Supplementary Appendix to: On the (Non-)Differentiability of the Optimal Value Function When the Optimal Solution Is Unique

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Abstract

This supplementary material contains detailed proofs and additional examples.

Supplementary Appendix

B.1. Proof of Strict Monotonicity in Example 5.1

In this section, we verify the positivity of the partial derivatives of the utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ given in Example 5.1:

$$u(x_1, x_2) = 6(x_1 + x_2 - 2) + (x_1 + x_2 - 2)^2 + u^0(x_1, x_2) \tag{1}$$

with

$$u^0(x_1, x_2) = \begin{cases} -\dfrac{8(x_1-1)^2(x_2-1)^2}{(x_1+x_2-2)^3} & \text{if } (x_1, x_2) \in [0,1)^2, \\ \dfrac{8(x_1-1)^2(x_2-1)^2}{(x_1+x_2-2)^3} & \text{if } (x_1, x_2) \in (1,\infty)^2, \\ -(x_1-1)^2(x_2-1)^2 & \text{otherwise.} \end{cases} \tag{2}$$

The function u^0 is differentiable in each x_i, $i = 1, 2$, at all $(x_1, x_2) \in \mathbb{R}^2_+$, where the partial derivatives are given by

$$u^0_{x_i}(x_1, x_2) = \begin{cases} -\dfrac{8(x_i-1)(x_j-1)^2(-x_i+2x_j-1)}{(x_1+x_2-2)^4} & \text{if } (x_1, x_2) \in [0,1)^2, \\ \dfrac{8(x_i-1)(x_j-1)^2(-x_i+2x_j-1)}{(x_1+x_2-2)^4} & \text{if } (x_1, x_2) \in (1,\infty)^2, \\ -2(x_i-1)(x_j-1)^2 & \text{otherwise.} \end{cases}$$

Proposition B.1. *For the function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ defined by* (1)–(2), *$u_{x_i}(x_1, x_2) > 0$ for all $i = 1, 2$ and all $(x_1, x_2) \in \mathbb{R}^2_+$.*

By symmetry, we only show this for $i = 1$.

Observation 1. $\frac{\partial}{\partial x_1}\left[6(x_1+x_2-2)+(x_1+x_2-2)^2\right] \geq 2$ *for all* $(x_1, x_2) \in \mathbb{R}^2_+$.

Proof. For all $x_1, x_2 \geq 0$,

$$\frac{\partial}{\partial x_1}\left[6(x_1+x_2-2)+(x_1+x_2-2)^2\right] = 6 + 2(x_1+x_2-2) \geq 2,$$

as stated. □

Observation 2. $u^0_{x_1}(x_1, x_2) > -2$ *for all* $(x_1, x_2) \in [0, 1)^2$.

Proof. Evaluating $u^0_{x_1}$ on each line segment $x_2 = c(x_1 - 1) + 1$ in $[0, 1)^2$, where $c > 0$, we have

$$u^0_{x_1}(x_1, c(x_1-1)+1) = -\frac{8c^2(2c-1)}{(c+1)^4}.$$

This value is nonnegative for $c \leq \frac{1}{2}$. For $c > \frac{1}{2}$, by the inequality of arithmetic and geometric means we have

$$\begin{aligned}\frac{8c^2(2c-1)}{(c+1)^4} &= \frac{8}{5}\frac{5 \times c \times c \times (2c-1)}{(c+1)^4} \\ &\leq \frac{8}{5}\left[\frac{1}{4}\left(\frac{5}{c+1}+\frac{c}{c+1}+\frac{c}{c+1}+\frac{2c-1}{c+1}\right)\right]^4 \\ &= \frac{8}{5} < 2.\end{aligned}$$

Hence, $u^0_{x_1}(x_1, x_2) > -2$ for all $(x_1, x_2) \in [0, 1)^2$. □

Observation 3. $u^0_{x_1}(x_1, x_2) > -2$ *for all* $(x_1, x_2) \in (1, \infty)^2$.

Proof. Evaluating $u^0_{x_1}$ on each half line $x_2 = c(x_1 - 1) + 1$ in $(1, \infty)^2$, where $c > 0$, we have

$$u^0_{x_1}(x_1, c(x_1-1)+1) = \frac{8c^2(2c-1)}{(c+1)^4}.$$

This value is nonnegative for $c \geq \frac{1}{2}$. For $c < \frac{1}{2}$, by the inequality of arithmetic and geometric means we have

$$\begin{aligned}-\frac{8c^2(2c-1)}{(c+1)^4} &= 2\frac{1 \times 2c \times 2c \times (1-2c)}{(c+1)^4} \\ &\leq 2\left[\frac{1}{4}\left(\frac{1}{c+1}+\frac{2c}{c+1}+\frac{2c}{c+1}+\frac{1-2c}{c+1}\right)\right]^4 \\ &= 2\left(\frac{1}{2}\right)^4 = \frac{1}{8} < 2.\end{aligned}$$

Hence, $u^0_{x_1}(x_1, x_2) > -2$ for all $(x_1, x_2) \in (1, \infty)^2$. □

Observation 4. $u_{x_1}(x_1, x_2) \geq 4$ *for all* $(x_1, x_2) \in ([0, 1] \times [1, \infty)) \cup ([1, \infty) \times [0, 1])$.

Proof. For $(x_1, x_2) \in ([0,1] \times [1,\infty)) \cup ([1,\infty) \times [0,1])$, we have

$$\begin{aligned} u_{x_1}(x_1, x_2) &= 6 + 2(x_1 + x_2 - 2) - 2(x_1 - 1)(x_2 - 1)^2 \\ &= 2x_2[1 - (x_1 - 1)(x_2 - 2)] + 4. \end{aligned}$$

We claim that $(x_1 - 1)(x_2 - 2) \le 1$ for all $(x_1, x_2) \in ([0,1] \times [1,\infty)) \cup ([1,\infty) \times [0,1])$. Indeed, $(x_1-1)(x_2-2) \le 0$ for $(x_1, x_2) \in ([0,1]\times[2,\infty)) \cup ([1,\infty)\times[0,1])$, while for $(x_1, x_2) \in [0,1] \times [1,2]$, $(x_1 - 1)(x_2 - 2)$ is nonincreasing, so that $(x_1 - 1)(x_2 - 2) \le (0 - 1)(1 - 2) = 1$, as claimed. □

Proof of Proposition B.1. By Observations 1–4, $u_{x_1}(x_1, x_2) > 0$, and $u_{x_2}(x_1, x_2) > 0$ by symmetry, for all $(x_1, x_2) \in \mathbb{R}^2_+$. □

B.2. Totally Differentiable Version of Examples 2.1 and 5.1

While the differentiability of the objective function f with respect to the choice variable x is never an issue (and not even defined) in our main analysis, we can modify the function in Example 2.1 to be (totally) differentiable in (x, α) keeping the other properties, as we show in Example B.1. Correspondingly, in Example B.2 we present a modified version of the utility function in Example 5.1 that is differentiable in (x_1, x_2).

Example B.1. Let $X = \mathbb{R}$ and $A = \mathbb{R}$, and denote

$$\begin{aligned} R_1 &= \left\{(x,\alpha) \in \mathbb{R}^2 \;\middle|\; x < 0,\ x^2 + 3\alpha < 0\right\}, \\ R_2 &= \left\{(x,\alpha) \in \mathbb{R}^2 \;\middle|\; x \le 0,\ x^2 + 3\alpha \ge 0\right\}, \\ R_3 &= \left\{(x,\alpha) \in \mathbb{R}^2 \;\middle|\; x > 0,\ x^2 - 3\alpha < 0\right\}, \\ R_4 &= \left\{(x,\alpha) \in \mathbb{R}^2 \;\middle|\; x \ge 0,\ x^2 - 3\alpha \ge 0\right\}. \end{aligned}$$

Then define the continuous function $f\colon X \times A \to \mathbb{R}$ by

$$f(x,\alpha) = \begin{cases} \dfrac{1}{4\alpha^2} x^2 (x^2 + 3\alpha)^2 & \text{if } (x,\alpha) \in R_1, \\ -x^2(x^2 + 3\alpha)^2 & \text{if } (x,\alpha) \in R_2, \\ \dfrac{1}{4\alpha^2} x^2 (x^2 - 3\alpha)^2 & \text{if } (x,\alpha) \in R_3, \\ -x^2(x^2 - 3\alpha)^2 & \text{if } (x,\alpha) \in R_4. \end{cases}$$

Its partial derivatives are

$$f_x(x,\alpha) = \begin{cases} \dfrac{3}{2\alpha^2} x(x^2 + \alpha)(x^2 + 3\alpha) & \text{if } (x,\alpha) \in R_1, \\ -6x(x^2 + \alpha)(x^2 + 3\alpha) & \text{if } (x,\alpha) \in R_2, \\ \dfrac{3}{2\alpha^2} x(x^2 - \alpha)(x^2 - 3\alpha) & \text{if } (x,\alpha) \in R_3, \\ -6x(x^2 - \alpha)(x^2 - 3\alpha) & \text{if } (x,\alpha) \in R_4, \end{cases}$$

and

$$f_\alpha(x,\alpha) = \begin{cases} -\dfrac{1}{2\alpha^3}x^4(x^2+3\alpha) & \text{if } (x,\alpha)\in R_1, \\ -6x^2(x^2+3\alpha) & \text{if } (x,\alpha)\in R_2, \\ -\dfrac{1}{2\alpha^3}x^4(x^2-3\alpha) & \text{if } (x,\alpha)\in R_3, \\ 6x^2(x^2-3\alpha) & \text{if } (x,\alpha)\in R_4. \end{cases}$$

One can verify that

$$X^*(\alpha) = \begin{cases} \{-\sqrt{-\alpha}\} & \text{if } \alpha < 0, \\ \{\sqrt{\alpha}\} & \text{if } \alpha \geq 0, \end{cases}$$

and

$$v(\alpha) = |\alpha|,$$

which is not differentiable at $\alpha = 0$.

We claim that f is differentiable in (x,α) everywhere. The differentiability at $(x,\alpha) \neq (0,0)$ is clear. To establish the differentiability at $(x,\alpha) = (0,0)$, take any $\varepsilon > 0$. Since $f(0,0) = 0$ and $f_x(0,0) = f_\alpha(0,0) = 0$, we want to find a $\delta > 0$ such that $|f(x,\alpha)| \leq \varepsilon|(x,\alpha)|$ whenever $|(x,\alpha)| \leq \delta$, where $|(x,\alpha)| = \sqrt{x^2+\alpha^2}$.

For C^1 functions $g^1(x,\alpha) = -x^2(x^2+3\alpha)^2$ and $g^2(x,\alpha) = -x^2(x^2-3\alpha)^2$ on $\mathbb{R}^2$, let $\delta_0 > 0$ be such that $|\nabla g^k(x,\alpha)| \leq \varepsilon$, $k = 1,2$, whenever $|(x,\alpha)| \leq \delta_0$. Then let $\delta = \min\{\delta_0, 4\varepsilon/9\} > 0$, and take any (x,α) with $|(x,\alpha)| \leq \delta$. If $(x,\alpha) \in R_2 \cup R_4$, we have

$$|f(x,\alpha)| \leq \max_{k=1,2,\,\theta\in[0,1]} |\nabla g^k(\theta x, \theta\alpha)||(x,\alpha)| \leq \varepsilon|(x,\alpha)|,$$

where the first inequality follows from the mean value theorem. If $(x,\alpha) \in R_1$, let $c\alpha = x^2$ with $-3 < c < 0$. Then we have

$$\begin{aligned} 0 \leq f(x,\alpha) = \frac{1}{4}(c+3)^2x^2 \leq \frac{9}{4}x^2 &\leq \frac{9}{4}|(x,\alpha)|^2 \\ &\leq \frac{9}{4}\delta|(x,\alpha)| \leq \varepsilon|(x,\alpha)|. \end{aligned}$$

If $(x,\alpha) \in R_3$, again let $c\alpha = x^2$ with $0 < c < 3$. Then we similarly have

$$\begin{aligned} 0 \leq f(x,\alpha) = \frac{1}{4}(c-3)^2x^2 \leq \frac{9}{4}x^2 &\leq \frac{9}{4}|(x,\alpha)|^2 \\ &\leq \frac{9}{4}\delta|(x,\alpha)| \leq \varepsilon|(x,\alpha)|. \end{aligned}$$

Thus, we have shown that f is differentiable at $(x,\alpha) = (0,0)$.

Finally, one can verify that f_α is not continuous in (x,α) at $(x,\alpha) = (0,0)$ (for $(x,\alpha) \in R_1 \cup R_3$, let $c\alpha = x^2$). □

The partial differentiability condition (b) in Proposition 5.7 can be strengthened to (total) differentiability (where $X = \mathbb{R}^n_+$).

Proposition B.2. *There exists a continuous utility function $u\colon X \to \mathbb{R}$ such that for some $\bar{p} \in \mathbb{R}^n_{++}$,*

(a) *$D(\bar{p}, w)$ and $H(\bar{p}, u)$ are singletons for all w and all u and are continuous (as single-valued functions) in w and u, respectively,*

(b) *u is differentiable in x at all $x \in X$, and*

(c) *$u_{x_i}(x) > 0$ for all $i = 1, \dots, n$ and all $x \in X$,*

but $v(\bar{p}, w)$ and $e(\bar{p}, u)$ are not differentiable in w and u at some $\bar{w}$ and some $\bar{u}$, respectively.

Using the function in Example B.1, we provide such a utility function with $n = 2$.

Example B.2. Given the function f in Example B.1, define the continuous function $u^0\colon \mathbb{R}^2_+ \to \mathbb{R}$ by

$$u^0(x_1, x_2) = f\left(x_1 - 1, \frac{1}{2}(x_1 + x_2 - 2)\right),$$

and then define the utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ by

$$u(x_1, x_2) = a_1(x_1 + x_2) + a_6(x_1 + x_2)^6 + u^0(x_1, x_2),$$

where a_1 and a_6 are positive constants.

(1) Being the composition of the differentiable functions $(x_1, x_2) \mapsto \left(x_1 - 1, \frac{1}{2}(x_1 + x_2 - 2)\right)$ and f, u^0 is differentiable in (x_1, x_2), and so is u.

(2) For sufficiently large a_1 and a_6, $u_{x_i}(x_1, x_2) > 0$, $i = 1, 2$, for all $(x_1, x_2) \in \mathbb{R}^2_+$. This can be verified as follows. Denote

$$Q_j = \left\{(x_1, x_2) \in \mathbb{R}^2_+ \;\middle|\; \left(x_1 - 1, \frac{1}{2}(x_1 + x_2 - 2)\right) \in R_j\right\}$$

for $j = 1, \dots, 4$. Since $u^0_{x_i}(x_1, x_2)$, $i = 1, 2$, is written as

$$u^0_{x_i}(x_1, x_2) = \frac{(\text{polynomial of } (x_1, x_2) \text{ of degree } 6)}{(x_1 + x_2 - 2)^3}$$

on $Q_1 \cup Q_3$ and

$$u^0_{x_i}(x_1, x_2) = (\text{polynomial of } (x_1, x_2) \text{ of degree } 5)$$

on $Q_2 \cup Q_4$, we can take an $a_6 > 0$ and a $b > 2$ large enough that

$$\frac{u^0_{x_i}(x_1, x_2)}{6a_6(x_1 + x_2)^5} > -1$$

for all $i = 1, 2$ and all $(x_1, x_2) \in \mathbb{R}^2_+$ such that $x_1 + x_2 > b$. Next, one can verify that $u^0_{x_i}(x_1, x_2)$, $i = 1, 2$, is bounded on any bounded set. So let $a_1 > 0$ be such that

$$u^0_{x_i}(x_1, x_2) > -a_1$$

for all $i = 1, 2$ and all $(x_1, x_2) \in \mathbb{R}^2_+$ such that $x_1 + x_2 \leq b$. With these a_1 and a_6 so constructed, we have

$$u_{x_i}(x_1, x_2) = a_1 + 6a_6(x_1 + x_2)^5 + u^0_{x_1}(x_1, x_2) > 0$$

for all $i = 1, 2$ and all $(x_1, x_2) \in \mathbb{R}^2_+$.

(3) By (2), u is strictly increasing, so that the optimal consumption lies on the budget line.

Fix the price vector to $\bar{p} = (\frac{1}{2}, \frac{1}{2})$. By substituting the budget equality $\frac{1}{2}x_1 + \frac{1}{2}x_2 = w$, the indirect utility $v(\bar{p}, w)$ for the utility function u is given by

$$v(\bar{p}, w) = \max_{0 \leq x_1 \leq 2w} a_1(x_1 + x_2) + a_6(x_1 + x_2)^6 + f(x_1 - 1, w - 1).$$

As in Example B.1, the unique optimal solution is given by $x_1^* = 1 - \sqrt{1 - w}$ if $w < 1$ and $x_1^* = 1 + \sqrt{w - 1}$ if $w \geq 1$ (note that $0 \leq x_1^* \leq 2w$ in each case). Thus, the indirect utility function with $p = \bar{p}$ is given by

$$v(\bar{p}, w) = 2a_1 + 64a_6 + |w - 1|,$$

which is not differentiable at $w = 1$. The expenditure function $e(p, u)$ with $p = \bar{p}$ is given by the inverse function of $w \mapsto v(\bar{p}, w)$ and is not differentiable in u at $u = 0$. □