, \forall\, i \in I : \sigma_i(t_i) &= \zeta_i(t_i) \ \forall\, t_i \in T_i \setminus T_i', \\
\sigma_i(t_i) &\in \Delta([\overline{a}_i^k, \overline{a}_i^{k-1}]) \ \forall\, t_i \in T_i'\}.
\end{aligned}$$

Consider the maximization problems:

$$\max\ J_P^k(\sigma) \quad \text{s.t. } \sigma \in \Sigma_{\xi,T'}^{k,-}, \tag{3.2}$$

$$\max\ J_P^k(\sigma) \quad \text{s.t. } \sigma \in \Sigma_{\zeta,T'}^{k,+}. \tag{3.3}$$

Since J_P^k is continuous, and $\Sigma_{\xi,T'}^{k,-}$ and $\Sigma_{\zeta,T'}^{k,+}$ are compact, the above maximization problems admit solutions.

Lemma 3.3. (1) *For each* $k = 1, \dots, m$ *and for any* $P \in \Delta_0(T)$*, any simple event* $T' \subset T$*, and any* $\xi, \zeta \in \Sigma$ *such that* $\xi(t) \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^k])$ *and* $\zeta(t) \in \prod_i \Delta([\overline{a}_i^k, \overline{a}_i^0])$ *for all* $t \in T$*: there exists a solution* $\sigma^{k,-}$ *to the maximization problem* (3.2) *such that*

$$\sigma_i^{k,-}(t_i) = \min BR_{v^k}^i(\sigma_{-i}^{k,-} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \tag{3.4}$$

for all $i \in I$ *and all* $t_i \in T_i'$*; and there exists a solution* $\sigma^{k,+}$ *to the maximization problem* (3.3) *such that*

$$\sigma_i^{k,+}(t_i) = \max BR_{v^k}^i(\sigma_{-i}^{k,+} | [\overline{a}_i^k, \overline{a}_i^{k-1}])(t_i) \tag{3.5}$$

for all $i \in I$ *and all* $t_i \in T_i'$*.*

(2) *For each* $k = 1, \dots, m$*, there exists* $\kappa^k > 0$ *such that for any* $P \in \Delta_0(T)$*, any simple event* $T' \subset T$*, and any* $\xi, \zeta \in \Sigma$ *such that* $\xi(t) \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^k])$ *and* $\zeta(t) \in \prod_i \Delta([\overline{a}_i^k, \overline{a}_i^0])$ *for all* $t \in T$*: any solution* σ *to the maximization problem* (3.2) *satisfies*

$$\sigma_P(\underline{a}^k) \geq 1 - \kappa^k P(T \setminus T');$$

and any solution σ *to the maximization problem* (3.3) *satisfies*

$$\sigma_P(\overline{a}^k) \geq 1 - \kappa^k P(T \setminus T').$$

Proof. (1) We only show the existence of a solution that satisfies (3.4) (the existence of a solution that satisfies (3.5) is proved similarly). First note that for each i,

$$\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) v^k(\sigma(t_i, t_{-i})) = \Big(\sum\nolimits_{t'_{-i} \in T_{-i}} P(t_i, t'_{-i})\Big) \sum\nolimits_{h \in A_i} \sigma_i(h|t_i) U_i^k(h, \sigma_{-i})(t_i) \tag{3.6}$$

for all $t_i \in T_i'$, where

$$U_i^k(h, \sigma_{-i})(t_i) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\, v^k\big((h, \sigma_{-i}(t_{-i})), (t_i, t_{-i})\big).$$

Therefore, any solution to (3.2), σ^k, satisfies, for all $i \in I$,

$$\sigma_i^k(h|t_i) > 0 \Rightarrow h \in BR_{v^k}^i(\sigma_{-i}^k | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \tag{3.7}$$

for all $t_i \in T_i'$.

Since J_P^k is continuous on $\Sigma_{\xi,T'}^{k,-}$, the set of maximizers is a nonempty, closed, and hence compact, subset of $\Sigma_{\xi,T'}^{k,-}$. Hence, a minimal optimal solution (with respect to the order $\precsim$ on Σ) exists by Zorn's lemma (see Lemma A.2.2 in OTH (2003)). Let $\sigma^{k,-}$ be such a minimal solution.

Take any $i \in I$, and consider the strategy σ_i given by

$$\sigma_i(t_i) = \begin{cases} \xi_i(t_i) & \text{for all } t_i \in T_i \setminus T_i' \\ \min BR^i_{v^k}(\sigma^{k,-}_{-i}|[\underline{a}^{k-1}_i, \underline{a}^k_i])(t_i) & \text{for all } t_i \in T_i'. \end{cases}$$

By the definition of σ_i together with equation (3.7), we have $\sigma_i \precsim \sigma^{k,-}_i$. On the other hand, by equation (3.6)

$$J^k_P(\sigma_i, \sigma^{k,-}_{-i}) \geq J^k_P(\sigma^{k,-}),$$

meaning that $(\sigma_i, \sigma^{k,-}_{-i}) \in \Sigma^{k,-}_{\xi,T'}$ is also optimal. Hence, the minimality of $\sigma^{k,-}$ implies that $\sigma_i(t_i) = \sigma^{k,-}_i(t_i)$ for all $t_i \in T_i$. Thus, we have (3.4).

(2) Let $v^k_{\max} = v^k(\underline{a}^k) = v^k(\overline{a}^k)$, $\overline{v}^k = \max_{a \in A \setminus [\underline{a}^k, \overline{a}^k]} v^k(a)$, and $\underline{v}^k = \min_{a \in A} v^k(a)$. Note that $v^k_{\max} > \overline{v}^k \geq \underline{v}^k$. Set $\kappa^k = (v^k_{\max} - \underline{v}^k)/(v^k_{\max} - \overline{v}^k)$. Then, the same argument in the proof of Theorem 3 in Ui (2001) will establish the conclusion. Let $\tilde{\sigma} \in \Sigma^{k,-}_{\xi,T'}$ be such that, $\tilde{\sigma}(\underline{a}^k|t) = 1$ for all $t \in T'$. Let σ be any solution to the maximization problem (3.2). Hence we have

$$\begin{aligned} J^k_P(\sigma) \geq J^k_P(\tilde{\sigma}) &= \sum_{t \in T'} \sum_{a \in A} P(t)\tilde{\sigma}(a|t)v^k(a) + \sum_{t \in T \setminus T'} \sum_{a \in A} P(t)\tilde{\sigma}(a|t)v^k(a) \\ &= P(T')v^k_{\max} + \sum_{t \in T \setminus T'} \sum_{a \in A} P(t)\tilde{\sigma}(a|t)v^k(a) \\ &\geq P(T')v^k_{\max} + [1 - P(T')]\underline{v}^k. \end{aligned}$$

We also have

$$\begin{aligned} J^k_P(\sigma) &= \sum_{a \in A} \left[\sum_{t \in T} P(t)\sigma(a|t) \right] v(a) \\ &= \sum_{a \in A} \sigma_P(a)v(a) \\ &= \sigma_P(\underline{a}^k)v^k_{\max} + \sum_{a \neq \underline{a}^k} \sigma_P(a)v(a) \\ &\leq \sigma_P(\underline{a}^k)v^k_{\max} + (1 - \sigma_P(\underline{a}^k))\overline{v}^k. \end{aligned}$$

Combining the above inequalities, we have:

$$\sigma_P(\underline{a}^k)v^k_{\max} + (1 - \sigma_P(\underline{a}^k))\overline{v}^k \geq P(T')v^k_{\max} + [1 - P(T')]\underline{v}^k$$

and thus,

$$\sigma_P(\underline{a}^k) \geq 1 - \frac{v^k_{\max} - \underline{v}^k}{v^k_{\max} - \overline{v}^k} P(T \setminus T'),$$

as claimed. ∎

We will need the following lemma, the proof of which mimics that of Lemma B in Kajii and Morris (1997).

Lemma 3.4. *Given any simple event $S \subset T$, let*

$$T_i' = S_i \cap \{t_i \in T_i \mid P(S_{-i}|t_i) \geq 1 - \eta\}$$

for $i \in I$, and $T' = \prod_{i \in I} T_i'$. Then,

$$1 - P(T') \leq \gamma(1 - P(S)),$$

where $\gamma = 1 + N(1-\eta)/\eta > 0$.

Proof. Let $B_i = \{t_i \in T_i \mid P(S_{-i}|t_i) \geq 1 - \eta\}$ and $B = \prod_{i \in I} B_i$. By Kajii and Morris (1997, Lemma A), we have

$$P(S \cap (B_i^c \times T_{-i})) \leq \frac{1-\eta}{\eta} P((B_i^c \times T_{-i}) \setminus S)$$

for all $i \in I$. Note then that

$$P(S \setminus B) \leq \sum_{i \in I} P(S \cap (B_i^c \times T_{-i})) \leq N \frac{1-\eta}{\eta} P((B_{i'}^c \times T_{-i'}) \setminus S)$$

for some $i' \in I$. We therefore have

$$\begin{aligned} 1 - P(T') &= P(S \setminus B) + P(T \setminus S) \\ &\leq N \frac{1-\eta}{\eta} P((B_{i'}^c \times T_{-i'}) \setminus S) + P(T \setminus S) \\ &\leq N \frac{1-\eta}{\eta} P(T \setminus S) + P(T \setminus S) \\ &= \gamma P(T \setminus S), \end{aligned}$$

as claimed. ▌

In the following, we let $\sigma^{0,-}, \sigma^{0,+} \in \Sigma$ be such that $\sigma^{0,-}(t) = \underline{a}^0$ and $\sigma^{0,+}(t) = \overline{a}^0$ for all $t \in T$, respectively.

Lemma 3.5. *There exist $c^1, \dots, c^m > 0$ such that for any $P \in \Delta_0(T)$ and any simple event $T^0 \subset T$, there exist $\sigma^{1,-}, \dots, \sigma^{m,-}, \sigma^{1,+}, \dots, \sigma^{m,+} \in \Sigma$ and simple events $T^1, \dots, T^{m-1} \subset T$ with $T^0 \supset T^1 \supset \cdots \supset T^{m-1}$ such that for each $k = 1, \dots, m$,*

($*_k^-$) *for all $i \in I$, $\sigma_i^{k,-}(t_i) = \sigma_i^{k-1,-}(t_i)$ for all $t_i \in T_i \setminus T_i^{k-1}$,*

$$\sigma_i^{k,-}(t_i) = \min BR^i_{v^k}(\sigma_{-i}^{k,-} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \quad \text{for all } t_i \in T_i^{k-1} \tag{3.8}$$

and

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \sigma_{-i}^{k,-}([\underline{a}_{-i}^{k-1}, a_{-i}^*]|t_{-i}) \geq 1 - \eta \quad \text{for all } t_i \in T_i^{k-1}, \tag{3.9}$$

and $\sigma_P^{k,-}(\underline{a}^k) \geq 1 - c^k P(T \setminus T^0)$.

and

$(*_k^+)$ *for all* $i \in I$, $\sigma_i^{k,+}(t) = \sigma_i^{k-1,+}(t)$ *for all* $t_i \in T_i \setminus T_i^{k-1}$,

$$\sigma_i^{k,+}(t_i) = \max BR^i_{v^k}(\sigma_{-i}^{k,+}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t_i) \quad \textit{for all } t_i \in T_i^{k-1} \tag{3.10}$$

and

$$\sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}^{k,+}([a_{-i}^*, \overline{a}_{-i}^{k-1}]|t_{-i}) \geq 1-\eta \quad \textit{for all } t_i \in T_i^{k-1}, \tag{3.11}$$

and $\sigma_P^{k,+}(\overline{a}^k) \geq 1 - c^k P(T \setminus T^0)$.

Proof. Let $\kappa^1, \ldots, \kappa^m > 0$ be as in Lemma 3.3(2) and γ as in Lemma 3.4. Set $c^k = (2\gamma)^{k-1}\kappa^1 \cdots \kappa^k$ for $k = 1, \ldots, m$. Fix any $P \in \Delta_0(T)$ and any simple event $T^0 \subset T$. First, by Lemma 3.3 for (3.2) and (3.3) with $k = 1$, $\xi = \sigma^{0,-}$, $\zeta = \sigma^{0,+}$, and $T' = T^0$, we have $\sigma^{1,-}$ and $\sigma^{1,+}$ that satisfy $(*_1^-)$ and $(*_1^+)$, respectively.

Next, for $k \geq 2$ assume that there exist $T^1, \ldots, T^{k-2}$, $\sigma^{1,-}, \ldots, \sigma^{k-1,-}$, and $\sigma^{1,+}, \ldots, \sigma^{k-1,+}$ that satisfy $(*_1^-), \ldots, (*_{k-1}^-)$ and $(*_1^+), \ldots, (*_{k-1}^+)$, respectively. We can assume that there is no redundancy in $T^1, \ldots, T^{k-2}$ (if $k \geq 3$); i.e., for all $\ell = 2, \ldots, k-1$, if $\underline{a}_i^\ell = \underline{a}_i^{\ell-1}$ and $\overline{a}_i^\ell = \overline{a}_i^{\ell-1}$, then $T_i^{\ell-1} = T_i^{\ell-2}$. Let

$$S_i^{k-1} = T_i^{k-2} \cap \{t_i \in T_i \mid \sigma_i^{k-1,-}(t_i) = \underline{a}_i^{k-1} \text{ and } \sigma_i^{k-1,+}(t_i) = \overline{a}_i^{k-1}\}$$

for each $i \in I$, and $S^{k-1} = \prod_{i\in I} S_i^{k-1}$. Let also

$$T_i^{k-1} = S_i^{k-1} \cap \{t_i \in T_i \,|\, P(S_{-i}^{k-1}|t_i) \geq 1-\eta\} \tag{3.12}$$

for each $i \in I$, and $T^{k-1} = \prod_{i\in I} T_i^{k-1}$. Note that $T^{k-1} \subset T^{k-2}$.

Now consider the maximization problems (3.2) and (3.3) with $\xi = \sigma^{k-1,-}$, $\zeta = \sigma^{k-1,+}$, and $T' = T^{k-1}$. Then by Lemma 3.3, we have $\sigma^{k,-}$ and $\sigma^{k,+}$ that satisfy (3.8) and (3.10), and $\sigma_P^{k,-}(\underline{a}^k) \geq 1 - \kappa^k P(T \setminus T^{k-1})$ and $\sigma_P^{k,+}(\overline{a}^k) \geq 1 - \kappa^k P(T \setminus T^{k-1})$, respectively. Since $\sigma_{-i}^{k,-}([\underline{a}_{-i}^{k-1}, a_{-i}^*]|t_{-i}) = \sigma_{-i}^{k,+}([a_{-i}^*, \overline{a}_{-i}^{k-1}]|t_{-i}) = 1$ for all $t_{-i} \in S_{-i}^{k-1}$ (by the definition of S_{-i}^{k-1} and the maximization problems), it follows that

$$\begin{aligned}&\sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}^{k,-}([\underline{a}_{-i}^{k-1}, a_{-i}^*]|t_{-i}) \\ &\qquad \geq \sum_{t_{-i}\in S_{-i}^{k-1}} P(t_{-i}|t_i)\sigma_{-i}^{k,-}([\underline{a}_{-i}^{k-1}, a_{-i}^*]|t_{-i}) = P(S_{-i}^{k-1}|t_i) \geq 1-\eta\end{aligned}$$

for all $i \in I$ and all $t_i \in T_i^{k-1}$, where the last inequality follows from the definition of T_i^{k-1}, (3.12). This means that $\sigma^{k,-}$ satisfies (3.9). Note that since $\sigma^{k-1,-}$ and $\sigma^{k-1,+}$ are pure strategies, $\sigma_P^{k-1,-}(\underline{a}^{k-1}) = P(\{t \in T \mid \sigma^{k-1,-}(t) = \underline{a}^{k-1}\})$ and $\sigma_P^{k-1,+}(\overline{a}^{k-1}) = P(\{t \in T \mid \sigma^{k-1,+}(t) = \overline{a}^{k-1}\})$. Since, by the no-redundancy assumption, for all $t \in T \setminus T^{k-2}$, there exists an $i \in I$ such that $\sigma_i^{k-1,-}(t_i) < \underline{a}_i^{k-1}$ or $\sigma_i^{k-1,+}(t_i) > \overline{a}_i^{k-1}$, it follows that $S^{k-1} = \{t \in T \mid \sigma^{k-1,-}(t) = \underline{a}^{k-1}$ and $\sigma^{k-1,+}(t) = \overline{a}^{k-1}\}$. Hence,

$$\begin{aligned} P(T \setminus S^{k-1}) \leq P(T \setminus \{t \in T \mid \sigma^{k-1,-}(t) = \underline{a}^{k-1}\}) \\ + P(T \setminus \{t \in T \mid \sigma^{k-1,+}(t) = \overline{a}^{k-1}\}) \\ = \big(1 - \sigma_P^{k-1,-}(\underline{a}^{k-1})\big) + \big(1 - \sigma_P^{k-1,+}(\overline{a}^{k-1})\big) \\ \leq 2c^{k-1} P(T \setminus T^0). \end{aligned} \tag{3.13}$$

Thus, we have

$$\begin{aligned} \sigma_P^{k,-}(\underline{a}^k) \geq 1 - \kappa^k P(T \setminus T^{k-1}) \geq 1 - \kappa^k \times \gamma P(T \setminus S^{k-1}) \\ \geq 1 - \kappa^k \gamma \times 2c^{k-1} P(T \setminus T^0) = 1 - c^k P(T \setminus T^0), \end{aligned}$$

where the first inequality follows from Lemma 3.3, the second inequality follows from Lemma 3.4, and the third inequality follows from (3.13). The same argument applies to $\sigma^{k,+}$. ▮

Lemma 3.6. *For every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for any ε-elaboration $(\mathbf{u}, P)$ with $\varepsilon \leq \bar{\varepsilon}$, there exist $\sigma^-, \sigma^+ \in \Sigma$ and simple events $T^1, \ldots, T^{m-1} \subset T$ with $T^{\mathbf{g}} = T^0 \supset T^1 \supset \cdots \supset T^{m-1} \supset T^m = \emptyset$ such that*

($*^-$) *for all $i \in I$, $\sigma_i^-(t_i) = \underline{a}_i^0$ for all $t_i \in T_i \setminus T_i^{g_i}$,*

$$\sigma_i^-(t_i) = \min BR^i_{v^k}(\sigma_{-i}^- | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \quad \textit{for all } t_i \in T_i^{k-1} \setminus T_i^k \tag{3.14}$$

and

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \sigma_{-i}^-([\underline{a}_{-i}^{k-1}, a_{-i}^*] | t_{-i}) \geq 1 - \eta \quad \textit{for all } t_i \in T_i^{k-1} \tag{3.15}$$

for each $k = 1, \ldots, m$, and $\sigma_P^-(a^*) \geq 1 - \delta$,

and

($*^+$) *for all $i \in I$, $\sigma_i^+(t_i) = \overline{a}_i^0$ for all $t_i \in T_i \setminus T_i^{g_i}$,*

$$\sigma_i^+(t_i) = \max BR^i_{v^k}(\sigma_{-i}^+ | [\overline{a}_i^k, \overline{a}_i^{k-1}])(t_i) \quad \textit{for all } t_i \in T_i^{k-1} \setminus T_i^k \tag{3.16}$$

and

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \sigma_{-i}^+([a_{-i}^*, \overline{a}_{-i}^{k-1}] | t_{-i}) \geq 1 - \eta \quad \textit{for all } t_i \in T_i^{k-1} \tag{3.17}$$

for each $k = 1, \ldots, m$, and $\sigma_P^+(a^*) \geq 1 - \delta$.

Proof. Take $c^1, \dots, c^m > 0$ as in Lemma 3.5. Given any $\delta > 0$, let $\bar{\varepsilon} = \delta / c^m$. Fix any ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$ with $\varepsilon \leq \bar{\varepsilon}$, and let $T^0 = T^{\mathbf{g}}$. Then take $\sigma^{0,-}, \dots, \sigma^{m,-}$ and $\sigma^{0,+}, \dots, \sigma^{m,+}$ that satisfy $(*_k^-)$ and $(*_k^+)$ for $k = 1, \dots, m$, respectively, with $T^1, \dots, T^{m-1} \subset T$. Set $\sigma^- = \sigma^{m,-}$ and $\sigma^+ = \sigma^{m,+}$. We only verify that σ^- satisfies $(*^-)$.

By construction, we have (3.15) for each $k = 1, \dots, m$. We also have $\sigma_P^-(a^*) \geq 1 - \delta$ by $(*_m^-)$.

Consider any $k = 1, \dots, m-1$. Note from (3.12) that

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \sigma_{-i}^{k,-}(\underline{a}_{-i}^k | t_{-i}) \geq 1 - \eta,$$

for all $t_i \in T_i^k$. It follows by the choice of η that for all $i \in I$,

$$\sigma_i^{k,-}(t_i) = \min BR_{v^k}^i(\sigma_{-i}^{k,-} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) = \underline{a}_i^k$$

for all $t_i \in T_i^k$ $(\subset T_i^{k-1})$, so that $\sigma^{k,-}(t) = \underline{a}^k$ and hence $\sigma^-(t) \in [\underline{a}^k, a^*]$ for all $t \in T^k$. Note also that $\sigma^-(t) = \sigma^{k,-}(t)$ for all $t \in T \setminus T^k$. Since $v^k(a) = v^k(a')$ for all $a, a' \in [\underline{a}^k, \overline{a}^k]$, it follows that for all $i \in I$ and all $t_i \in T_i^{k-1}$, $BR_{v^k}^i(\sigma_{-i}^- | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) = BR_{v^k}^i(\sigma_{-i}^{k,-} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i)$. Therefore, for all $i \in I$ and all $t_i \in T_i^{k-1} \setminus T_i^k$,

$$\begin{aligned}\sigma_{-i}^-(t_i) = \sigma_{-i}^{k,-}(t_i) &= \min BR_{v^k}^i(\sigma_{-i}^{k,-} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \\ &= \min BR_{v^k}^i(\sigma_{-i}^- | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i),\end{aligned}$$

which means that σ^- satisfies (3.14). ▮

Proof of Theorem 3.1. Suppose that v^k's are monotone potential functions for a^* relative to $B_{2\eta}([\underline{a}^{k-1}, \overline{a}^{k-1}])$. Let $\delta > 0$ be given Take $\bar{\varepsilon}$ as in Lemma 3.6. Fix any ε-elaboration $(\mathbf{u}, P)$ with $\varepsilon \leq \bar{\varepsilon}$, and take σ^-, σ^+, and $T^0, T^1, \dots, T^m$ that satisfy $(*^-)$ and $(*^+)$, respectively. Let $\tilde{\Sigma} = \{\sigma \in \Sigma \mid \sigma^- \precsim \sigma \precsim \sigma^+\}$. We will show that $\tilde{\beta}(\sigma) = \beta(\sigma) \cap \tilde{\Sigma}$ is nonempty for any $\sigma \in \tilde{\Sigma}$, where β is the best response correspondence of $(\mathbf{u}, P)$ defined in (3.1). Then, since $\tilde{\Sigma}$ is convex and compact, it follows from Kakutani's fixed point theorem that the nonempty-, convex-, and compact-valued upper semi-continuous correspondence β has a fixed point $\sigma^* \in \tilde{\beta}(\sigma^*) \subset \tilde{\Sigma}$, which is a Bayesian Nash equilibrium of $(\mathbf{u}, P)$ and satisfies $\sigma^- \precsim \sigma^* \precsim \sigma^+$. Since both σ^- and σ^+ satisfy $\sigma_P^-(a^*) \geq 1 - \delta$ and $\sigma_P^+(a^*) \geq 1 - \delta$, respectively, σ^* satisfies $\sigma_P^*(a^*) \geq 1 - 2\delta$.

Take any $\sigma \in \tilde{\Sigma}$. For $t_i \in T_i \setminus T_i^0$, $BR_{g_i}^i(\sigma)(t_i) \subset [\sigma_i^-(t_i), \sigma_i^+(t_i)]$ holds. Consider any $k = 1, \dots, m$. Note that

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \sigma_{-i}([\underline{a}_{-i}^{k-1}, \overline{a}_{-i}^{k-1}] | t_{-i}) \geq 1 - 2\eta$$

for all $i \in I$ and all $t_i \in T_i^{k-1}$.

Suppose first that $g_i|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$ are supermodular for all $i \in I$. Then, for all $i \in I$,

$$\begin{aligned}\min BR^i_{v^k}(\sigma^-_{-i}|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i) &\le \max BR^i_{g_i}(\sigma^-_{-i}|[\underline{a}_i^{k-1},\overline{a}_i^k])(t_i)\\ &\le \max BR^i_{g_i}(\sigma_{-i}|[\underline{a}_i^{k-1},\overline{a}_i^k])(t_i)\end{aligned}$$

for all $t_i \in T_i^{k-1} \setminus T_i^k$, where the second inequality follows from the assumption that v^k is a monotone potential function relative to $B_{2\eta}([\underline{a}^{k-1},\overline{a}^{k-1}])$, and the third inequality follows from the supermodularity of $g_i|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$. Similarly, for all $i \in I$,

$$\begin{aligned}\max BR^i_{v^k}(\sigma^+_{-i}|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i) &\ge \min BR^i_{g_i}(\sigma^+_{-i}|[\underline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\\ &\ge \min BR^i_{g_i}(\sigma_{-i}|[\underline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\end{aligned}$$

for all $t_i \in T_i^{k-1} \setminus T_i^k$.

Suppose next that $v^k|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$ are supermodular for all $i \in I$. Then, for all $i \in I$,

$$\begin{aligned}\min BR^i_{v^k}(\sigma^-|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i) &\le \min BR^i_{v^k}(\sigma|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i)\\ &\le \max BR^i_{g_i}(\sigma|[\underline{a}_i^{k-1},\overline{a}_i^k])(t_i)\end{aligned}$$

for all $t_i \in T_i^{k-1} \setminus T_i^k$, where the second inequality follows from the supermodularity of $v^k|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$, and the third inequality follows from the assumption that v^k is a monotone potential function relative to $B_{2\eta}([\underline{a}^{k-1},\overline{a}^{k-1}])$. Similarly, for all $i \in I$,

$$\begin{aligned}\max BR^i_{v^k}(\sigma^+|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i) &\ge \max BR^i_{v^k}(\sigma|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\\ &\ge \min BR^i_{g_i}(\sigma|[\underline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\end{aligned}$$

for all $t_i \in T_i^{k-1} \setminus T_i^k$.

Therefore, in each case, we have for all $t_i \in T_i^{k-1} \setminus T_i^k$,

$$\begin{aligned}\max BR^i_{g_i}(\sigma|[\underline{a}_i^{k-1},\overline{a}_i^k])(t_i) &\ge \min BR^i_{v^k}(\sigma^-|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i),\\ \min BR^i_{g_i}(\sigma|[\underline{a}_i^k,\overline{a}_i^{k-1}])(t_i) &\le \max BR^i_{v^k}(\sigma^+|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i).\end{aligned}$$

Since

$$\sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}([\underline{a}_{-i}^{k-1},\overline{a}_{-i}^{k-1}]|t_{-i}) \ge 1-2\eta$$

for all $i \in I$ and all $t_i \in T_i^{k-1}$ and hence

$$BR^i_{g_i}(\sigma)(t_i) \cap [\underline{a}_i^{k-1},\overline{a}_i^{k-1}] \neq \emptyset$$

by the choice of η, it follows that

$$\begin{aligned}&BR^i_{g_i}(\sigma)(t_i)\\ &\qquad \cap \left[\min BR^i_{v^k}(\sigma^-|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i), \max BR^i_{v^k}(\sigma^+|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\right] \neq \emptyset.\end{aligned}$$

This implies the nonemptiness of $\tilde{\beta}(\sigma)$. ▮

By Proposition 2.4, we immediately have the following.

Corollary 3.7. *If a^* is an iterated strict* **p***-dominant equilibrium of* **g** *with* $\sum_{i \in I} p_i < 1$*, then a^* is robust to all elaborations in* **g**.

3.3 Uniqueness of Robust Equilibrium and Iterated p-Dominance

Our first theorem, together with our results provided in Subsection 2.3, shows that an iterated **p**-dominant equilibrium with low **p** is actually robust to incomplete information. In this subsection, we prove a stronger result: when an iterated strict **p**-dominant equilibrium with low **p** exists, it is the *unique* robust equilibrium.

Proposition 3.8. *An iterated strict* **p***-dominant equilibrium of* **g** *with* $\sum_{i \in I} p_i < 1$ *is the unique robust equilibrium in* **g**.

This proposition is a corollary to the following lemma.

Lemma 3.9. *Suppose that a^* is an iterated strict* **p***-dominant equilibrium of* **g** *with* $\sum_{i \in I} p_i \leq 1$*. Then, for all $\varepsilon > 0$, there exists an ε-elaboration where the strategy profile σ^* such that $\sigma^*(t) = a^*$ for all $t \in T$ is the unique Bayesian Nash equilibrium.*

Proof. Let a^* be an iterated strict **p**-dominant equilibrium with $\sum_{i \in I} p_i \leq 1$ and $(S^0, \ldots, S^m)$ an associated sequence. Let $q_i = (p_i / \sum_{j \in I} p_j) \geq p_i$ for each $i \in I$ (we can assume without loss of generality that $p_i > 0$ for all i). Note that $\sum_{i \in I} q_i = 1$. Now let $T_i = \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ for each $i \in I$. For each $\varepsilon > 0$, we construct an ε-elaboration $(\mathbf{u}, P^\varepsilon)$ as follows. Define $P^\varepsilon \in \Delta_0(T)$ by

$$P^\varepsilon(t_1, \ldots, t_N) = \begin{cases} \varepsilon(1-\varepsilon)^\tau q_i & \text{if } t_i = \tau + 1 \text{ and } t_j = \tau \text{ for all } j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

and $u_i \colon A \times T \to \mathbb{R}$ for each $i \in I$ by

$$u_i(a; t) = \begin{cases} g_i(a) & \text{if } t_i \neq 0, \\ 1 & \text{if } t_i = 0 \text{ and } a_i = a_i^*, \\ 0 & \text{if } t_i = 0 \text{ and } a_i \neq a_i^*. \end{cases}$$

Fix any $\varepsilon > 0$, and let us now study the set of Bayesian Nash equilibria of $(\mathbf{u}, P^\varepsilon)$.

Consider the sequence of modified incomplete information games $\{(\mathbf{u}|_{S^k}, P^\varepsilon)\}_{k=0}^{m-1}$ where in $(\mathbf{u}|_{S^k}, P^\varepsilon)$, the set of actions available to player $i \in I$ is S_i^k and player i's payoff function $u_i|_{S_i^k} \colon S^k \times T \to \mathbb{R}$ is given by the restriction of u_i to $S^k \times T$. We want to show that any Bayesian Nash equilibrium of $(\mathbf{u}, P^\varepsilon)$, σ^*, satisfies $\sigma^*(t) = a^*$ for all $t \in T$.

First note that if σ^* is a Bayesian Nash equilibrium of $(\mathbf{u}, P^\varepsilon)$ such that for $k = 0, \ldots, m-1$, $\mathrm{supp}(\sigma^*(t)) \subset S^k$ for all $t \in T$, then σ^* is an equilibrium of $(\mathbf{u}|_{S^k}, P^\varepsilon)$. It is therefore sufficient to show that for each $k = 0, \ldots, m-1$, any Bayesian Nash equilibrium σ^* of $(\mathbf{u}|_{S^{k-1}}, P^\varepsilon)$ is such that $\mathrm{supp}(\sigma^*(t)) \subset S^k$ for all $t \in T$. We proceed by induction.

Let σ^* be a Bayesian Nash equilibrium of $(\mathbf{u}|_{S^{k-1}}, P^\varepsilon)$. We show that for all $i \in I$, $\sum_{a_i \in S_i^k} \sigma_i^*(a_i|\tau) = 1$ for all $\tau \geq 0$. By construction, for all $i \in I$, $\sum_{a_i \in S_i^k} \sigma_i^*(a_i|0) = 1$. Our inductive hypothesis is that for all $i \in I$, $\sum_{a_i \in S_i^k} \sigma_i^*(a_i|\tau) = 1$. Take any $i \in I$ and consider the type $t_i = \tau + 1$. By construction of the type space, we have

$$\begin{aligned} P^\varepsilon((t_1, \ldots, t_{i-1}, t_{i+1}, \ldots t_N)|\tau + 1) &= \frac{\varepsilon(1-\varepsilon)^\tau q_i}{\varepsilon(1-\varepsilon)^\tau q_i + \sum_{j \neq i} \varepsilon(1-\varepsilon)^{\tau+1} q_j} \\ &> q_i \geq p_i \end{aligned}$$

if $t_j = \tau$ for all $j \neq i$. Thus by the inductive hypothesis, each agent i assigns a probability strictly above p_i to the other players playing actions in S_{-i}^k. But since S^k is a strict $\mathbf{p}$-best response set of $\mathbf{g}|_{S^{k-1}}$ and since $\tau + 1 \in T_i^{u_i}$, this implies that $\sum_{a_i \in S_i^k} \sigma_i^*(a_i|\tau + 1) = 1$. Thus our inductive hypothesis holds for $\tau + 1$. ∎

Proof of Proposition 3.8. If a^* is an iterated strict $\mathbf{p}$-dominant equilibrium with $\sum_{i \in I} p_i < 1$, then it is an iterated MP-maximizer with supermodular monotone potential functions by Proposition 2.4 and hence is robust to all elaborations by Theorem 3.1. But by Lemma 3.9, no action profile other than a^* is played in any robust equilibrium. ∎

4 Stability under Perfect Foresight Dynamics

4.1 Perfect Foresight Paths and Stability Concepts

Given the game $\mathbf{g}$, we consider the following dynamic societal game. Society consists of N continua of agents, one for each role in $\mathbf{g}$. In each population, agents are identical and anonymous. At each point in time, one agent is selected randomly from each population and matched to form an N-tuple and play $\mathbf{g}$. Agents cannot switch actions at every point in time. Instead, every agent must make a commitment to a particular action for a random time interval. Time instants at which each agent can switch actions follow a Poisson process with the arrival rate $\lambda > 0$. The processes are independent across agents. We choose without loss of generality the unit of time in such a way that $\lambda = 1$.

The action distribution in population $i \in I$ at time $t \in \mathbb{R}_+$ is denoted by $\phi_i(t) = (\phi_{ih}(t))_{h \in A_i} \in \Delta(A_i)$, where $\phi_{ih}(t)$ is the fraction of agents who are committing to action $h \in A_i$ at time t. Let $\phi(t) = (\phi_i(t))_{i \in I} \in \prod_{i \in I} \Delta(A_i)$ and $\phi_{-i}(t) = (\phi_j(t))_{j \neq i} \in \prod_{j \neq i} \Delta(A_j)$. Due to the assumption that the switching times follow independent Poisson processes with arrival rate $\lambda = 1$, $\phi_{ih}(\cdot)$ is Lipschitz continuous with Lipschitz constant 1, which implies in particular that it is differentiable at almost all $t \geq 0$.

Definition 4.1. A path $\phi \colon \mathbb{R}_+ \to \prod_{i \in I} \Delta(A_i)$ is said to be *feasible* if it is Lipschitz continuous, and for all $i \in I$ and almost all $t \geq 0$, there exists $\alpha_i(t) \in \Delta(A_i)$ such that

$$\dot{\phi}_i(t) = \alpha_i(t) - \phi_i(t). \tag{4.1}$$

Denote by Φ^i the set of feasible paths for population i, and let $\Phi = \prod_{i \in I} \Phi^i$ and $\Phi^{-i} = \prod_{j \neq i} \Phi^j$. For $x \in \prod_{i \in I} \Delta(A_i)$, the set of feasible paths starting from x is denoted by $\Phi_x = \prod_i \Phi^i_x$. We endow Φ_x with the topology of uniform convergence on compact intervals.[10] The set Φ_x is convex, and compact with respect to this topology.

We define $\phi_i \precsim \psi_i$ for $\phi_i, \psi_i \in \Phi^i$ by $\phi_i(t) \precsim \psi_i(t)$ for all $t \geq 0$; $\phi \precsim \psi$ for $\phi, \psi \in \Phi$ by $\phi_i \precsim \psi_i$ for all $i \in I$; and $\phi_{-i} \precsim \psi_{-i}$ for $\phi_{-i}, \psi_{-i} \in \Phi^{-i}$ by $\phi_j \precsim \psi_j$ for all $j \neq i$. Note that if $\phi(0) \precsim \psi(0)$ and $\dot{\phi}(t) + \phi(t) \precsim \dot{\psi}(t) + \psi(t)$ for almost all $t \geq 0$, then $\phi \precsim \psi$.

A revising agent in population i anticipates the future evolution of the action distribution, and commits to an action that maximizes his expected discounted payoff. The expected discounted payoff of committing to action $h \in A_i$ at time t with a given anticipated path $\phi_{-i} \in \Phi^{-i}$ is given by

$$\begin{aligned} V_{ih}(\phi_{-i})(t) &= (1+\theta) \int_0^\infty \int_t^{t+s} e^{-\theta(z-t)} g_i(h, \phi_{-i}(z))\, dz\, e^{-s} ds \\ &= (1+\theta) \int_t^\infty e^{-(1+\theta)(s-t)} g_i(h, \phi_{-i}(s))\, ds, \end{aligned}$$

where $\theta > 0$ is a common discount rate. Following Matsui and Matsuyama (1995), we view $\theta/\lambda = \theta$ as the *degree of friction*.

Let $BR^i_{g_i} \colon \Phi^{-i} \times \mathbb{R}_+ \to A_i$ be defined for each i by

$$BR^i_{g_i}(\phi_{-i})(t) = \arg\max\{V_{ih}(\phi_{-i})(t) \mid h \in A_i\}.$$

Note that for each $i \in I$, the correspondence $BR^i_{g_i}$ is upper semi-continuous since V_i is continuous.

[10] This topology is metrizable by the metric d_r defined by

$$d_r(\phi, \phi') = \sup_{t \geq 0} e^{-rt} |\phi(t) - \phi'(t)|$$

for $r > 0$.

Definition 4.2. A feasible path ϕ is said to be a *perfect foresight path* in $\mathbf{g}$ if for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \in BR^i_{g_i}(\phi_{-i})(t).$$

Let $\beta^i_x \colon \Phi^{-i}_x \to \Phi^i_x$ be defined by

$$\beta^i_x(\phi_{-i}) = \{\psi_i \in \Phi^i_x \mid \dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \in BR^i_{g_i}(\phi_{-i})(t) \ \text{ a.e.}\}, \quad (4.2)$$

and $\beta_x \colon \Phi_x \to \Phi_x$ be given by $\beta_x(\phi) = \prod_i \beta^i_x(\phi_{-i})$. A perfect foresight path ϕ with $\phi(0) = x$ is a fixed point of $\beta_x \colon \Phi_x \to \Phi_x$, i.e., $\phi \in \beta_x(\phi)$. Verify that β_x is nonempty-, convex-, and compact-valued and upper semi-continuous (see, e.g., OTH (2003, Remark 2.1)). The existence of perfect foresight paths then follows from Kakutani's fixed point theorem.

Following Matsui and Matsuyama (1995) and OTH (2003), we employ the following stability concepts.

Definition 4.3. (a) $a^* \in A$ is *globally accessible* in $\mathbf{g}$ if for any $x \in \prod_i \Delta(A_i)$, there exists a perfect foresight path from x that converges to a^*.

(b) $a^* \in A$ is *absorbing* in $\mathbf{g}$ if there exists $\varepsilon > 0$ such that any perfect foresight path from any $x \in B_\varepsilon(a^*)$ converges to a^*.

(c) $a^* \in A$ is *linearly absorbing* in $\mathbf{g}$ if there exists $\varepsilon > 0$ such that for any $x \in B_\varepsilon(a^*)$, the linear path to a^* is a unique perfect foresight path from x.

Given $\theta > 0$, we write for any function $f \colon A \to \mathbb{R}$

$$BR^i_f(\phi_{-i}|S_i)(t) = (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} f(h, \phi_{-i}(s))\, ds, ,$$

where $S_i \subset A_i$, $\phi_{-i} \in \Phi_{-i}$, and $t \geq 0$. Note that this can be written as

$$BR^i_f(\phi_{-i}|S_i)(t) = br^i_f(\pi^{t_i}_i(\phi_{-i})|S_i)$$

where $\pi^{t_i}_i(\phi_{-i}) \in \Delta(A_{-i})$ is given by

$$\pi^{t_i}_i(\phi_{-i})(a_{-i}) = (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} \left(\prod\nolimits_{j\neq i} \phi_{ja_j}(s)\right) ds.$$

Thus, if $f|_{S_i \times A_{-i}}$ is supermodular, then whenever $\phi_{-i} \precsim \phi'_{-i}$, we have

$$\min BR^i_f(\phi_{-i}|S_i)(t) \leq \min BR^i_f(\phi'_{-i}|S_i)(t),$$
$$\max BR^i_f(\phi_{-i}|S_i)(t) \leq \max BR^i_f(\phi'_{-i}|S_i)(t).$$

4.2 Global Accessibility of Iterated MP-Maximizer

In this subsection, we move to our second main result. We show that under the same monotonicity conditions as in the incomplete information case, an iterated MP-maximizer is selected by the perfect foresight dynamics approach.

In addition, as will become clear, by exploiting the similarity between the mathematical structures of incomplete information elaborations and perfect foresight dynamics, we provide a proof of this result that is strongly related to the proof of our first main result.

Theorem 4.1. *Suppose that* $\mathbf{g}$ *has an iterated MP-maximizer* a^* *with associated intervals* $(S^k)_{k=0}^m$ *and monotone potential functions* $(v^k)_{k=1}^m$*. If for each* $k = 1, \ldots, m$*,* $g_i|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$ *or* $v^k|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$*, then there exists* $\bar{\theta} > 0$ *such that* a^* *is globally accessible in* $\mathbf{g}$ *for all* $\theta \in (0, \bar{\theta})$*.*

Due to Lemma 2.2, we immediately have the following.

Corollary 4.2. *Suppose that* $\mathbf{g}$ *has an iterated strict MP-maximizer* a^* *with associated intervals* $(S^k)_{k=0}^m$ *and strict monotone potential functions* $(v^k)_{k=1}^m$*. If for each* $k = 1, \ldots, m$*,* $g_i|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$ *or* $v^k|_{S^{k-1}}$ *is supermodular, then there exists* $\bar{\theta} > 0$ *such that* a^* *is globally accessible in* $\mathbf{g}$ *for all* $\theta \in (0, \bar{\theta})$*.*

Suppose that a^* is an iterated MP-maximizer of $\mathbf{g}$ with monotone potential functions $(v^k)_{k=1}^m$ that are relative to $B_\eta(S^{k-1})$ respectively for $k = 1, \ldots, m$, where $\eta > 0$ is sufficiently small so that for all $i \in I$ and all $k = 1, \ldots, m$,

$$br_{g_i}^i(\pi_i) \cap S_i^k \neq \emptyset,$$

and therefore,

$$br_{g_i}^i(\pi_i | S_i^k) \subset br_{g_i}^i(\pi_i)$$

hold for $\pi_i \in B_\eta(S_{-i}^k)$ (see Lemma 2.3). For each $k = 0, 1, \ldots, m$ and $i \in I$, write $S_i^k = [\underline{a}_i^k, \overline{a}_i^k]$, where $0 = \underline{a}_i^0 \leq \underline{a}_i^1 \leq \cdots \leq \underline{a}_i^m = a_i^* = \overline{a}_i^m \leq \cdots \leq \overline{a}_i^1 \leq \overline{a}_i^0 = n_i$.

For each $k = 1, \ldots, m$, define $J_\theta^k \colon \Phi \to \mathbb{R}$ to be

$$J_\theta^k(\phi) = \int_0^\infty \theta e^{-\theta t} v^k(\phi(t))\, dt,$$

and for any $x \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^{k-1}])$ and $y \in \prod_i \Delta([\overline{a}_i^{k-1}, \overline{a}_i^0])$, let

$$\begin{aligned}\Phi_x^{k,-} = \{\phi \in \Phi \mid \phi(0) = x,& \\ &\dot{\phi}_i(t) + \phi_i(t) \in \Delta([\underline{a}_i^{k-1}, \underline{a}_i^k]) \ \forall i \in I, \ \text{a.a.}\, t \geq 0\},\\ \Phi_y^{k,+} = \{\phi \in \Phi \mid \phi(0) = y,& \\ &\dot{\phi}_i(t) + \phi_i(t) \in \Delta([\overline{a}_i^k, \overline{a}_i^{k-1}]) \ \forall i \in I, \ \text{a.a.}\, t \geq 0\}.\end{aligned}$$

Consider the maximization problems:

$$\max J_\theta^k(\phi) \quad \text{s.t. } \phi \in \Phi_x^{k,-}, \tag{4.3}$$

$$\max J_\theta^k(\phi) \quad \text{s.t. } \phi \in \Phi_y^{k,+}. \tag{4.4}$$

Since J_θ^k is continuous, and $\Phi_x^{k,-}$ and $\Phi_y^{k,+}$ are compact, the above maximization problems admit solutions.

Lemma 4.3. (1) *For each $k = 1, \dots, m$, and for any $\theta > 0$ and any $x \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^k])$ and $y \in \prod_i \Delta([\overline{a}_i^k, \overline{a}_i^0])$: there exists a solution to the maximization problem* (4.3), *$\phi^{k,-}$, such that*

$$\dot{\phi}_i^{k,-}(t) = \min BR^i_{v^k}(\phi_{-i}^{k,-} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t) - \phi_i^{k,-}(t) \tag{4.5}$$

for all $i \in I$ and almost all $t \geq 0$; there exists a solution to the maximization problem (4.4), *$\phi^{k,+}$, such that*

$$\dot{\phi}_i^{k,+}(t) = \min BR^i_{v^k}(\phi_{-i}^{k,+} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t) - \phi_i^{k,+}(t) \tag{4.6}$$

for all $i \in I$ and almost all $t \geq 0$.

(2) *For each $k = 1, \dots, m$, there exists $\bar{\theta}^k > 0$ such that for any $\theta \in (0, \bar{\theta}^k)$ and any $x \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^k])$ ($y \in \prod_i \Delta([\overline{a}_i^k, \overline{a}_i^0])$, resp.), any solution to the maximization problem* (4.3) *(*(4.4)*, resp.) converges to $\underline{a}^k$ ($\overline{a}^k$, resp.).*

Proof. (1) We only show the existence of a solution that satisfies (4.5) (the existence of a solution that satisfies (4.6) is proved similarly). First note that for each $i \in I$,

$$\begin{aligned}(1+\theta)e^{-\theta t} v^k(\phi(t)) &= \sum_{h \in A_i} e^t \phi_{ih}(t) \frac{d}{dt}\left(-e^{-(1+\theta)t} V_{ih}^k(\phi_{-i})(t)\right) \\ &= \frac{d}{dt}\left(-e^{-\theta t} \textstyle\sum_{h \in A_i} \phi_{ih}(t) V_{ih}^k(\phi_{-i})(t)\right) \\ &\qquad + e^{-\theta t} \textstyle\sum_{h \in A_i} \left(\dot{\phi}_{ih}(t) + \phi_{ih}(t)\right) V_{ih}^k(\phi_{-i})(t)\end{aligned}$$

for almost all $t \geq 0$, where

$$V_{ih}^k(\phi_{-i})(t) = (1+\theta) \int_t^\infty e^{-(1+\theta)(s-t)} v^k(h, \phi_{-i}(s))\, ds.$$

Therefore, any solution to (4.3), ϕ^k, satisfies

$$\dot{\phi}_{ih}^k(t) > -\phi_{ih}^k \Rightarrow h \in BR^i_{v^k}(\phi_{-i}^k | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t) \tag{4.7}$$

for all $i \in I$ and almost all $t \geq 0$. It then follows from Lemma A.1.3 in OTH (2003) that there exists a feasible path $\phi_i^{k,-}$ that satisfies (4.5).

(2) We show that there exists $\bar{\theta}^k > 0$ such that for any $\theta \in (0, \bar{\theta}^k)$, any solution to (4.3) ((4.4), resp.) approaches arbitrarily close to $\underline{a}^k$ ($\overline{a}^k$, resp.). Here, $\bar{\theta}^k$ can be taken independently of x and y. Then, by following the proofs of Lemmas 3 and 4 in HS (1999) (see also Theorem 4.1 in HS (2002)) for the potential game $v|_{[\underline{a}^0, \underline{a}^k]}$, one can show that once any feasible path that satisfies (4.7) gets close enough to the potential maximizer $\underline{a}^k$, it must converge to $\underline{a}^k$. A dual argument applies to solutions to (4.4)

Let $v^k_{\max} = v^k(\underline{a}^k) = v^k(\overline{a}^k)$, $\overline{v}^k = \max_{a \in A \setminus [\underline{a}^k, \overline{a}^k]} v^k(a)$, and $\underline{v}^k = \min_{a \in A} v^k(a)$. Note that $v^k_{\max} > \overline{v}^k \geq \underline{v}^k$. Let ϕ be any solution to (4.3), and ψ the linear path from x to $\underline{a}^k$: i.e., for all $i \in I$ and $t \geq 0$, $\psi_{ih}(t) = 1 - (1 - x_{ih})e^{-t}$ if $h = \underline{a}^k_i$ and $\psi_{ih}(t) = x_{ih}e^{-t}$ otherwise. Denote $\phi(a|t) = \prod_{i \in I} \phi_{ia_i}(t)$ and $\psi(a|t) = \prod_{i \in I} \psi_{ia_i}(t)$. We first have

$$
\begin{aligned}
J^k_\theta(\phi) &\geq J^k_\theta(\psi) \\
&= \int_0^\infty \theta e^{-\theta t} \psi(\underline{a}^k|t)\, dt\, v^k_{\max} + \sum_{a \neq \underline{a}^k} \int_0^\infty \theta e^{-\theta t} \psi(a|t)\, dt\, v^k(a) \\
&\geq \int_0^\infty \theta e^{-\theta t} \psi(\underline{a}^k|t)\, dt\, v^k_{\max} + \left[1 - \int_0^\infty \theta e^{-\theta t} \psi(\underline{a}^k|t)\, dt\right] \underline{v}^k \\
&= v^k_{\max} - \left[1 - \int_0^\infty \theta e^{-\theta t} \prod_{i \in I} \left\{1 - \left(1 - x_{i\underline{a}^k_i}\right) e^{-t}\right\} dt\right] \left(v^k_{\max} - \underline{v}^k\right) \\
&\geq v^k_{\max} - \left[1 - \int_0^\infty \theta e^{-\theta t} \left(1 - e^{-t}\right)^N dt\right] \left(v^k_{\max} - \underline{v}^k\right).
\end{aligned}
$$

We also have

$$
\begin{aligned}
J^k_\theta(\phi) &= \int_0^\infty \theta e^{-\theta t} \phi(\underline{a}^k|t)\, dt\, v^k_{\max} + \sum_{a \neq \underline{a}^k} \int_0^\infty \theta e^{-\theta t} \phi(a|t)\, dt\, v^k(a) \\
&\leq \int_0^\infty \theta e^{-\theta t} \phi(\underline{a}^k|t)\, dt\, v^k_{\max} + \left[1 - \int_0^\infty \theta e^{-\theta t} \phi(\underline{a}^k|t)\, dt\right] \overline{v}^k.
\end{aligned}
$$

Combining these inequalities, we have

$$
\int_0^\infty \theta e^{-\theta t} \phi(\underline{a}^k|t)\, dt \geq 1 - \frac{v^k_{\max} - \underline{v}^k}{v^k_{\max} - \overline{v}^k} \left[1 - \int_0^\infty \theta e^{-\theta t} \left(1 - e^{-t}\right)^N dt\right].
$$

The integral in the right hand side converges to one as θ goes to zero. Therefore, given $\delta > 0$ we have $\overline{\theta}^k > 0$ such that for all $\theta \in (0, \overline{\theta}^k)$,

$$
\int_0^\infty \theta e^{-\theta t} \phi(\underline{a}^k|t)\, dt \geq 1 - \delta,
$$

which implies that there exists $t \geq 0$ such that $\phi(\underline{a}^k|t) \geq 1 - \delta$, and hence, $\phi_{i\underline{a}^k_i}(t) \geq 1 - \delta$ for all $i \in I$. ▮

In the following, we set $T^0 = 0$, and $\phi^{0,-}$ and $\phi^{0,+}$ to be such that $\phi^{0,-}(t) = \underline{a}^0$ and $\phi^{0,+}(t) = \overline{a}^0$ for all $t \geq 0$, respectively.

Lemma 4.4. *There exists $\bar{\theta} > 0$ such that for any $\theta \in (0, \bar{\theta})$, there exist $T^1, \dots, T^{m-1}$ with $T^1 \leq \cdots \leq T^{m-1} < \infty$ and feasible paths $\phi^{1,-}, \dots, \phi^{m,-}$ and $\phi^{1,+}, \dots, \phi^{m,+}$ such that for each $k = 1, \dots, m$,*

$(*_k^-)$ $\phi^{k,-}(t) = \phi^{k-1,-}(t)$ *for all* $t \in [0, T^{k-1}]$, $\phi^{k,-}(T^{k-1}) \in B_\eta(\underline{a}^{k-1})$,

$$\dot{\phi}_i^{k,-}(t) = \min BR^i_{v^k}(\phi_{-i}^{k,-} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t) - \phi_i^{k,-}(t)$$

for all $i \in I$ *and almost all* $t \in [T^{k-1}, \infty)$, *and* $\lim_{t \to \infty} \phi^{k,-}(t) = \underline{a}^k$,

and

$(*_k^+)$ $\phi^{k,+}(t) = \phi^{k-1,+}(t)$ *for all* $t \in [0, T^{k-1}]$, $\phi^{k,+}(T^{k-1}) \in B_\eta(\overline{a}^{k-1})$,

$$\dot{\phi}_i^{k,+}(t) = \max BR^i_{v^k}(\phi_{-i}^{k,+} | [\overline{a}_i^k, \overline{a}_i^{k-1}])(t) - \phi_i^{k,+}(t)$$

for all $i \in I$ *and almost all* $t \in [T^{k-1}, \infty)$, *and* $\lim_{t \to \infty} \phi^{k,+}(t) = \overline{a}^k$.

Proof. Take $\bar{\theta}^1, \dots, \bar{\theta}^m$ as in Lemma 4.3, and set $\bar{\theta} = \min\{\bar{\theta}^1, \dots, \bar{\theta}^m\}$. Fix any $\theta \in (0, \bar{\theta})$. First, by Lemma 4.3 for (4.3) and (4.4) with $k = 1$, $x = \underline{a}^0$, and $y = \overline{a}^0$, we have feasible paths $\phi^{1,-}$ and $\phi^{1,+}$ that satisfy $(*_1^-)$ and $(*_1^+)$, respectively.

Next, for $k \geq 2$ assume that there exist $T^0, \dots, T^{k-2}$, $\phi^{1,-}, \dots, \phi^{k-1,-}$, and $\phi^{1,+}, \dots, \phi^{k-1,+}$ that satisfy $(*_1^-), \dots, (*_{k-1}^-)$ and $(*_1^+), \dots, (*_{k-1}^+)$. Let $T^{k-1} \geq T^{k-2}$ be such that $\phi^{k-1,-}(t) \in B_\eta(\underline{a}^{k-1})$ and $\phi^{k-1,+}(t) \in B_\eta(\overline{a}^{k-1})$ for all $t \geq T^{k-1}$. Then, consider the maximization problems:

$$\max\ J_\theta^k(\phi) \quad \text{s.t. } \phi \in \Phi_{T^{k-1}}^{k,-}, \tag{4.8}$$

$$\max\ J_\theta^k(\phi) \quad \text{s.t. } \phi \in \Phi_{T^{k-1}}^{k,+}, \tag{4.9}$$

where

$$\begin{aligned} \Phi_{T^{k-1}}^{k,-} = \{\phi \in \Phi \mid \phi(t) = \phi^{k-1,-}(t)\ \forall t \in [0, T^{k-1}], \\ \dot{\phi}_i(t) + \phi_i(t) \in \Delta([\underline{a}_i^{k-1}, \underline{a}_i^k])\ \forall i \in I,\ \text{a.a.}\, t \in [T^{k-1}, \infty)\}, \\ \Phi_{T^{k-1}}^{k,+} = \{\phi \in \Phi \mid \phi(t) = \phi^{k-1,+}(t)\ \forall t \in [0, T^{k-1}], \\ \dot{\phi}_i(t) + \phi_i(t) \in \Delta([\overline{a}_i^k, \overline{a}_i^{k-1}])\ \forall i \in I,\ \text{a.a.}\, t \in [T^{k-1}, \infty)\}. \end{aligned}$$

Observe that (4.8) and (4.9) are equivalent to (4.3) with $x = \phi^{k-1,-}(T^{k-1})$ and (4.4) with $y = \phi^{k-1,+}(T^{k-1})$, respectively. Therefore, by Lemma 4.3 we have feasible paths $\phi^{k,-}$ and $\phi^{k,+}$ that satisfy $(*_k^-)$ and $(*_k^+)$, respectively. ▮

Let $T^m = \infty$.

Lemma 4.5. *There exists* $\bar{\theta} > 0$ *such that for any* $\theta \in (0, \bar{\theta})$*, there exist* $T^1, \dots, T^{m-1}$ *with* $T^1 \leq \cdots < T^{m-1} \leq \infty$ *and feasible paths* ϕ^- *and* ϕ^+ *such that*

($*^-$) $\phi^-(0) = \underline{a}^0$, $\lim_{t\to\infty} \phi^-(t) = a^*$*, and for each* $k = 1, \dots, m$, $\phi^-(t) \in B_\eta([\underline{a}^{k-1}, a^*])$ *for all* $t \in [T^{k-1}, \infty)$ *and*

$$\dot{\phi}_i^-(t) = \min BR^i_{v^k}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) - \phi_i^-(t)$$

for all $i \in I$ *and almost all* $t \in [T^{k-1}, T^k)$,

and

($*^+$) $\phi^+(0) = \overline{a}^0$, $\lim_{t\to\infty} \phi^+(t) = a^*$*, and for each* $k = 1, \dots, m$, $\phi^+(t) \in B_\eta([a^*, \overline{a}^{k-1}])$ *for all* $t \in [T^{k-1}, \infty)$ *and*

$$\dot{\phi}_i^+(t) = \max BR^i_{v^k}(\phi^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t) - \phi_i^+(t)$$

for all $i \in I$ *and almost all* $t \in [T^{k-1}, T^k)$.

Proof. Take $\bar{\theta}$ as in Lemma 4.4. Fix any $\theta \in (0, \bar{\theta})$, and let $\phi^{1,-}, \dots, \phi^{m,-}$, and $\phi^{1,+}, \dots, \phi^{m,+}$ satisfy ($*_k^-$) and ($*_k^+$) for $k = 1, \dots, m$, respectively. Set $\phi^- = \phi^{m,-}$ and $\phi^+ = \phi^{m,+}$. We only verify that ϕ^- satisfies ($*^-$).

For each $k = 1, \dots, m$, we have $\phi_i^-(t) \in B_\eta([\underline{a}_i^{k-1}, a_i^*])$ for all $i \in I$ and all $t \geq T^{k-1}$. We also have $\lim_{t\to\infty} \phi^-(t) = a^*$. Observe that T^k's can be taken sufficiently large so that for each $k = 1, \dots, m-1$ and $i \in I$, $\phi_{ih}^-(t) = \phi_{ih}^{k,-}(t) = e^{-(t-T^k)} \phi_{ih}^{k,-}(T^k)$ for all $h \notin [\underline{a}_i^k, a_i^*]$ and all $t \geq T^k$. Note that by construction, $\phi^-(t) = \phi^{k,-}(t)$ for all $t \leq T^k$. Since $v^k(a) = v^k(a')$ for all $a, a' \in [\underline{a}^k, \overline{a}^k]$, it follows that for each $k = 1, \dots, m-1$ and $i \in I$, $BR^i_{v^k}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) = BR^i_{v^k}(\phi^{k,-}_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t)$ for all $t \leq T^k$. ∎

Proof of Theorem 4.1. Suppose that v^k's are monotone potential functions for a^* relative to $B_\eta([\underline{a}^{k-1}, \overline{a}^{k-1}])$. Take $\bar{\theta}$ as in Lemma 4.5. Fix any $\theta \in (0, \bar{\theta})$ and let ϕ^- and ϕ^+ satisfy ($*^-$) and ($*^+$), respectively.

Fix any $x \in \prod_i \Delta(A_i)$. Let β_x be the best response correspondence defined in (4.2). Let $\tilde{\Phi}_x = \{\phi \in \Phi_x \mid \phi^- \precsim \phi \precsim \phi^+\}$. We will show that $\tilde{\beta}_x(\phi) = \beta_x(\phi) \cap \tilde{\Phi}_x$ is nonempty for any $\phi \in \tilde{\Phi}_x$. Then, since $\tilde{\Phi}_x$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^* \in \tilde{\beta}_x(\phi^*) \subset \tilde{\Phi}_x$, which is a perfect foresight path in $\mathbf{g}$ and satisfies $\phi^- \precsim \phi^* \precsim \phi^+$. Since both ϕ^- and ϕ^+ converge to a^*, ϕ^* also converges to a^*.

Take any $\phi \in \tilde{\Phi}_x$. Consider any $k = 1, \dots, m$. Note that $\phi(t) \in B_\eta([\underline{a}^{k-1}, \overline{a}^{k-1}])$ for all $t \geq T^{k-1}$.

Suppose first that $g_i|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$ are supermodular for all $i \in I$. Then, for all $i \in I$,

$$\begin{aligned}\min BR^i_{v^k}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) &\leq \max BR^i_{g_i}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t) \\ &\leq \max BR^i_{g_i}(\phi_{-i}|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t)\end{aligned}$$

for all $t \in [T^{k-1}, T^k)$, where the second inequality follows from the assumption that v^k is a monotone potential function relative to $B_\eta([\underline{a}^{k-1}, \overline{a}^{k-1}])$, and the third inequality follows from the supermodularity of $g_i|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$. Similarly, for all $i \in I$,

$$\begin{aligned}\max BR^i_{v^k}(\phi^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t) &\geq \min BR^i_{g_i}(\phi^+_{-i}|[\underline{a}_i^k, \overline{a}_i^{k-1}])(t) \\ &\geq \min BR^i_{g_i}(\phi_{-i}|[\underline{a}_i^k, \overline{a}_i^{k-1}])(t)\end{aligned}$$

for all $t \in [T^{k-1}, T^k)$.

Suppose next that $v^k|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$ are supermodular for all $i \in I$. Then, for all $i \in I$,

$$\begin{aligned}\min BR^i_{v^k}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) &\leq \min BR^i_{v^k}(\phi_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) \\ &\leq \max BR^i_{g_i}(\phi_{-i}|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t)\end{aligned}$$

for all $t \in [T^{k-1}, T^k)$, where the second inequality follows from the supermodularity of $v^k|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$, and the third inequality follows from the assumption that v^k is a monotone potential function relative to $B_\eta([\underline{a}^{k-1}, \overline{a}^{k-1}])$. Similarly, for all $i \in I$,

$$\begin{aligned}\max BR^i_{v^k}(\phi^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t) &\geq \max BR^i_{v^k}(\phi_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t) \\ &\geq \min BR^i_{g_i}(\phi_{-i}|[\underline{a}_i^k, \overline{a}_i^{k-1}])(t)\end{aligned}$$

for all $t \in [T^{k-1}, T^k)$.

Therefore, in each case, we have for all $t \in [T^{k-1}, T^k)$,

$$\begin{aligned}\max BR^i_{g_i}(\phi_{-i}|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t) &\geq \min BR^i_{v^k}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t), \\ \min BR^i_{g_i}(\phi_{-i}|\underline{a}_i^k, \overline{a}_i^{k-1}])(t) &\leq \max BR^i_{v^k}(\phi^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t).\end{aligned}$$

Since $\phi(t) \in B_\eta(\Delta([\underline{a}^{k-1}, \overline{a}^{k-1}]))$ for all $t \geq T^{k-1}$ and hence

$$BR^i_{g_i}(\phi_{-i})(t) \cap [\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \neq \emptyset$$

by the choice of η, it follows that

$$\begin{aligned}&BR^i_{g_i}(\phi_{-i})(t) \\ &\quad \cap \left[\min BR^i_{v^k}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t), \max BR^i_{v^k}(\phi^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t)\right] \neq \emptyset,\end{aligned}$$

Let $\tilde{F}_i(\phi_{-i})(t)$ be the convex hull of the above set. Then the differential inclusion

$$\dot{\psi}(t) \in \tilde{F}(\phi)(t) - \psi(t), \qquad \psi(0) = x$$

has a solution ψ (see OTH (2003, Remark 2.1)). Since $\tilde{F}_i(\phi_{-i})(t) \subset F_i(\phi_{-i})(t) = \{\alpha_i \in \Delta(A_i) \,|\, \alpha_{ih} > 0 \Rightarrow h \in BR^i_{g_i}(\phi_{-i})(t)\}$, we have $\psi \in \beta_x(\phi)$. By the construction of ϕ^-, ϕ^+, and ψ, we have $\phi^- \precsim \psi \precsim \phi^+$. Thus, we have $\psi \in \tilde{\beta}_x(\phi) = \beta_x(\phi) \cap \tilde{\Phi}_x$, implying the nonemptiness of $\tilde{\beta}_x(\phi)$. ∎

By Proposition 2.4, we immediately have the following.

Corollary 4.6. *If a^* is an iterated strict* $\mathbf{p}$*-dominant equilibrium of* $\mathbf{g}$ *with* $\sum_{i\in I} p_i < 1$*, then there exists $\bar{\theta} > 0$ such that a^* is globally accessible in* $\mathbf{g}$ *for all $\theta \in (0, \bar{\theta})$.*

4.3 Linear Absorption of Iterated Strict MP-Maximizer

In this subsection, we prove that under the same monotonicity condition as in the informational robustness and the global accessibility results, an iterated strict MP-maximizer is linearly absorbing (regardless of the degree of friction), and therefore, it is the unique equilibrium that is globally accessible and linearly absorbing for any small degree of friction.

Theorem 4.7. *Suppose that* $\mathbf{g}$ *has an iterated strict MP-maximizer a^* with associated intervals $(S^k)_{k=0}^m$ and strict monotone potential functions $(v^k)_{k=1}^m$. If for each $k = 1, \dots, m$, $g_i|_{S_i^{k-1}\times A_{-i}}$ is supermodular for all $i \in I$ or $v^k|_{S^{k-1}}$ is supermodular, then a^* is linearly absorbing in* $\mathbf{g}$ *for all $\theta > 0$.*

We will use the following result due to Hofbauer and Sorger (2002) and OTH (2003).

Lemma 4.8. *Suppose that $v|_S$ is a potential game with a unique potential maximizer $a^* \in S$. Then, a^* is absorbing in $v|_S$ for all $\theta > 0$. If in addition, $v|_S$ is supermodular, then a^* is linearly absorbing in $v|_S$ for all $\theta > 0$.*

Suppose that a^* is an iterated strict MP-maximizer of $\mathbf{g}$ with associated intervals $(S^k)_{k=0}^m$ and strict monotone potential functions $(v^k)_{k=1}^m$. Due to Lemma 2.2, we can have $(\tilde{v}^k)_{k=1}^m$ and $\eta > 0$ such that for each $k = 1, \dots, m$, $\tilde{v}^k : A \to \mathbb{R}$ is a strict monotone potential function relative to $B_\eta(\Delta(S^{k-1}))$. For each $k = 0, 1, \dots, m$ and $i \in I$, write $S_i^k = [\underline{a}_i^k, \overline{a}_i^k]$, where $0 = \underline{a}_i^0 \le \underline{a}_i^1 \le \cdots \le \underline{a}_i^m = a_i^* = \overline{a}_i^m \le \cdots \le \overline{a}_i^1 \le \overline{a}_i^0 = n_i$. In defining such $(\tilde{v}^k)_{k=1}^m$ and $\eta > 0$, we extend v^k $(k = 1, \dots, m)$ to A so that $[\underline{a}^{k-1}, \underline{a}^k]$ and $[\overline{a}^k, \overline{a}^{k-1}]$ are strict best response sets in the games $\tilde{v}^k|_{[\underline{a}^0,\underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k,\overline{a}^0]}$, respectively, and take $\eta > 0$ to be sufficiently small so that for all $k = 1, \dots, m$ and all $i \in I$,

$$br^i_{\tilde{v}^k}(\pi_i|[\underline{a}_i^0, \underline{a}_i^k]) \subset [\underline{a}_i^{k-1}, \underline{a}_i^k]$$

for all $\pi_i \in B_\eta(\Delta([\underline{a}_{-i}^{k-1}, \underline{a}_{-i}^k]))$ and

$$br^i_{\tilde{v}^k}(\pi_i|[\overline{a}_i^k, \overline{a}_i^0]) \subset [[\overline{a}_i^k, \overline{a}_i^{k-1}]$$

for all $\pi_i \in B_\eta(\Delta([\overline{a}_{-i}^k, \overline{a}_{-i}^{k-1}]))$. In the case where $v^k|_{[\underline{a}^{k-1},\overline{a}^{k-1}]}$ is supermodular, v^k is extended so that $\tilde{v}^k|_{[\underline{a}^0,\underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k,\overline{a}^0]}$ are supermodular. We assume without loss of generality that in each potential game $\tilde{v}^k|_{[\underline{a}^0,\underline{a}^k]}$ ($\tilde{v}^k|_{[\overline{a}^k,\overline{a}^0]}$, resp.), any perfect foresight path from $B_\eta(\underline{a}^k)$ ($B_\eta(\overline{a}^k)$, resp.)

converges (linearly, in the case where the game is also supermodular) to $\underline{a}^k$ ($\overline{a}^k$, resp.).

For an interval $S \subset A$, we say that a feasible path ϕ is an *S-perfect foresight path* if for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \in BR^i_{g_i}(\phi_{-i}|S_i)(t). \tag{4.10}$$

Note that if ϕ is an S-perfect foresight path with $\phi(0) = x$, then for all $i \in A_i$ and all $h \notin S_i$, $\phi_{ih}(t) = x_{ih}e^{-t}$ for all $t \geq 0$.

Lemma 4.9. *For each $k = 1, \ldots, m$, if $g_i|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$ is supermodular for all $i \in I$ or $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$ are supermodular, then* (1) *for any $[\underline{a}^{k-1}, \overline{a}^{k-1}]$-perfect foresight path ϕ^* with $\phi^*(0) \in B_\eta(\Delta([\underline{a}^k, \overline{a}^k]))$,*

$$\lim_{t\to\infty} \sum_{h \in [\underline{a}_i^k, \overline{a}_i^k]} \phi^*_{ih}(t) = 1 \tag{4.11}$$

for all $i \in I$, and (2) *there exists $\eta^k \in (0, \eta]$ such that for any $[\underline{a}^{k-1}, \overline{a}^{k-1}]$-perfect foresight path ϕ^* with $\phi^*(0) \in B_{\eta^k}(\Delta([\underline{a}^k, \overline{a}^k]))$,*

$$BR^i_{g_i}(\phi^*_{-i}|[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}])(t) \subset [\underline{a}_i^k, \overline{a}_i^k] \tag{4.12}$$

for all $i \in I$ and $t \geq 0$.

Proof. (1) Take any $x \in B_\eta(\Delta([\underline{a}^k, \overline{a}^k]))$ and any $[\underline{a}^{k-1}, \overline{a}^{k-1}]$-perfect foresight path ϕ^* with $\phi^*(0) = x$. Note that $\phi^*(t) \in B_\eta(\Delta([\underline{a}^{k-1}, \overline{a}^{k-1}]))$ for all $t \geq 0$. Let

$$x_i^{k,-} = \eta \underline{a}_i^0 + (1-\eta)\underline{a}_i^k, \qquad x_i^{k,+} = \eta \overline{a}_i^0 + (1-\eta)\overline{a}_i^k,$$

and denote $x^{k,-} = \left(x_i^{k,-}\right)_{i\in I}$ and $x^{k,+} = \left(x_i^{k,+}\right)_{i\in I}$. We will find perfect foresight paths $\phi^{k,-}$ and $\phi^{k,+}$ for $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$, respectively, such that $\phi^{k,-}(0) = x^{k,-}$, $\phi^{k,+}(0) = x^{k,+}$, and $\phi^{k,-}(t) \precsim \phi^*(t) \precsim \phi^{k,+}(t)$ for all $t \geq 0$. Then, since the potential maximizer $\underline{a}^k$ ($\overline{a}^k$, resp.) is absorbing in $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ ($\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$, resp.), and hence $\phi^{k,-}$ ($\phi^{k,+}$, resp.) converges to $\underline{a}^k$ ($\overline{a}^k$, resp.), ϕ^* must satisfy (4.11).

The argument below follows that in OTH (2003, Appendix A.3). We show the existence of $\phi^{k,-}$; the existence of $\phi^{k,+}$ can be shown similarly. Let $\tilde{\Phi}_{x^{k,-}}$ be the set of feasible paths $\phi \in \Phi_{x^{k,-}}$ such that for all $i \in I$ and all $t \geq 0$, $\phi_i(t) \in \Delta([\underline{a}_i^0, \underline{a}_i^k])$, $\phi_i(t) \precsim \phi_i^*(t)$, and $\phi_{ih}(t) = x_{ih}^{k,-}e^{-t}$ for all $h < \underline{a}_i^{k-1}$. Consider the best response correspondence $\beta^-_{\tilde{v}^k}$ for the stage game $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$. We will show that $\tilde{\beta}^-_{\tilde{v}^k}(\phi) = \beta^-_{\tilde{v}^k}(\phi) \cap \tilde{\Phi}_{x^{k,-}}$ is nonempty for any $\phi \in \tilde{\Phi}_{x_\varepsilon^-}$. Then, since $\tilde{\Phi}_{x^{k,-}}$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^{k,-} \in \tilde{\beta}^-_{\tilde{v}^k}(\phi^{k,-}) \subset \tilde{\Phi}_{x^{k,-}}$, as desired.

Take any $\phi \in \tilde{\Phi}_{x^{k,-}}$. Note that $\phi(t) \in B_\eta(\Delta([\underline{a}^{k-1}, \underline{a}^k]))$ for all $t \geq 0$, and therefore $BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}^0_i, \underline{a}^k_i])(t) = BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}^{k-1}_i, \underline{a}^k_i])(t)$ by the choice of η. In the case where $g_i|_{[\underline{a}^{k-1}_i, \overline{a}^{k-1}_i] \times A_{-i}}$ is supermodular for all $i \in I$, we have, for all $i \in I$ and all $t \geq 0$,

$$\begin{aligned}\min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}^{k-1}_i, \underline{a}^k_i])(t) &\leq \min BR^i_{g_i}(\phi_{-i}|[\underline{a}^{k-1}_i, \overline{a}^k_i])(t) \\ &\leq \min BR^i_{g_i}(\phi^*_{-i}|[\underline{a}^{k-1}_i, \overline{a}^k_i])(t),\end{aligned}$$

where the first inequality follows from the assumption that $\tilde{v}^k$ is a strict monotone potential relative to $B_\eta(\Delta([\underline{a}^{k-1}, \overline{a}^{k-1}]))$ and the second inequality follows from the supermodularity of $g_i|_{[\underline{a}^{k-1}_i, \overline{a}^{k-1}_i] \times A_{-i}}$. In the case where v^k is supermodular, we have, for all $i \in I$ and all $t \geq 0$,

$$\begin{aligned}\min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}^{k-1}_i, \underline{a}^k_i])(t) &\leq \min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}^{k-1}_i, \underline{a}^k_i])(t) \\ &\leq \min BR^i_{g_i}(\phi^*_{-i}|[\underline{a}^{k-1}_i, \overline{a}^k_i])(t),\end{aligned}$$

where the first inequality follows from the supermodularity of $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and the second inequality follows from the assumption that $\tilde{v}^k$ is a strict monotone potential relative to $B_\eta(\Delta([\underline{a}^{k-1}, \overline{a}^{k-1}]))$. Therefore, in each case, we have, for all $i \in I$ and all $t \geq 0$,

$$\min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}^0_i, \underline{a}^k_i])(t) \leq \min BR^i_{g_i}(\phi^*_{-i}|[\underline{a}^{k-1}_i, \overline{a}^k_i])(t).$$

It follows that the solution ψ to

$$\dot{\psi}_i(t) = \min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}^0_i, \underline{a}^k_i])(t) - \psi_i(t), \qquad \psi(0) = x^{k,-}_i,$$

which is a best response to ϕ in the game $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$, satisfies $\psi \in \tilde{\Phi}_{x^{k,-}}$. This implies the nonemptiness of $\tilde{\beta}^-_{\tilde{v}^k}(\phi)$.

(2) If $g_i|_{[\underline{a}^{k-1}_i, \overline{a}^{k-1}_i] \times A_{-i}}$ is supermodular for all $i \in I$, then arguments analogous to those in OTH (2003, Appendix A.1) show that (1) implies (2). If $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$ are supermodular, then $\underline{a}^k$ ($\overline{a}^k$, resp.) is linearly absorbing in $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ ($\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$, resp.) and hence $\phi^{k,-}$ ($\phi^{k,+}$, resp.) converges linearly to $\underline{a}^k$ ($\overline{a}^k$, resp.). Therefore, for all i and all $h \notin [\underline{a}^k_i, \overline{a}^k_i]$, $\phi^*(t) = x_{ih} e^{-t}$ for all $t \geq 0$. Since $[\underline{a}^k, \overline{a}^k]$ is a strict best response set in $\mathbf{g}$, it follows that ϕ^* must satisfy (4.12). ∎

Proof of Theorem 4.7. Suppose that $\tilde{v}^k$'s are strict monotone potential functions relative to $B_\eta(\Delta([\underline{a}^{k-1}, \overline{a}^{k-1}]))$ and that for each $k = 1, \ldots, m$, $g_i|_{[\underline{a}^{k-1}_i, \overline{a}^{k-1}_i] \times A_{-i}}$ is supermodular for all $i \in I$ or $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$ are supermodular. Take $\eta^1, \ldots, \eta^m$ as in Lemma 4.9, and let $\varepsilon = \min\{\eta^1, \ldots, \eta^m\}$.

Fix any $x \in B_\varepsilon(a^*)$ and any perfect foresight path ϕ^* in $\mathbf{g}$ with $\phi^*(0) = x$. It is sufficient to prove that for all $k = 1, \ldots, m$,

$$BR^i_{g_i}(\phi^*_{-i}|[\underline{a}^{k-1}_i, \overline{a}^{k-1}_i])(t) \subset [\underline{a}^k_i, \overline{a}^k_i] \qquad (*_k)$$

holds for all $i \in I$ and all $t \geq 0$, which can be done by applying Lemma 4.9 iteratively. Indeed, since ϕ^* is an $[\underline{a}^0, \overline{a}^0]$-perfect foresight path, $(*_1)$ is true by Lemma 4.9. If $(*_1)$–$(*_{k-1})$ are true, then ϕ^* is an $[\underline{a}^{k-1}, \overline{a}^{k-1}]$-perfect foresight path, so that $(*_k)$ is also true by Lemma 4.9. ▮

By Proposition 2.4, we immediately have the following.

Corollary 4.10. *If a^* is an iterated strict $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ with $\sum_{i \in I} p_i < 1$, then a^* is linearly absorbing in $\mathbf{g}$ for all $\theta > 0$.*

5 Discussions

5.1 Set-Valued Concepts

In this subsection, we consider set-valued concepts of robustness to incomplete information and stability under perfect foresight dynamics.

5.1.1 Robust Sets to Incomplete Information

For $\sigma \in \Sigma$ and $S \subset A$, denote $\sigma_P(S) = \sum_{t \in T} \sum_{a \in S} P(t)\sigma(a|t)$.

Definition 5.1. A set of action profiles $S^* \subset A$ is *robust to all elaborations* in $\mathbf{g}$ if for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \leq \bar{\varepsilon}$, any ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$ has a Bayesian Nash equilibrium σ such that $\sigma_P(S^*) \geq 1 - \delta$.

The proof of Theorem 3.1 in fact shows the following.

Theorem 5.1. *Suppose that $\mathbf{g}$ has an iterated MP-maximizer set S^* with associated intervals $(S^k)_{k=0}^m$ and monotone potential functions $(v^k)_{k=1}^m$. If for each $k = 1, \dots, m$, $g_i|_{S_i^{k-1} \times A_{-i}}$ is supermodular for all $i \in I$ or $v^k|_{S_i^{k-1} \times A_{-i}}$ is supermodular for all $i \in I$, then S^* is robust to all elaborations in $\mathbf{g}$.*

Remark 5.1. Denote by $\mathcal{C} \subset \Delta(A)$ the set of all correlated equilibria of $\mathbf{g}$. If S^* is robust to all elaborations, then the set of correlated equilibria of $\mathbf{g}$ that assign probability one to S^*, $\mathcal{E}_{S^*} = \{\mu \in \mathcal{C} \mid \mu(S^*) = 1\}$, is nonempty and robust to all elaborations in the sense of Morris and Ui (2005, Definition 2).

5.1.2 Stable Sets under Perfect Foresight Dynamics

We say that a feasible path ϕ converges to S if $\lim_{t \to \infty} \sum_{h \in S_i} \phi_{ih}(t) = 1$ for all $i \in I$, and that a feasible path ϕ is a linear path from x to S if $\phi(0) = x$ and $\sum_{h \notin S_i} \phi_{ih}(t) = \sum_{h \notin S_i} x_{ih} e^{-t}$ for all $i \in I$.

Definition 5.2. (a) A set of action profiles $S^* \subset A$ is *globally accessible* in $\mathbf{g}$ if for any $x \in \prod_i \Delta(A_i)$, there exists a perfect foresight path from x that converges to S^*.

(b) A set of action profiles $S^* \subset A$ is *absorbing* in $\mathbf{g}$ if there exists $\varepsilon > 0$ such that for any $x \in B_\varepsilon(S^*)$, any unique perfect foresight path from x converges to S^*.

(c) A set of action profiles $S^* \subset A$ is *linearly absorbing* in $\mathbf{g}$ if there exists $\varepsilon > 0$ such that for any $x \in B_\varepsilon(S^*)$, any unique perfect foresight path from x is a linear path to S^*.

Here we do not require minimality, contrary to the set-valued stability concepts under perfect foresight dynamics defined in Matsui and Oyama (2006), Oyama (2002), and Tercieux (2004).

The proofs of Theorems 4.1 and 4.7 in fact show the following.

Theorem 5.2. *Suppose that* $\mathbf{g}$ *has an iterated MP-maximizer set* S^* *with associated intervals* $(S^k)_{k=0}^m$ *and monotone potential functions* $(v^k)_{k=1}^m$*. If for each* $k = 1, \dots, m$*,* $g_i|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$ *or* $v^k|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$*, then there exists* $\bar{\theta} > 0$ *such that* S^* *is globally accessible in* $\mathbf{g}$ *for all* $\theta \in (0, \bar{\theta})$*.*

Theorem 5.3. *Suppose that* $\mathbf{g}$ *has an iterated strict MP-maximizer set* S^* *with strict monotone potential functions* $(v^k)_{k=1}^m$*. If* $\mathbf{g}$ *is supermodular or* v^k *is supermodular for all* $k = 1, \dots, m$*, then* S^* *is linearly absorbing in* $\mathbf{g}$ *for all* $\theta > 0$*.*

Remark 5.2. If S^* is absorbing and globally accessible, then $\prod_{i \in I} \Delta(S_i^*)$ contains a unique globally accessible set in the sense of Oyama (2002, Definition 7). If S^* is linearly absorbing, then $\prod_{i \in I} \Delta(S_i^*)$ is closed under perfect foresight and hence contains a PF-stable set, and if in addition, S^* is also globally accessible, then $\prod_{i \in I} \Delta(S_i^*)$ contains all PF-stable sets in the sense of Matsui and Oyama (2006, Definition 3.3).

5.2 Iterated GP-Maximizer

In this subsection, we apply our iterative construction to the concept of generalized potential maximizer (GP-maximizer, in short) introduced by Morris and Ui (2005).

We say that $\mathcal{A}_i \subset 2^{A_i} \setminus \{\emptyset\}$, $i \in I$, is a covering of A_i if $\bigcup_{X_i \in \mathcal{A}_i} X_i = A_i$ and that $\mathcal{A} \subset 2^A \setminus \{\emptyset\}$ is a product covering of A if $\mathcal{A} = \{\prod_{i \in I} X_i \mid X_i \in \mathcal{A}_i \text{ for } i \in I\}$ for some covering $\mathcal{A}_i$ of A_i for each $i \in I$. For a product covering $\mathcal{A}$, write $\mathcal{A}_{-i} = \{\prod_{j \neq i} X_j \mid X_j \in \mathcal{A}_j \text{ for } j \neq i\}$. Given $\mathcal{A}_{-i}$ and $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, define $\Delta_{\Lambda_i}(A_{-i}) \subset \Delta(A_{-i})$ by

$$\Delta_{\Lambda_i}(A_{-i}) = \left\{ \lambda_i \in \Delta(A_{-i}) \;\middle|\; \lambda_i(B_{-i}) \geq v_i^{\Lambda_i}(B_{-i}) \text{ for all } B_{-i} \subset A_{-i} \right\},$$

where

$$v_i^{\Lambda_i}(B_{-i}) = \sum_{X_{-i} \in \mathcal{A}_i, X_{-i} \subset B_{-i}} \Lambda_i(X_{-i}).$$

For a function $F\colon \mathcal{A} \to \mathbb{R}$, which is extended to $\mathcal{A}_i \times \Delta(\mathcal{A}_{-i})$ in the usual way, write

$$BR^i_F(\Lambda_i) = \arg\max\{F(X_i, \Lambda_i) \mid X_i \in \mathcal{A}_i\}$$

for $\Lambda_i \in \Delta(\mathcal{A}_{-i})$.

Definition 5.3. Let S^*_i be a nonempty subset of A_i for each $i \in I$, and $S^* = \prod_{i\in I} S^*_i$. The set S^* is a *GP-maximizer* of $\mathbf{g}$ if there exist a product covering $\mathcal{A}$ containing S^* and a function $F\colon \mathcal{A} \to \mathbb{R}$ with $F(S^*) > F(X)$ for all $X \in \mathcal{A} \setminus \{S^*\}$ such that for all $i \in I$, all $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, and all $\lambda_i \in \Delta_{\Lambda_i}(A_i)$,

$$X_i \cap br^i_{g_i}(\lambda_i) \neq \emptyset$$

for every

$$X_i \in BR^i_F(\Lambda_i)$$

that is maximal in $BR^i_F(\Lambda_i)$ with respect to set inclusion. Such a function F is called a *generalized potential function.*

Morris and Ui (2005, Theorem 5) establish the informational robustness of GP-maximizer.

Proposition 5.4. *Suppose that* $\mathbf{g}$ *has a GP-maximizer* S^* *with respect to a product covering* $\mathcal{A}$*. If* $A_i \in \mathcal{A}_i$ *for all* $i \in I$*, then* S^* *is robust to all elaborations in* $\mathbf{g}$.

In the proof of their Proposition 2, Morris and Ui (2005) show that with the supermodularity condition, an MP-maximizer is indeed a GP-maximizer. Suppose that an interval S^* is an MP-maximizer of $\mathbf{g}$ with a monotone potential function v, and let $\mathcal{P}_i = \{S^*_i\} \cup \{\{a_i\} \mid a_i \notin S^*_i\}$ and $\mathcal{P} = \{\prod_{i\in I} X_i \mid X_i \in \mathcal{P}_i \text{ for } i \in I\}$. Then let $\mathcal{A}$ be the covering induced by $\mathcal{P}$: i.e.,

$$\mathcal{A}_i = \{[a_i, a'_i] \mid a_i \leq \min S^*_i,\ \max S^*_i \leq a'_i\}$$

for each $i \in I$. Finally, assume that v is $\mathcal{P}$-measurable, and define $F\colon \mathcal{A} \to \mathbb{R}$ by

$$F([a, a']) = v(a) + v(a'). \tag{5.1}$$

Note that $F(S^*) > F(S)$ for all $S \in \mathcal{A} \setminus S^*$.

Proposition 5.5. *Suppose that* $\mathbf{g}$ *has an MP-maximizer set* S^* *with a* $\mathcal{P}$*-measurable monotone potential function* v*. If* $\mathbf{g}$ *or* v *is supermodular, then* S^* *is a GP-maximizer of* $\mathbf{g}$ *with* F *defined by* (5.1) *being a generalized potential function.*

Now we want to consider iteration of GP-maximizer keeping its robustness to incomplete information. To this end, we employ a refinement of GP-maximizer, as done in the case of MP-maximizer.

Definition 5.4. Let S_i^* be a nonempty subset of A_i for each $i \in I$, and $S^* = \prod_{i \in I} S_i^*$. The set S^* is a *strict GP-maximizer* of $\mathbf{g}$ if there exist a product covering $\mathcal{A}$ containing S^* and a function $F\colon \mathcal{A} \to \mathbb{R}$ with $F(S^*) > F(X)$ for all $X \in \mathcal{A} \setminus \{S^*\}$ such that for all $i \in I$, all $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, and all $\lambda_i \in \Delta_{\Lambda_i}(A_i)$,

$$X_i \supset br^i_{g_i}(\lambda_i)$$

for every

$$X_i \in BR^i_F(\Lambda_i)$$

that is maximal in $BR^i_F(\Lambda_i)$ with respect to set inclusion. Such a function F is called a *strict generalized potential function*.

Analogously for Proposition 5.5, one can show the following.

Proposition 5.6. *Suppose that* $\mathbf{g}$ *has a strict MP-maximizer set* S^* *with a* $\mathcal{P}$*-measurable strict monotone potential function* v*. If* $\mathbf{g}$ *or* v *is supermodular, then* S^* *is a strict GP-maximizer of* $\mathbf{g}$ *with* F *defined by* (5.1) *being a strict generalized potential function.*

Applying our iterative construction to strict GP-maximizer leads us to the following concept.

Definition 5.5. An interval S^* is an *iterated strict GP-maximizer set* of $\mathbf{g}$ if there exists a sequence of intervals $S^0, S^1, \dots, S^m$ with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ such that S^k is a strict GP-maximizer set of $\mathbf{g}|_{S^{k-1}}$ for each $k = 1, \dots, m$.

We conclude this subsection with a set of conjectures.

Conjecture 1. *Suppose that* $\mathbf{g}$ *has an iterated strict GP-maximizer* S^* *with associated intervals* $(S^k)_{k=0}^m$ *and coverings* $(\mathcal{S}^k)_{k=0}^{m-1}$*. If for each* $k = 1, \dots, m-1$, $S_i^k \in \mathcal{S}_i^k$ *for all* $i \in I$*, then* S^* *is robust to all elaborations in* $\mathbf{g}$.

Conjecture 2. *Suppose that* $\mathbf{g}$ *has an iterated strict GP-maximizer* S^* *with associated intervals* $(S^k)_{k=0}^m$ *and coverings* $(\mathcal{S}^k)_{k=0}^{m-1}$*. If for each* $k = 1, \dots, m-1$, $S_i^k \in \mathcal{S}_i^k$ *is induced by an ordered partition on* S_i^k *for all* $i \in I$*, then there exists* $\bar{\theta} > 0$ *such that* S^* *is globally accessible in* $\mathbf{g}$ *for all* $\theta \in (0, \bar{\theta})$.

For the linear absorption, on the other hand, we conjecture that we will need an additional structure, such as the supermodularity of $\mathbf{g}$.

It is beyond the scope of the present paper to prove (or disprove) these conjectures, and we leave them for future research.

6 Conclusion

For any given set-valued solution concept, in principle, it is possible to consider iterative elimination of actions outside the solution set. In this paper, we applied such an iterative construction to two refinements of Nash equilibrium: p-dominant equilibrium (Morris, Rob, and Shin (1995) and Kajii and Morris (1997)) or p-best response set (Tercieux (2004)); and potential maximizer (Monderer and Shapley (1996)) or MP-maximizer (Morris and Ui (2005)). We showed that the iterative construction preserves their robustness to incomplete information (Kajii and Morris (1997)) as well as stability under perfect foresight dynamics (Matsui and Matsuyama (1995)): *iterated p-dominant equilibria* as well as *iterated MP-maximizers* (under some monotonicity conditions) are both robust to incomplete information and globally accessible (for a small degree of friction) and linearly absorbing under perfect foresight dynamics. We also proposed simple procedures, for some special classes of games, to find an iterated p-dominant equilibrium or an iterated MP-maximizer. In particular, we introduced iterated pairwise p-dominance and iterated risk-dominance for general supermodular games and two-player supermodular coordination games, respectively. Generally, finding an MP-maximizer or iterated MP-maximizer is a difficult task; no full characterization (i.e., necessary and sufficient condition) has been known for a game to have an MP-maximizer and hence an iterated MP-maximizer (unless the game is a simple game such as a 2×2 game). We see these simpler procedures as natural first steps to check whether our main theorems apply.

We provided numerical examples to show that for the p-dominance approach, our iterative construction strictly generalizes the existing results. In Example 2.5, for instance, the game has no (p_1, p_2)-dominant equilibrium such that $p_1 + p_2 < 1$. Hence, the existing results relying on the notion of p-dominance do not allow to conclude regarding the robustness of equilibria of this game. In contrast, this game has an iterated strict (p, p)-dominant equilibrium for some $p < 1/2$ and hence our results show that it is robust to incomplete information and stable under perfect foresight dynamics. Nonetheless, it should be noted that it is left an open question whether iterated MP-maximizer is a strictly more general concept than MP-maximizer. An MP-maximizer, by definition, is an iterated MP-maximizer, whereas we have not found any example that has an iterated MP-maximizer but no MP-maximizer. To assess the usefulness of the iterative construction for the potential maximization approach, this question is of importance. Investigation of conditions under which the iterative construction provides strictly more general results than a given refinement is yet to be conducted in the future.

Appendix

A.1 Proof of Lemma 2.2

Let $S^*, S \subset A$, and $v\colon S \to \mathbb{R}$ be as in the statement. For $i \in I$ and $a_i \in A_i$, let

$$\Pi^-_{ia_i}(g_i) = \{\pi_i \in \Delta(A_{-i}) \mid \min br^i_{g_i}(\pi_i|[\min S_i, \max S^*_i]) \le a_i\},$$
$$\Pi^+_{ia_i}(g_i) = \{\pi_i \in \Delta(A_{-i}) \mid \max br^i_{g_i}(\pi_i|[\min S^*_i, \max S_i]) \ge a_i\};$$

and for $f \in \mathbb{R}^A$,

$$\widehat{\Pi}^-_{ia_i}(f) = \{\pi_i \in \Delta(A_{-i}) \mid \min br^i_f(\pi_i|[\min S_i, \min S^*_i]) \le a_i\},$$
$$\widehat{\Pi}^+_{ia_i}(f) = \{\pi_i \in \Delta(A_{-i}) \mid \max br^i_f(\pi_i|[\max S^*_i, \max S_i]) \ge a_i\}.$$

Observe that $\Pi^-_{ia_i}(g_i)$ and $\widehat{\Pi}^-_{ia_i}(f)$ ($\Pi^+_{ia_i}(g_i)$ and $\widehat{\Pi}^+_{ia_i}(f)$, resp.) are closed (in $\Delta(A_{-i})$) due to the lower (upper, resp.) semi-continuity of $\min br^i_{g_i}$ and $\min br^i_f$ ($\max br^i_{g_i}$ and $\max br^i_f$, resp.). Note that these sets may be empty. Here we give a characterization of strict MP-maximizer in terms of these sets.

Lemma A.1.1. *S^* is a strict MP-maximizer set of $\mathbf{g}|_S$ with a strict monotone potential function v if and only if $S^* = \arg\max_{a\in S} v(a)$, and for all $i \in I$,*

$$\Pi^-_{ia_i}(g_i) \cap \Delta(S_{-i}) \subset \widehat{\Pi}^-_{ia_i}(v) \cap \Delta(S_{-i})$$

for all $a_i \in [\min S_i, \min S^*_i]$ and

$$\Pi^+_{ia_i}(g_i) \cap \Delta(S_{-i}) \subset \widehat{\Pi}^+_{ia_i}(v) \cap \Delta(S_{-i})$$

for all $a_i \in [\max S^*_i, \max S_i]$.

Now, extend v arbitrarily to A (i.e., consider a function defined on A that coincides with v on S, and denote it again by v) satisfying $S^* = \arg\max_{a\in S} v(a)$. In the case where v is supermodular, extend v so that $v|_A$ is supermodular.

For $\gamma > 0$, define $c_\gamma\colon A \to \mathbb{R}$ by

$$c_\gamma(a) = \gamma \sum_{i\in I} |a_i - S^*_i|,$$

where

$$|a_i - S^*_i| = \begin{cases} 0 & \text{if } a_i \in S^*_i, \\ \min S^*_i - a_i & \text{if } a_i < \min S^*_i, \\ a_i - \max S^*_i & \text{if } a_i > \max S^*_i. \end{cases}$$

Observe that if $h < k \le \min S_i^*$ or $h > k \ge \max S_i^*$, then for all $a_{-i} \in A_{-i}$,

$$c_\gamma(k, a_{-i}) - c_\gamma(h, a_{-i}) = -|k-h|\gamma \le -\gamma.$$

Fix any $\gamma > 0$ such that $\gamma < \big(\max_{a \in A} v(a) - \max_{a \notin S^*} v(a)\big) / \sum_{i \in I} n_i$. Then define $\tilde{v} \colon A \to \mathbb{R}$ by

$$\tilde{v}(a) = v(a) + c_\gamma(a). \tag{A.1}$$

By the choice of γ, $S^* = \arg\max_{a \in A} \tilde{v}(a)$. Verify also that if $v|_S$ is supermodular, then so is $\tilde{v}$. The following lemma shows that the transformation above expands $\widehat{\Pi}^-_{ia_i}(v)$ and $\widehat{\Pi}^+_{ia_i}(v)$.

Lemma A.1.2. *Given $v \colon A \to \mathbb{R}$, let $\tilde{v} \colon A \to \mathbb{R}$ be defined by* (A.1)*. For each $i \in I$ and $a_i \in [\min S_i, \min S_i^*]$, there exists an open set $U^-_{ia_i} \subset \Delta(A_{-i})$ such that*

$$\widehat{\Pi}^-_{ia_i}(v) \subset U^-_{ia_i} \subset \widehat{\Pi}^-_{ia_i}(\tilde{v}).$$

Similarly, for each $i \in I$ and $a_i \in [\max S_i^*, \max S_i]$, there exists an open set $U^+_{ia_i} \subset \Delta(A_{-i})$ such that

$$\widehat{\Pi}^+_{ia_i}(v) \subset U^+_{ia_i} \subset \widehat{\Pi}^+_{ia_i}(\tilde{v}).$$

Proof. Fix $i \in I$ and $a_i \in [\min S_i, \min S_i^*]$. Take any $\pi_i \in \widehat{\Pi}^-_{ia_i}(v)$: i.e., $\min br^i_v(\pi_i | [\min S_i, \min S_i^*]) \le a_i$. Take $\varepsilon(\pi_i) > 0$ such that if $\pi_i' \in B_{\varepsilon(\pi_i)}(\pi_i)$, then

$$\max_{h,k \in A_i} \big| \big(\tilde{v}(k, \pi_i') - \tilde{v}(h, \pi_i')\big) - \big(\tilde{v}(k, \pi_i) - \tilde{v}(h, \pi_i)\big) \big| < \gamma.$$

Let us show that $B_{\varepsilon(\pi_i)}(\pi_i) \subset \widehat{\Pi}^-_{ia_i}(\tilde{v})$. Take any $\pi_i' \in B_{\varepsilon(\pi_i)}(\pi_i)$, and let $\underline{a}_i = \min br^i_{\tilde{v}}(\pi_i' | [\min S_i, \min S_i^*])$. We want to show that $\underline{a}_i \le a_i$. It is sufficient to show that $\underline{a}_i \le \min br^i_v(\pi_i | [\min S_i, \min S_i^*])$. If $h < \underline{a}_i$, then

$$\begin{aligned} v(\underline{a}_i, \pi_i) - v(h, \pi_i) &= \big(\tilde{v}(\underline{a}_i, \pi_i) - c_\gamma(\underline{a}_i, \pi_i)\big) - \big(\tilde{v}(h, \pi_i) - c_\gamma(h, \pi_i)\big) \\ &= \tilde{v}(\underline{a}_i, \pi_i) - \tilde{v}(h, \pi_i) + (\underline{a}_i - h)\gamma \\ &\ge \tilde{v}(\underline{a}_i, \pi_i) - \tilde{v}(h, \pi_i) + \gamma \\ &> \tilde{v}(\underline{a}_i, \pi_i') - \tilde{v}(h, \pi_i') > 0. \end{aligned}$$

This means that $\underline{a}_i \le \min br^i_v(\pi_i | [\min S_i, \min S_i^*])$, which implies that $\pi_i' \in \widehat{\Pi}^-_{ia_i}(\tilde{v})$.

Then set $U^-_{ia_i} = \bigcup_{\pi_i \in \widehat{\Pi}^-_{ia_i}(v)} B_{\varepsilon(\pi_i)}(\pi_i)$. ∎

Proof of Lemma 2.2. Given $v \colon A \to \mathbb{R}$, let $\tilde{v} \colon A \to \mathbb{R}$ be defined by (A.1). Then, $S^* = \arg\max_{a \in A} \tilde{v}(a)$; and if $v|_S$ is supermodular, then so is $\tilde{v}$. For each $i \in I$ and $a_i \in [\min S_i, \min S_i^*]$ such that $\Pi^-_{ia_i}(g_i) \neq \emptyset$, take an open set $U^-_{ia_i}$ as in Lemma A.1.2. Note that $\Pi^-_{ia_i}(g_i) \cap \Delta(S_{-i}) \subset U^-_{ia_i}$. Since $\Pi^-_{ia_i}(g_i)$

and $\Delta(S_{-i})$ are closed in a compact set $\Delta(A_{-i})$, there exists $\eta^{-}(i, a_i) > 0$ such that

$$\Pi^{-}_{ia_i}(g_i) \cap B_{\eta^{-}(i,a_i)}(\Delta(S_{-i})) \subset U^{-}_{ia_i}.$$

Apply the same argument to each $i \in I$ and $a'_i \in [\max S^*_i, \max S_i]$ such that $\Pi^{+}_{ia'_i}(g_i) \neq \emptyset$ to obtain $\eta^{+}(i, a'_i) > 0$ such that

$$\Pi^{+}_{ia'_i}(g_i) \cap B_{\eta^{+}(i,a'_i)}(\Delta(S_{-i})) \subset U^{+}_{ia'_i},$$

where $U^{+}_{ia'_i}$ is as in Lemma A.1.2.

Finally, set $\eta = \min_{i,a_i} \eta^{-}(i, a_i) \wedge \min_{i,a'_i} \eta^{+}(i, a'_i)$. ∎

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