Rationalizable Foresight Dynamics*

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Abstract

This paper proposes and studies the rationalizable foresight dynamics. A normal form game is repeatedly played in a random matching fashion by a continuum of agents who make decisions at stochastic points in time. A *rationalizable foresight path* is a feasible path of action distribution along which each agent takes an action that maximizes his expected discounted payoff against another path which is in turn a rationalizable foresight path. We consider a set-valued stability concept under this dynamics and compare it with the corresponding concept under the perfect foresight dynamics.

JEL Classification: C72, C73.

Keywords: rationalizability, Nash equilibrium, rationalizable foresight, perfect foresight, stability under rationalizable foresight (RF-stability), stability under perfect foresight (PF-stability).

Contents

1 Introduction

This paper considers a dynamic adjustment process in a society with a continuum of overlapping rational agents who repeatedly play a stage game. Agents make irreversible decisions (e.g., career or sector choices, as considered in Matsuyama (1991)) upon entry, so that beliefs about the future states of the society are of relevance in making decisions. In this society, rationality is common knowledge, but beliefs may or may not be coordinated among the agents. If the beliefs are correctly coordinated, then the future course of behavior pattern follows a perfect foresight path as defined by Matsui and Matsuyama (1995). If, on the other hand, the beliefs are not necessarily coordinated, then the actual decisions need not be best responses to each other as postulated in the standard (static) rationalizability due to Bernheim (1984) and Pearce (1984). Incorporating rationalizability into the above dynamic framework, the present paper proposes and studies the *rationalizable foresight dynamics*.

In bold strokes, the main concept of rationalizable foresight path is inductively defined in the following way. First, given the set of (physically) feasible paths of action distribution, Φ^0, define its subset Φ^1 as the set of paths along which each entrant takes an optimal action against *some* path in Φ^0. Here, we allow different agents to optimize against different paths. Then given this newly defined set Φ^1, define its subset Φ^2 of paths along which each agent takes an optimal action against some path in Φ^1. We repeat this procedure inductively and take the limit of these sets: this limit is called the set of *rationalizable foresight paths*. Indeed, along a rationalizable foresight path each agent optimizes against another—typically different—rationalizable foresight path.

We use the rationalizable foresight paths to define stability concepts. A set of action distributions is said to be a *stable set under rationalizable foresight*, or an *RF-stable set*, if it is a minimal nonempty and closed set from which no rationalizable foresight path leaves. If an RF-stable set is a singleton, its element is said to be a *stable state under rationalizable foresight*, or an *RF-stable state*.

We study the relationship between the stability concepts under rationalizable foresight and the corresponding concepts under perfect foresight. A perfect foresight path is a feasible path of action distribution along which every entrant takes a best response to this path itself. A stable set and a stable state under perfect foresight (a *PF-stable set* and a *PF-stable state*) are defined analogously to those under rationalizable foresight. A perfect foresight path is a rationalizable foresight path, but not vice versa. Therefore, for a given state, it is easier to escape from the state under rationalizable foresight than under perfect foresight. This immediately implies that every RF-stable state is a PF-stable state. The converse, however, is not true in general. We construct an example in which stability under rationalizable foresight provides a sharper prediction than that under perfect foresight. In this example, the set of RF-stable states is a proper subset of the set of PF-stable states, and no other stable set exists. Along a rationalizable foresight path that escapes from a PF-stable state, agents constantly misforecast the future.

In our analysis, inertia induced by the irreversibility of decisions plays a key role.[1] If there were no inertia, the behavior pattern might jump around, and there would be no hope for providing sharp predictions. To see this, we introduce a variant of the standard rationalizability for static societal games where a continuum of agents are randomly matched to play a given normal form game only once. In this environment, a *rationalizable action distribution* is defined in such a way that every action taken by some agents, i.e., contained in the support of this action distribution, is a best response to another rationalizable action distribution. We then show that as inertia vanishes, the unique RF-stable set of the dynamic societal game converges to the set of rationalizable action distributions of the static societal game.

We then give a full characterization of the stability concepts for the class of 2×2 games. In the games with two strict Nash equilibria, the risk-dominant equilibrium constitutes a unique RF-stable set as a singleton if the degree of friction is sufficiently small; otherwise, each of the strict equilibria constitutes an RF-stable set, and therefore, there are two RF-stable sets. In the games with a unique symmetric Nash equilibrium, which is completely mixed, the unique RF-stable set consists of some non-equilibrium states as well as the equilibrium state (independently of the degree of friction). This fact is in stark contrast with the fact that the PF-stable set consists only of the equilibrium state. When no RF-stable state (singleton RF-stable set) exists, the prediction generated by rationalizable foresight may be more obscure than that generated by perfect foresight.

The stability concepts of the present paper share the basic idea with those of Gilboa and Matsui (1991). In that paper, a best response path is defined to be a feasible path along which agents take best responses to the current action distribution. An action distribution is accessible from another (under the best response dynamics) if there exists a best response path from the latter to the former. A set of action distributions is a stable set under the best response dynamics if it is closed under accessibility and any two elements are mutually accessible.[2] The assumption that an action increases its frequency only when it is a best response to the current action distribution implicitly assumes myopia of decision makers. Indeed, the dynamics studied

[1] We deal with a specific environment in which an adjustment cost is infinite after one chooses his action. We could modify this assumption to accommodate other environments such as the one with a finite adjustment cost.

[2] In Gilboa and Matsui (1991), the set-valued and the point-valued stability concepts under the best response dynamics are called 'cyclically stable set' and 'socially stable strategy', respectively.

in the present paper becomes similar to the best response dynamics if the agents are sufficiently impatient.

The perfect foresight dynamics has been studied by Krugman (1991) and Matsuyama (1991) in the context of development economics,[3] and Matsui and Matsuyama (1995) for societal games.[4] While discussing the possibility of escapes from locally stable states by way of agents' forward looking abilities, these papers consider the perfect foresight paths, and therefore, do not bear the idea of miscoordination of beliefs.

The rest of the paper is organized as follows. Section 2 gives our basic framework. Section 3 defines rationalizable and perfect foresight dynamics together with corresponding stability concepts and examines their properties. Section 4 completely characterizes RF-stable sets for the class of symmetric 2×2 games. Section 5 concludes the paper.

2 Framework

We consider a symmetric two-player game with $n \geq 2$ actions. The set of actions and the payoff matrix, which are common to both players, are given by $A = \{a_1, \dots, a_n\}$ and $(u_{ij}) \in \mathbb{R}^{n \times n}$, respectively, where u_{ij} $(i, j = 1, \dots, n)$ is the payoff received by a player taking action a_i against an opponent playing action a_j. In the sequel, we represent such a game by (u_{ij}). The set of mixed strategies is identified with the $(n-1)$-dimensional simplex, denoted by Δ, which is a subset of the n-dimensional real space endowed with a norm $|\cdot|$. We say that $x^* = (x_1^*, \dots, x_n^*) \in \Delta$ is an equilibrium state if (x^*, x^*) is a Nash equilibrium, i.e., $x_i^* > 0 \Rightarrow \sum_k x_k^* u_{ik} \geq \sum_k x_k^* u_{jk}$ for all j. We denote by $[a_i]$ the element of Δ that assigns one to the ith coordinate (and zero to the others).

The above game is played repeatedly in a society with a continuum of identical anonymous agents. At each point in time, agents are matched randomly to form pairs and play the game. Each agent is replaced by his successor according to a Poisson process with parameter $\lambda > 0$. These processes are independent across the agents. Thus, during a time interval $[t, t+h)$, approximately a fraction λh of agents are replaced by the same size of entrants.[5] Each agent is entitled to choose his action only upon entry to the society, i.e., he cannot change his action once it is chosen. An interpretation of this assumption is that there exists a large switching cost.[6]

[3]See also Matsuyama (1992) and Kaneda (1995).

[4]See Hofbauer and Sorger (1999, 2002), Oyama (2002), and Oyama, Takahashi, and Hofbauer (2003) for more recent developments.

[5]There are some technical problems concerning the law of large numbers with a continuum of i.i.d. random variables, as first pointed out by Feldman and Gilles (1985) and Judd (1985). Boylan (1992), Gilboa and Matsui (1992), and Alós-Ferrer (1999) discuss these issues in the context of random matching games and offer some possible solutions.

[6]Here we follow the interpretation in Matsuyama (1991). Another interpretation, which

A path of action distribution is described by a function $\phi\colon [0,\infty) \to \Delta$, where $\phi(t) = (\phi_1(t), \dots, \phi_n(t))$ is the action distribution of the society at time t, with $\phi_i(t)$ denoting the fraction of the agents playing action a_i. The assumption of Poisson replacement motivates the following feasibility concept.

Definition 2.1. A path of action distribution $\phi\colon [0,\infty) \to \Delta$ is *feasible* if it is Lipschitz continuous with Lipschitz constant λ and satisfies the condition that for almost all t, there exists $\alpha(t) \in \Delta$ such that

$$\dot{\phi}(t) = \lambda(\alpha(t) - \phi(t)). \tag{2.1}$$

The above condition is equivalent to the condition that for all $i = 1, \dots, n$,

$$\dot{\phi}_i(t) \geq -\lambda\phi_i(t) \text{ a.e.} \tag{2.2}$$

Note, for example, that $\dot{\phi}_i(t) = -\lambda\phi_i(t)$ implies that (almost) all the entrants at time t take actions other than a_i. Note also that $\dot{\phi}_i(t) \geq -\lambda\phi_i(t)$ for all i together with $\phi(t) \in \Delta$ implies $\dot{\phi}_i(t) \leq \lambda(1 - \phi_i(t))$ for all i. The set of feasible paths is denoted by Φ^0.

An entrant anticipates a future path of action distribution and chooses an action that maximizes the expected discounted payoff. For a given anticipated path ϕ, the expected discounted payoff for an entrant at time t from taking action a_i is calculated as

$$\begin{aligned} V_i(\phi)(t) &= (\lambda+\theta)\int_0^\infty \int_t^{t+s} e^{-\theta(z-t)} \sum_{k=1}^n \phi_k(z)u_{ik}\,dz\,\lambda e^{-\lambda s}\,ds \\ &= (\lambda+\theta)\int_t^\infty e^{-(\lambda+\theta)(s-t)} \sum_{k=1}^n \phi_k(s)u_{ik}\,ds, \end{aligned}$$

where $\theta > 0$ is the common rate of time preference, while $\lambda + \theta$ is viewed as the *effective* discount rate. Note that this expression is well defined whenever $\lambda + \theta > 0$ since $\phi_k(\cdot)$ is bounded for each k. We write $V_i(\cdot) = V_i(\cdot)(0)$.

Given a feasible path ϕ, let $BR(\phi)(t)$ be the set of best responses in pure strategies to ϕ at time t, i.e.,

$$BR(\phi)(t) = \{a_i \in A \mid V_i(\phi)(t) \geq V_j(\phi)(t) \text{ for all } j\}.$$

We write $BR(\cdot) = BR(\cdot)(0)$. Note that two games (u_{ij}) and (v_{ij}) are equivalent in terms of their best response properties if there exist $\alpha > 0$ and $(w_j) \in \mathbb{R}^n$ such that $u_{ij} = \alpha v_{ij} + w_j$ holds for all i and j. The analyses below are invariant under positive affine transformations of this form.

Finally, we denote the *degree of friction* by $\delta = \theta/\lambda > 0$.

appears in Matsui and Matsuyama (1995), is that each agent lives forever and revises his action only occasionally at random points in time which follow a Poisson process with the parameter λ, and his belief may change as well when his revision opportunity arises.

3 Rationalizable Foresight and Stability Concepts

3.1 Rationalizable and Perfect Foresight Paths

The behavior pattern of the society is governed by the beliefs of the agents therein and the way they act under these beliefs. We consider a situation in which the agents form their beliefs in a *rationalizable* manner. In particular, they do not necessarily coordinate their beliefs with each other. To express this idea, we introduce the concept of *rationalizable foresight path.* For comparison, we also consider the concept of *perfect foresight path* defined in Matsui and Matsuyama (1995), which embodies the concept of equilibrium in the present dynamic framework.

Rationalizable foresight paths are defined inductively as follows. First, let Φ^0 be the set of all feasible paths, i.e., the set of Lipschitz continuous paths satisfying (2.2). Then for a given positive integer k, let Φ^k be the set of the paths in Φ^{k-1} along which every entrant, knowing the current action distribution, takes a best response to *some* path in Φ^{k-1}. Formally, define Φ^k as

$$\Phi^k = \big\{\phi \in \Phi^{k-1} \,|\, \forall\, i : \big[\dot{\phi}_i(t) > -\lambda\phi_i(t) \\ \Rightarrow \exists\, \psi \in \Phi^{k-1} : \psi(t) = \phi(t) \text{ and } a_i \in BR(\psi)(t)\big] \text{ a.e.}\big\}.$$

In this definition, $\dot{\phi}_i(t) > -\lambda\phi_i(t)$ implies that at least some positive fraction of the entire population take a_i at time t upon entry.

From this definition, it is easy to verify that $\Phi^k \subset \Phi^{k-1}$ holds and that $\Phi^k = \Phi^{k-1}$ implies $\Phi^{k+1} = \Phi^k$. Let $\Phi^* = \bigcap_{k=0}^{\infty} \Phi^k$.

Definition 3.1. A path in Φ^* is a *rationalizable foresight path.*

A path in Φ^* is rationalizable in the sense that each agent can construct an infinite hierarchy of beliefs which are consistent with the "rationality hypothesis". Note that every path that rests at a one-shot equilibrium state is always in Φ^k's, and therefore, it is in Φ^*. The existence of one-shot equilibrium states thus implies the nonemptiness of Φ^*. The following claim simply states this observation.

Claim 3.1. *If $x^* \in \Delta$ is an equilibrium state, then the path ϕ such that $\phi(t) = x^*$ for all t is a rationalizable foresight path.*

A perfect foresight path is defined to be a feasible path to which every entrant takes a best response.

Definition 3.2. A feasible path ϕ is a *perfect foresight path* if for all $i = 1, \ldots, n$ and almost all $t \geq 0$, $\dot{\phi}_i(t) > -\lambda\phi_i(t)$ implies $a_i \in BR(\phi)(t)$.

Here we state an immediate, but important observation for reference. It parallels the fact that in one-shot games, Nash equilibrium strategies are rationalizable (Bernheim (1984) and Pearce (1984)).

Claim 3.2. *A perfect foresight path is a rationalizable foresight path.*

For each initial action distribution, there exists at least one perfect (*a fortiori* rationalizable) foresight path (see, e.g., Oyama, Takahashi, and Hofbauer (2003)), while it need not be unique: in particular, there may exist a perfect (and rationalizable) foresight path that escapes even from a strict Nash equilibrium state when the degree of friction is sufficiently small, as demonstrated in Matsui and Matsuyama (1995) and others.

3.2 RF-Stability and PF-Stability

The action distribution is subject to fluctuation through, say, belief changes. Still, it is conceivable that the action distribution stays in a certain set and never leaves it no matter how beliefs may change. We introduce stability concepts under rationalizable foresight to incorporate this point and compare them with the corresponding concepts under perfect foresight.

Definition 3.3. $F \subset \Delta$ is *closed under rationalizable foresight* (resp. *perfect foresight*), if it satisfies the following properties:

(i) F is nonempty and closed; and

(ii) for any rationalizable foresight path (resp. perfect foresight path) ϕ, if $\phi(0) \in F$, then $\phi(t) \in F$ for all $t > 0$.

$F^* \subset \Delta$ is a *stable set under rationalizable foresight* (resp. *perfect foresight*), or an *RF-stable set* (resp. *PF-stable set*), if it is a minimal set that is closed under rationalizable foresight (resp. perfect foresight).

$x^* \in \Delta$ is a *stable state under rationalizable foresight* (resp. *perfect foresight*), or an *RF-stable state* (resp. *PF-stable state*) if $\{x^*\}$ is an RF-stable set (resp. PF-stable set).

The entire state space Δ is closed under rationalizable (and perfect) foresight. Therefore, the definition would be vacuous unless we require minimality. The requirement of minimality indeed refines the stability concept in the following sense. Suppose that F is closed under rationalizable foresight but not minimal (the same argument can be applied to perfect foresight). Let $\bar{F} = F \setminus \bigcup\{U(F^*) \mid F^* \text{ is an RF-stable set}\}$, where U stands for open neighborhood. Then, there exists a rationalizable foresight path that leaves $\bar{F}$ and reaches a neighborhood of some RF-stable set. For if not, we can show the existence of another RF-stable set in $\bar{F}$ in the same manner as the existence proof since $\bar{F}$ would be closed under rationalizable foresight. From RF-stable sets, on the other hand, no rationalizable foresight path reaches $\bar{F}$.

Furthermore, from any closed, proper subset F^{**} of an RF-stable set F^*, there exists a rationalizable foresight path that leaves it and reaches $F^* \setminus F^{**}$. In this sense, any RF-stable set is "connected" under rationalizable foresight paths.

Closedness in condition (i) expresses the idea that if a foresight path approaches arbitrarily close to a state, then it should be regarded as being "reached". Indeed, without closedness, our solution concepts would fail to exclude some unreasonable outcomes, due to the fact that the boundary is never reached in finite time from interior points. To see this, consider

$$(u_{ij}) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \tag{3.1}$$

and suppose $\delta = 1$. In this example, the unique RF-stable (and PF-stable) set is $\{[a_1]\}$. The set $\Delta(\{a_2, a_3\}) = \{x \in \Delta \mid x_1 = 0\}$ is neither RF-stable nor PF-stable, since there is a perfect (*a fortiori* rationalizable) foresight path from $[a_2]$ to $[a_1]$. If we removed closedness, then $int(\Delta(\{a_2, a_3\})) = \{x \in \Delta \mid x_1 = 0,\ x_2, x_3 > 0\}$ would become RF-stable as well as PF-stable. Indeed, from any point in $int(\Delta(\{a_2, a_3\}))$, there is no rationalizable (*a fortiori* perfect) foresight path that leaves the set, even though there is one that converges to $[a_2]$.

The existence of stable sets is proved in a similar manner as demonstrated in Matsui (1992) for cyclically stable sets. Let $\mathcal{F}^*$ be the family of all the sets that are closed under rationalizable foresight, which is partially ordered with respect to set inclusion. Take any totally ordered subset of $\mathcal{F}^*$, and denote it by $\{F_\alpha\}_{\alpha \in \Lambda}$. Then it has a lower bound in $\mathcal{F}^*$, since $F = \bigcap_{\alpha \in \Lambda} F_\alpha$ is also in $\mathcal{F}^*$; for if not, there exists a rationalizable foresight path that leaves F, which implies there exists some F_α that violates the condition (ii). It therefore follows from Zorn's lemma that $\mathcal{F}^*$ has a minimal element. The same logic applies to PF-stable sets.

Theorem 3.1. *Every game has at least one RF-stable set and at least one PF-stable set.*

On the other hand, RF-stable states and PF-stable states do not always exist as shown in the following game:

$$(u_{ij}) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

In this game, there are four Nash equilibrium states, $[a_1], [a_2], [a_3]$, and $\bar{x} = (1/3, 1/3, 1/3)$. From $[a_i]$ there is a perfect (and rationalizable) foresight path that converges to $[a_{i+1}]$ (mod 3), and from $\bar{x}$ there is one that converges to $[a_i]$ for all $i = 1, 2, 3$.

We next present an example in which a unique RF-stable set contains non-equilibrium states.

Example 3.1 A 2×2 Game

Consider the following 2×2 game:

$$(u_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.3}$$

which is a special case of the class of games studied in Subsection 4.2 ($b = c$, i.e., the unique equilibrium state $(\hat{x}_1, \hat{x}_2)$ is $(1/2, 1/2)$). First note that the unique PF-stable set is $\{(1/2, 1/2)\}$ for any degree of friction δ.

The unique RF-stable set, on the other hand, is written as

$$F^*(\delta) = \{(\alpha_1, \alpha_2) \in \Delta \,|\, \alpha^*(\delta) \leq \alpha_1 \leq 1 - \alpha^*(\delta)\}$$

(Fig. 1). Here, $\alpha^* = \alpha^*(\delta)$ is given by the following condition: when the current action distribution is $(\alpha^*, 1 - \alpha^*)$, new entrants are indifferent between actions a_1 and a_2 against the path ϕ^* such that the action distribution will move towards $[a_1]$ until it reaches $(1 - \alpha^*, \alpha^*)$ and then stays there forever. Then the RF-stable set F^* has the endpoints $(1 - \alpha^*, \alpha^*)$ and $(\alpha^*, 1 - \alpha^*)$, containing non-equilibrium states as well as the equilibrium state $(1/2, 1/2)$. In order to reach a non-equilibrium state, one needs a rationalizable foresight path along which agents constantly misforecast the future. Such a path can be constructed in the following manner. From a state in F^*, suppose that newborns expect that the action distribution will move towards $[a_1]$ until it reaches $(1 - \alpha^*, \alpha^*)$, and will stay there. Under such an expectation, a_2 is a best response, and therefore, the newborn will take a_2. If the new generations keep believing that the path moves towards $[a_1]$, then they keep taking a_2 until it reaches $(\alpha^*, 1 - \alpha^*)$. Beyond this state, nobody is willing to take a_2 even if he expects the action distribution to move towards $[a_1]$. Notice that along this path, agents constantly misforecast the future. By a symmetric argument, a_1 is a best response to the path that moves towards $[a_2]$ until it reaches $(\alpha^*, 1 - \alpha^*)$, and stays there. In sum, at each state in F^*, the left-moving path and the right-moving path rationalize each other.

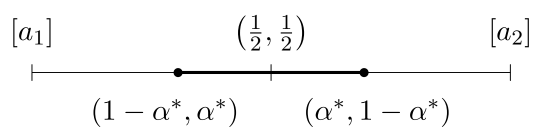


Figure 1: RF-stable set

To be precise, α^* is the unique solution to

$$\frac{1 + \delta}{2 + \delta}(1 - \alpha^*) + \frac{1}{2 + \delta} \frac{(\alpha^*)^{2+\delta}}{(1 - \alpha^*)^{1+\delta}} = \frac{1}{2}, \quad 0 < \alpha^* < \frac{1}{2}, \tag{3.4}$$

which is equivalent to $V_1(\phi^*) = V_2(\phi^*)$.

The smaller the degree of friction δ is, the larger is the RF-stable set $F^*(\delta)$, i.e., $F^*(\delta) \subset F^*(\delta')$ if $\delta > \delta'$. As δ goes to zero, $F^*(\delta)$ expands to the whole space Δ, and as δ goes to infinity, $F^*(\delta)$ shrinks to the singleton $\{(1/2, 1/2)\}$. Intuition behind this result is as follows. The farther the action distribution is away from $(1/2, 1/2)$, the larger is the instantaneous gain from taking the action that is not taken by the majority. If the degree of friction is small, the future state is relatively more important than the current state, and therefore, it is relatively easy for the action distribution to move around. On the other hand, if the degree of friction is sufficiently large, the gain from taking the action of the minority dominates any future loss, which implies that the path moves towards $(1/2, 1/2)$ without fail. □

We now discuss some properties of RF-stability in comparison with PF-stability. Since the set of rationalizable foresight paths contains the set of perfect foresight paths, an RF-stable set is closed under perfect foresight. By the same argument as in the proof of the existence of an RF-stable set, we have the following.

Theorem 3.2. (a) *Every RF-stable set contains at least one PF-stable set.*
(b) *An RF-stable state is a PF-stable state.*

The converse of Theorem 3.2(a) is not true in general, as seen in Example 3.1. The converse of Theorem 3.2(b) is not necessarily true, either, even when RF-stable sets and PF-stable sets are all singletons. Since the set of rationalizable foresight paths is larger than the set of perfect foresight paths, it is conceivable that there are some states from which the action distribution escapes under rationalizable foresight but not under perfect foresight. Using this logic, we present an example in which the rationalizable foresight dynamics yields a sharper prediction than the perfect foresight dynamics.

Example 3.2 A 3×3 Game

Consider the following 3×3 game:

$$(u_{ij}) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.5}$$

We show that for some degree of friction δ, the set of RF-stable states is a proper subset of the set of PF-stable states, and no other stable set exists.

Claim 3.3. *Let the stage game be given by* (3.5). *Then there exists a nonempty open set of δ for which*

(a) $\{[a_1]\}$ *and* $\{(0, 1/2, 1/2)\}$ *are the only PF-stable sets; and*

(b) $\{[a_1]\}$ *is the unique RF-stable set.*

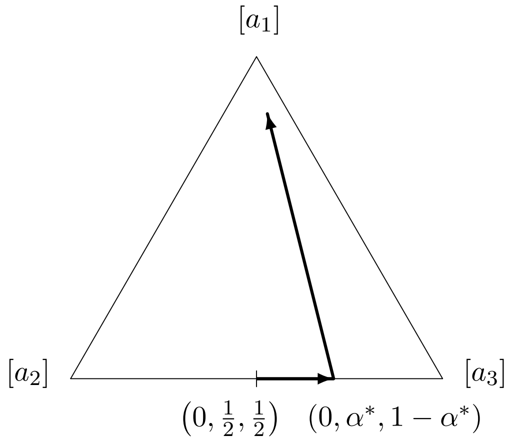


Figure 2: An escape path from $(0, 1/2, 1/2)$

Proof. See Appendix. ▮

While the formal proof is relegated to Appendix, an intuitive explanation for this statement is given below. As in Example 3.1, it can be verified that the equilibrium state $(0, 1/2, 1/2)$ is not an RF-stable state. But there might exist an RF-stable set that contains this state. A natural candidate is

$$\{(\alpha_1, \alpha_2, \alpha_3) \in \Delta \mid \alpha_1 = 0,\ \alpha^* \le \alpha_2 \le 1 - \alpha^*\},$$

where $\alpha^* \in (0, 1/2)$ is given by (3.4). We would like to show that such a set is not RF-stable for some appropriate degree of friction. Looking at the payoff matrix (3.5), one may realize that if the action distribution moves from $(0, 1/2, 1/2)$ towards $[a_3]$, it is more likely than otherwise that a_1 becomes a best response under the belief that other agents will take a_1 as well. A rationalizable foresight path performs this task (see Fig. 2), i.e., it brings the action distribution near $(0, \alpha^*, 1 - \alpha^*)$ as in Example 3.1. Once this state is reached, reasonably patient agents start taking a_1 under the new belief that others will do the same. A direct departure from $(0, 1/2, 1/2)$ is harder than that from $(0, \alpha^*, 1 - \alpha^*)$ since the agents taking a_1 incur more loss in the beginning along the path from $(0, 1/2, 1/2)$ to $[a_1]$ than along the path from $(0, \alpha^*, 1 - \alpha^*)$ to $[a_1]$. Thus, we can find an open interval of the degrees of friction for which $(0, 1/2, 1/2)$ remains PF-stable, but belongs to no RF-stable set. $\square$

3.3 Rationalizable Foresight and Rationalizability

In this subsection, we provide two characterizations of the set of rationalizable foresight paths. We also examine the relationship between static rationalizability and the rationalizable foresight dynamics.

We say that a set of feasible paths $\Phi' \subset \Phi^0$ has the *best response property* if for all $\phi \in \Phi'$ and for all i and almost all t such that $\dot{\phi}_i(t) > -\lambda\phi_i(t)$, there exists a feasible path $\psi \in \Phi'$ such that $\psi(t) = \phi(t)$ and $a_i \in BR(\psi)(t)$. We have the following.

Proposition 3.3. Φ^* *is the largest set that has the best response property.*

This proposition implies that along a feasible path, one can construct an infinite hierarchy of beliefs under which agents behave in a "rationalizable" manner if and only if this path is in Φ^*. This statement is nontrivial since $\phi \in \Phi^* = \bigcap_{k=0}^{\infty} \Phi^k$ merely implies that for all k, for all i, and for almost all t, there exists $\psi^{k-1} \in \Phi^{k-1}$ that satisfies a certain condition; that is to say, ψ^k's are different in general, and therefore, they might not be in Φ^*. In order to construct a desirable path $\psi \in \Phi^*$ by taking a subsequence of ψ^k's, we need to show that Φ^* is compact and the payoff function is continuous. The nontrivial step is to show that Φ^k's are compact with respect to an appropriate topology (compare to Bernheim (1984, Proposition 3.1) and Pearce (1984, Proposition 4)), which we demonstrate in Appendix.[7]

Proof. See Appendix. ∎

Another property of rationalizable foresight paths is that we can view a point in Δ as a state variable, so that irrespective of the past history, only the present action distribution determines a possible future course of evolution. This is obvious once we observe that the environment is stationary and that newborns' beliefs are not bound by the past history.

The above observation suggests another way of constructing Φ^*. First, define correspondences $H^0 \colon \Delta \to \Delta$ and $\Psi^0 \colon \Delta \to \Phi^0$ by

$$H^0(z) = \Delta,$$

and

$$\Psi^0(z) = \{\phi \in \Phi^0 \mid \phi(0) = z\}.$$

For $k = 1, 2, 3, \cdots$, define $H^k \colon \Delta \to \Delta$ and $\Psi^k \colon \Delta \to \Phi^0$ recursively by

$$H^k(z) = \left\{\alpha \in H^{k-1}(z) \mid \forall\, i : \left[\alpha_i > 0 \Rightarrow \exists\, \psi \in \Psi^{k-1}(z) : a_i \in BR(\psi)\right]\right\},$$

and

$$\Psi^k(z) = \{\phi \in \Psi^{k-1}(z) \mid \phi(0) = z \text{ and } \dot{\phi}(t) = \lambda(h(t) - \phi(t)), \quad h(t) \in H^k(\phi(t)) \ \text{ a.e.}\}.$$

[7] As in Pearce (1984, Proposition 4), one can show that for a given $z \in \Delta$, the iterative procedure of constructing $H^k(z)$'s, defined below, eventually "stops" in a finite game, i.e., for some $K = K(z)$, $H^k(z) = H^K(z)$ for all $k \geq K$. However, this is not the case for Φ^k, since K above can be varied across z, as seen in the proof of Proposition 4.3.

Finally, define $H^*\colon \Delta \to \Delta$ by

$$H^*(z) = \bigcap_{k=0}^{\infty} H^k(z).$$

We immediately have the following.

Proposition 3.4. *A feasible path ϕ is a rationalizable foresight path with initial state $x^0 \in \Delta$ if and only if $\phi(0) = x^0$, and*

$$\dot{\phi}(t) = \lambda(h(t) - \phi(t)), \quad h(t) \in H^*(\phi(t)) \quad a.e. \tag{3.6}$$

For each $z \in \Delta$, $H^*(z)$ is nonempty and compact, since $H^k(z)$'s are nonempty and compact, and $H^{k+1}(z) \subset H^k(z)$ holds. Moreover, H^* is upper semi-continuous and convex-valued as H^k's are. Therefore, by the existence theorem for differential inclusions (see, e.g., Theorem 2.1.4 in Aubin and Cellina (1984, p. 101)), for each $x^0 \in \Delta$, there exists at least one rationalizable foresight path ϕ with $\phi(0) = x^0$.

Now, we examine the relationship between static rationalizability and the rationalizable foresight dynamics. For this purpose, we allow, for a moment, the degree of friction $\delta = \theta/\lambda$ to be negative, while keeping $\lambda > 0$. Note that the dynamics is well-defined if the effective discount rate $\lambda + \theta$ is positive, i.e., $\delta > -1$. We show that as the effective discount rate goes to zero, i.e., as δ goes to -1, the RF-stable set becomes unique and coincides with the set of rationalizable action distributions.

Given a symmetric game (u_{ij}), we consider the static societal game in which a continuum of agents are randomly matched to play (u_{ij}) once and for all. We construct inductively the set of rationalizable action distributions analogously to the construction of H^* above.

Let $\bar{H}^0 = \Delta$. Given $\bar{H}^{k-1} \subset \Delta$ $(k = 1, 2, 3, \ldots)$, define $\bar{H}^k \subset \Delta$ to be

$$\bar{H}^k = \left\{x \in \bar{H}^{k-1} \,|\, \forall\, i : \left[x_i > 0 \Rightarrow \exists\, y \in \bar{H}^{k-1} : a_i \in \overline{BR}(y)\right]\right\},$$

where $\overline{BR}(y)$ is the set of one-shot best responses to y in pure strategies, i.e.,

$$\overline{BR}(y) = \left\{a_i \in A \;\middle|\; \sum_{k=1}^{n} y_k u_{ik} \geq \sum_{k=1}^{n} y_k u_{jk} \text{ for all } j\right\}.$$

Let $\bar{H}^* = \bigcap_{k=0}^{\infty} \bar{H}^k$, which is nonempty and closed.

Definition 3.4. An action distribution in $\bar{H}^*$ is a *rationalizable action distribution in the static societal game.*

Note that an action distribution is rationalizable in the static societal game if and only if every pure action in the support survives iterated strict dominance. Note also that $\bar{H}^*$ is the convex hull of the pure rationalizable action distributions.

We now have the following result.[8]

Proposition 3.5. *For all generic (symmetric) games, there exists $\bar{\delta} > -1$ such that for all $\delta \in (-1, \bar{\delta})$, an RF-stable set uniquely exists and coincides with $\bar{H}^*$.*

Proof. See Appendix. ∎

Consider an agent anticipating a path ψ that moves from x to y. If δ is close to -1, then he puts almost all weight on the distant future, i.e., $V_i(\psi)$ is approximated by $\sum_k y_k u_{ik}$. The proof of the proposition essentially utilizes this observation. Note that as δ goes to infinity, $V_i(\psi)$ converges to $\sum_k x_k u_{ik}$.

Recall that along a rationalizable foresight path, each agent believes that a single path of action distribution will realize with probability one, and chooses a pure action. Still, different actions can be observed in the society since there are a continuum of agents who may entertain different beliefs. In this way, mixed strategies and mixed beliefs in the standard rationalizability (Bernheim (1984) and Pearce (1984)) are replaced by the population distributions of actions and beliefs.

Note that our definition of rationalizable action distribution is different from the standard definition of rationalizable strategy. In the standard definition, $\bar{H}^k$'s are replaced by $\hat{H}^0 = \Delta$ and

$$\hat{H}^k = \big\{x \in \hat{H}^{k-1} \,|\, \exists\, y \in \operatorname{co} \hat{H}^{k-1}\, \forall\, i : \big[x_i > 0 \Rightarrow a_i \in \overline{BR}(y)\big]\big\}$$

for $k \geq 1$, where the symbol "co" stands for convex hull. Then the set of rationalizable strategies is given by $\hat{H}^* = \bigcap_{k=0}^{\infty} \hat{H}^k$. Since $\bar{H}^k$ is convex, we have $\hat{H}^k \subset \bar{H}^k$. Because of the difference between $\bar{H}^k$ and $\hat{H}^k$, the set of rationalizable action distributions in a large population may not be the same as the set of the standard rationalizable strategies. Let us consider the game:

$$(u_{ij}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 2 & 2 \end{pmatrix}. \tag{3.7}$$

The mixed strategy, or action distribution, $(1/2, 1/2, 0)$ is a rationalizable action distribution in the societal game, but not a standard rationalizable strategy. In the societal game, it is rationalizable since the half of the population believe that action a_1 will be chosen, while the other half believe that action a_2 will be chosen. On the other hand, it is not rationalizable in the standard definition since there is no mixed strategy to which $(1/2, 1/2, 0)$ is a best response.

[8] We say that a certain property holds for all generic (symmetric) games if the Lebesgue measure of the set of payoff matrices that do not satisfy this property is zero.

4 Complete Characterization for Symmetric 2×2 Games

This section completely characterizes RF-stable sets for the class of symmetric games with two actions:

$$(u_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.1}$$

If one action weakly dominates the other, the state in which every agent takes the dominant action constitutes the unique RF-stable set as well as the unique PF-stable set. Also, if $a = c$ and $b = d$ hold, then the entire space becomes a stable set under either dynamics.

There are two nontrivial cases to consider: (i) $a > c$ and $d > b$, i.e., coordination games; and (ii) $a < c$ and $d < b$, i.e., games with a unique symmetric Nash equilibrium.

4.1 Coordination Games

This subsection studies coordination games. In this case, (4.1) can be normalized without loss of generality to

$$(u_{ij}) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad a \geq d > 0. \tag{4.2}$$

Let $\mu = d/(a + d)$, where $\mu \leq 1/2$.

Proposition 4.1. *Let the stage game be given by* (4.2). *Then we have the following:*

(a) *If $\delta > (1 - 2\mu)/\mu$, then $\{[a_1]\}$ and $\{[a_2]\}$ are RF-stable sets, and no other RF-stable set exists.*

(b) *If $0 < \delta \leq (1 - 2\mu)/\mu$, then $\{[a_1]\}$ is the unique RF-stable set.*

If δ is large, both strict Nash equilibrium states are RF-stable. If δ is sufficiently small, then the rationalizable foresight dynamics selects $[a_1]$, the risk-dominant equilibrium, over $[a_2]$ provided that $\mu < 1/2$.

Proposition 4.1 is a special case of Proposition 4.2 below, a result for $n \times n$ games with (symmetric) pure Nash equilibria. To state the result, we review the notion of p-dominance (Morris, Rob, and Shin (1995)), which is a generalization of risk-dominance for games with more than two actions.[9]

[9] In symmetric 2×2 games, (a_i, a_i) is a p-dominant equilibrium for some $p < 1/2$ if and only if it is a risk-dominant equilibrium.

Definition 4.1. Action profile (a_i, a_i) is a *p-dominant equilibrium*[10] of symmetric $n \times n$ game (u_{ij}) if for all $\pi \in \Delta$ with $\pi_i > p$,

$$\sum_{k=1}^{n} \pi_k u_{ik} > \sum_{k=1}^{n} \pi_k u_{jk}$$

holds for all $j \neq i$.

Observe that in game (4.2), (a_1, a_1) is μ-dominant.

Proposition 4.2. *Suppose that (a_i, a_i) is a p-dominant equilibrium of the stage game.*

(a) *If $p < (1+\delta)/(2+\delta)$, then $\{[a_i]\}$ is an RF-stable set.*

(b) *If $p \leq 1/(2+\delta)$, then $\{[a_i]\}$ is the unique RF-stable set.*

The condition in (a) assures that there is no rationalizable foresight path away from $[a_i]$, while the condition in (b) implies that from any state in Δ, there is a rationalizable foresight path convergent to $[a_i]$. It is straightforward to see that the proof of Lemma 2 in Oyama (2002) in fact shows (a), while (b) follows from Lemma 1 in Oyama (2002) (and Claim 3.2) as well as (a).

4.2 Games with a Unique Symmetric Equilibrium

This subsection considers the case where $a < c$ and $d < b$. In this case, (4.1) can be normalized without loss of generality to

$$(u_{ij}) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \quad 0 < b \leq c. \tag{4.3}$$

Denote by $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \Delta$ the unique equilibrium state, i.e.,

$$(\hat{x}_1, \hat{x}_2) = \left(\frac{b}{b+c}, \frac{c}{b+c} \right).$$

Note that $0 < \hat{x}_1 \leq 1/2 \leq \hat{x}_2 < 1$.

First we note that $\{\hat{x}\}$ is the unique PF-stable set for any degree of friction. This follows from the facts that from any state in Δ, there exists a perfect foresight path that reaches $\hat{x}$ (independently of the friction), and that there exists no perfect foresight path that escapes from $\hat{x}$, which can be proved in the same way as the proof of Lemma A.2 in Appendix.

It therefore follows from Theorem 3.2 that there is a unique RF-stable set, which is of the form:

$$\{(x_1, x_2) \in \Delta \mid \hat{\alpha} \leq x_1 \leq 1 - \hat{\beta}\}$$

[10] This is called a *strict* (p, p)-dominant equilibrium in Kajii and Morris (1997, Definition 5.4).

with $\hat{\alpha} \leq \hat{x}_1$ and $\hat{\beta} \leq \hat{x}_2$. Our task is to determine the endpoints of this set, $(\hat{\alpha}, 1-\hat{\alpha})$ and $(1-\hat{\beta}, \hat{\beta})$. The state $(\hat{\alpha}, 1-\hat{\alpha})$ (resp. $(1-\hat{\beta}, \hat{\beta})$) is the closest to $[a_2]$ (resp. $[a_1]$) such that a_2 (resp. a_1) is a best response to the path that starts there, moves towards to $[a_1]$ (resp. $[a_2]$) at the maximum speed, and stays at $(1-\hat{\beta}, \hat{\beta})$ (resp. $(\hat{\alpha}, 1-\hat{\alpha})$) once reached.

For this purpose, let P and Q be two functions from $[0, \hat{x}_1] \times [0, \hat{x}_2]$ into $\mathbb{R}$ defined as:

$$\begin{aligned} P(\alpha, \beta) &= \frac{1+\delta}{2+\delta}(1-\alpha)^{2+\delta} + \frac{1}{2+\delta}\beta^{2+\delta} - \hat{x}_2(1-\alpha)^{1+\delta}, \\ Q(\alpha, \beta) &= \frac{1+\delta}{2+\delta}(1-\beta)^{2+\delta} + \frac{1}{2+\delta}\alpha^{2+\delta} - \hat{x}_1(1-\beta)^{1+\delta}. \end{aligned}$$

The sign of $P(\alpha, \beta)$ is identical with that of the payoff difference $V_1(\phi) - V_2(\phi)$ along the path ϕ that starts with $(\alpha, 1-\alpha)$, moves towards $[a_1]$ at the maximum speed, and stays at $(1-\beta, \beta)$ once reached. Similarly, $Q(\alpha, \beta)$ has the same sign as the payoff difference $V_2(\psi) - V_1(\psi)$ where ψ is the path that starts with $(1-\beta, \beta)$, moves towards $[a_2]$ at the maximum speed, and stays at $(\alpha, 1-\alpha)$ once reached.

Given $\delta > 0$, let $(\alpha^\star(\delta), \beta^\star(\delta)) \in (0, \hat{x}_1) \times (0, \hat{x}_2)$ be the unique solution to:

$$\begin{aligned} P(\alpha^\star(\delta), \beta^\star(\delta)) &= 0, \\ Q(\alpha^\star(\delta), \beta^\star(\delta)) &= 0, \end{aligned} \tag{4.4}$$

if it exists (see Fig. 3(a)).[11] If it does not, then let $\alpha^\star(\delta) = 0$ and solve the following system (see Fig. 3(b)):

$$\begin{aligned} P(0, \beta^\star(\delta)) &\leq 0, \\ Q(0, \beta^\star(\delta)) &= 0. \end{aligned} \tag{4.5}$$

In the latter case, we have a unique solution $\beta^\star(\delta) = 1 - \hat{x}_1(2+\delta)/(1+\delta) \in (0, \hat{x}_2)$ to (4.5).

[11] It can be verified that $P(\hat{x}_1, \hat{x}_2) = Q(\hat{x}_1, \hat{x}_2) = 0$ holds. We also have

$$\frac{\partial P}{\partial \alpha} = -(1+\delta)(1-\alpha)^\delta(\hat{x}_1 - \alpha),$$

which is negative for $\alpha \in (0, \hat{x}_1)$, and

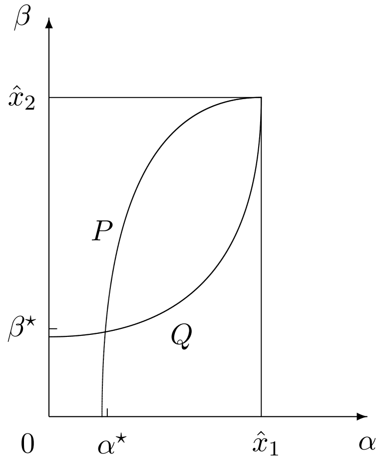
$$\frac{\partial P}{\partial \beta} = \beta^{1+\delta} > 0.$$

Let $\hat{\alpha}(\beta)$ satisfy $P(\hat{\alpha}(\beta), \beta) = 0$. Then, one can verify that $\hat{\alpha}$ is well defined for $\beta \in (0, \hat{x}_2)$, that

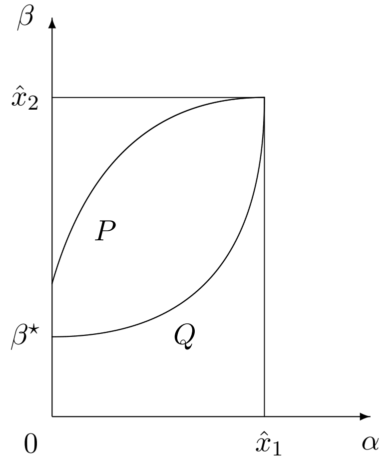
$$\frac{d\hat{\alpha}}{d\beta} = -\frac{(\partial P/\partial \beta)(\hat{\alpha}(\beta), \beta)}{(\partial P/\partial \alpha)(\hat{\alpha}(\beta), \beta)}$$

is positive, and that $\hat{\alpha}(\beta)$ is convex in β with $\hat{\alpha}'(0) = 0$ and $\lim_{\beta \to \hat{x}_2} \hat{\alpha}'(\beta) = \infty$.

Similarly, let $\hat{\beta}(\alpha)$ satisfy $Q(\alpha, \hat{\beta}(\alpha)) = 0$. In a similar manner, one can verify that $\hat{\beta}$ is well-defined, and that it is increasing and convex in α with $\hat{\beta}'(0) = 0$ and $\lim_{\alpha \to \hat{x}_1} \hat{\beta}'(\alpha) = \infty$.



(a) $\delta > \delta^\star$



(b) $\delta < \delta^\star$

Figure 3: Graphs of $P(\alpha, \beta) = 0$ and $Q(\alpha, \beta) = 0$

Let $\delta^\star \in [0, \infty)$ be the unique solution to

$$P\left(0, 1 - \hat{x}_1 \frac{2 + \delta^\star}{1 + \delta^\star}\right) = 0. \tag{4.6}$$

Indeed, it can be verified that $P(0, 1 - \hat{x}_1(2+\delta)/(1+\delta))$ is strictly increasing in δ, and has a nonpositive value, $-2(\hat{x}_2 - 1/2)(1 - \hat{x}_2)$, at $\delta = 0$, and a positive value, $1 - \hat{x}_2$, in the limit as $\delta \to \infty$. Observe that (4.4) has the unique solution in $(0, \hat{x}_1) \times (0, \hat{x}_2)$ if and only if $\delta > \delta^\star$. Observe also that $\delta^\star = 0$ if and only if $b = c$, i.e., the case considered in Example 3.1: in this case, $\alpha^\star(\delta) = \beta^\star(\delta) = \alpha^*(\delta)$.

Proposition 4.3. *Let the stage game be given by* (4.3). *Then we have the following:*

(a) *The unique RF-stable set is*

$$F^*(\delta) = \{(x_1, x_2) \in \Delta \,|\, \alpha^\star(\delta) \le x_1 \le 1 - \beta^\star(\delta)\}.$$

(b) (i) $\alpha^\star(\delta)$ *is increasing in* $\delta \in (\delta^\star, \infty)$ *and* $1 - \beta^\star(\delta)$ *is decreasing in* $\delta \in (0, \infty)$. (ii) $\alpha^\star(\delta) \searrow 0$ *as* $\delta \searrow \delta^\star$ *and* $1 - \beta^\star(\delta) \nearrow 2\hat{x}_1$ *as* $\delta \searrow 0$. (iii) $\alpha^\star(\delta) \nearrow \hat{x}_1$ *and* $1 - \beta^\star(\delta) \searrow \hat{x}_1$ *as* $\delta \nearrow \infty$.

Proof. See Appendix. ∎

As in Example 3.1, the RF-stable set $F^*(\delta)$ always contains non-equilibrium states, and one needs a rationalizable foresight path that is not a perfect foresight path in order to escape from the equilibrium state $\hat{x}$. The comparative statics result (b) shows that as the degree of friction δ goes to zero, $F^*(\delta)$ expands to $\{(x_1, x_2) \in \Delta \,|\, 0 \le x_1 \le 2\hat{x}_1\}$, whose midpoint is $\hat{x}$, while as δ goes to infinity, $F^*(\delta)$ collapses to the singleton $\{\hat{x}\}$.

5 Conclusion

We have proposed the rationalizable foresight dynamics and defined the stability concepts under the dynamics. We have then discussed its properties, including the existence of stable sets. By introducing the concept of rationalizable foresight, we have abandoned the requirement that agents' beliefs be coordinated as assumed in equilibrium theory. By incorporating inertia in decision making into the model, we have mitigated the poor performance of the theory of rationalizability as prediction device.

We have illustrated by an example that RF-stability can provide a sharper prediction than PF-stability. A key observation for this result is that in general, it is easier to escape from an action distribution under rationalizable foresight than under perfect foresight. Accordingly, there may exist a state from which a rationalizable foresight path escapes but no perfect foresight path does.

In our analysis, inertia plays a key role. If there is no inertia, then the behavior pattern may jump around, and there is no hope for providing sharp predictions. Indeed, in case where the effective discount rate $\lambda+\theta > 0$ is close to zero (i.e., the degree of friction δ is close to -1), the unique RF-stable set coincides with the set of rationalizable action distributions of the corresponding static societal game. Note here that mixed strategies and mixed beliefs in the standard rationalizability are replaced by the distributions of actions and beliefs.

We have limited our analysis to a special class of dynamic environments since our aim is to present a conceptual framework as opposed to providing a universal framework. How the present analysis can be extended to general situations remains to be seen in the future.

Appendix

A.1 Proof of Claim 3.3

In order to prove Claim 3.3, we need a few lemmata, which are given below.

Lemma A.1. *If $\delta > 1$, then $[a_1]$ is both PF-stable and RF-stable.*

Proof. This lemma follows from Proposition 4.2. ▮

Lemma A.2. *Any perfect foresight path ϕ with $\phi_2(0) = \phi_3(0)$ satisfies $\phi_2(t) = \phi_3(t)$ for all $t > 0$.*

Proof. Take any perfect foresight path ϕ with $\phi_2(0) = \phi_3(0)$. We suppose that $\phi_2(t^0) < \phi_3(t^0)$ for some $t^0 > 0$. Define $\underline{t}$ to be

$$\underline{t} = \inf\{t < t^0 \,|\, \forall\, s \in (t, t^0) : \phi_2(s) < \phi_3(s)\}.$$

Note that $\underline{t} < t^0$ and $\phi_2(\underline{t}) = \phi_3(\underline{t})$ due to the continuity of the perfect foresight path.

Claim A.1. There exists $t \in (\underline{t}, t^0)$ such that $V_2(t) \leq V_3(t)$.
If $V_2(t) > V_3(t)$ for all $t \in (\underline{t}, t^0)$, then

$$\phi_2(t^0) \geq \phi_2(\underline{t})\, e^{-\lambda(t-\underline{t})},$$
$$\phi_3(t^0) = \phi_3(\underline{t})\, e^{-\lambda(t-\underline{t})},$$

implying that $\phi_2(t^0) \geq \phi_3(t^0)$. This contradicts the definition of t^0, completing the proof of Claim A.1.

We denote by T^1 such a t in Claim A.1, i.e., $T^1 \in (\underline{t}, t^0)$, and

$$V_2(T^1) - V_3(T^1) \leq 0. \tag{A.1}$$

Claim A.2. There exists $t > t^0$ such that $\phi_2(t) \geq \phi_3(t)$.
Suppose the contrary. Then $\phi_2(t) < \phi_3(t)$ for all $t > T^1$. It follows that

$$V_2(T^1) - V_3(T^1) = (\lambda + \theta) \int_{T^1}^{\infty} e^{-(\lambda+\theta)(s-T^1)} \left(\phi_3(s) - \phi_2(s)\right) ds > 0.$$

This contradicts (A.1), completing the proof of Claim A.2.

Define $\bar{t}\,(> t^0)$ to be

$$\bar{t} = \sup\{t > t^0 \,|\, \forall\, s \in (t^0, t) : \phi_2(s) < \phi_3(s)\},$$

which is finite due to Claim A.2. Note again that $\phi_2(\bar{t}) = \phi_3(\bar{t})$.

Claim A.3. There exists $t \in (t^0, \bar{t})$ such that $V_2(t) \geq V_3(t)$.
If $V_2(t) < V_3(t)$ for all $t \in (t^0, \bar{t})$, then

$$\phi_2(\bar{t}) = \phi_2(t^0)\, e^{-\lambda(\bar{t}-t^0)},$$
$$\phi_3(\bar{t}) \geq \phi_3(t^0)\, e^{-\lambda(\bar{t}-t^0)},$$

implying that $\phi_2(t^0) \geq \phi_3(t^0)$. This contradicts the definition of t^0, completing the proof of Claim A.3.

We denote by T^2 such a t in Claim A.3, i.e., $T^2 \in (t^0, \bar{t})$, and

$$V_2(T^2) - V_3(T^2) \geq 0. \tag{A.2}$$

Since $\phi_2(t) < \phi_3(t)$ for all $t \in (T^1, T^2)$,

$$\begin{aligned}
V_2(T^1) - V_3(T^1) &= (\lambda + \theta) \int_{T^1}^{\infty} e^{-(\lambda+\theta)(s-T^1)} \left(\phi_3(s) - \phi_2(s)\right) ds \\
&= (\lambda + \theta) \int_{T^1}^{T^2} e^{-(\lambda+\theta)(s-T^1)} \left(\phi_3(s) - \phi_2(s)\right) ds \\
&\qquad + e^{-(\lambda+\theta)(T^2-T^1)}(V_2(T^2) - V_3(T^2)) \\
&> e^{-(\lambda+\theta)(T^2-T^1)}(V_2(T^2) - V_3(T^2)) \geq 0,
\end{aligned}$$

where the last inequality follows from (A.2). This contradicts (A.1). ∎

Lemma A.3. *If $\delta > 1$, then $(0, 1/2, 1/2)$ is a PF-stable state.*

Proof. Take any perfect foresight path ϕ with $\phi(0) = (0, 1/2, 1/2)$. Due to Lemma A.2, it must satisfy $\phi_2(t) = \phi_3(t)$ for all t. Hence,

$$\begin{aligned} V_1(\phi) &= (\lambda+\theta) \int_0^\infty e^{-(\lambda+\theta)s}\, \phi_1(s)\, ds \\ &\leq (\lambda+\theta) \int_0^\infty e^{-(\lambda+\theta)s}\, (1 - e^{-\lambda s})\, ds = \frac{1}{2+\delta}, \end{aligned}$$

and

$$\begin{aligned} V_2(\phi) = V_3(\phi) &= (\lambda+\theta) \int_0^\infty e^{-(\lambda+\theta)s}\, \phi_2(s)\, ds \\ &\geq (\lambda+\theta) \int_0^\infty e^{-(\lambda+\theta)s}\, \frac{1}{2}\, e^{-\lambda s}\, ds = \frac{1}{2} \cdot \frac{1+\delta}{2+\delta}, \end{aligned}$$

so that $V_1(\phi) < V_2(\phi) = V_3(\phi)$ for $\delta > 1$. It follows that $\phi_1(t) = 0$ and, therefore, $\phi_2(t) = \phi_3(t) = 1/2$. ∎

Lemma A.4. *There exists $\bar{\delta} > 1$ such that if $1 < \delta \leq \bar{\delta}$, then no RF-stable set contains $(0, 1/2, 1/2)$.*

Proof. Following the proof of Proposition 4.3, we can verify that there is a rationalizable foresight path from $(0, 1/2, 1/2)$ to $(0, \alpha^*(\delta), 1-\alpha^*(\delta))$. It is therefore sufficient to show that the linear path ϕ from $(0, \alpha^*(\delta), 1-\alpha^*(\delta))$ to $[a_1]$ is a rationalizable foresight path for a $\delta > 1$ sufficiently close to 1. Here, $\alpha^* = \alpha^*(\delta)$ is given by $g(\alpha^*, \delta) = 0$, where

$$g(\alpha, \delta) = \frac{1+\delta}{2+\delta}\,(1-\alpha) + \frac{1}{2+\delta}\, \frac{(\alpha)^{2+\delta}}{(1-\alpha)^{1+\delta}} - \frac{1}{2}, \quad 0 < \alpha < \frac{1}{2}. \tag{A.3}$$

One can verify that $g(\alpha, \delta) < 0$ if and only if $\alpha^* < \alpha < 1/2$. Along the path ϕ,

$$\begin{aligned} V_1(\phi) &= 1 - 2\alpha^*\, \frac{1+\delta}{2+\delta}, \\ V_2(\phi) &= (1-\alpha^*)\, \frac{1+\delta}{2+\delta}, \\ V_3(\phi) &= \alpha^*\, \frac{1+\delta}{2+\delta}. \end{aligned}$$

It is sufficient to demonstrate that there exists $\bar{\delta}$ such that if $1 < \delta \leq \bar{\delta}$, then $V_1(\phi) \geq V_2(\phi)$, or equivalently,

$$\alpha^* \leq \frac{1}{1+\delta}. \tag{A.4}$$

We find the range of $\delta\,(>1)$ such that $g(1/(1+\delta),\delta) \le 0$. Since

$$g\left(\frac{1}{1+\delta},\delta\right) = -\frac{1}{(1+\delta)(2+\delta)}\left\{\frac{(1+\delta)(2-\delta)}{2} - \delta^{-(1+\delta)}\right\},$$

$g(1/(1+\delta),\delta) \le 0$ if and only if

$$(1+\delta)(2-\delta)\delta^{1+\delta} - 2 \ge 0.$$

Write

$$h(\delta) = (1+\delta)(2-\delta)\delta^{1+\delta} - 2.$$

Then, we have

$$h'(\delta) = \delta^{\delta}\{\delta(1+\delta)(2-\delta)\log\delta - (-2-4\delta+2\delta^2+\delta^3)\}. \tag{A.5}$$

Since $h(1) = 0$ and $h'(1) > 0$, there exists $\bar{\delta} > 1$ such that for all $\delta \in (1,\bar{\delta}]$, $h(\delta) \ge 0$, i.e., $g(1/(1+\delta),\delta) \le 0$. Thus, for such a δ, the linear path from $(0,\alpha^*(\delta),1-\alpha^*(\delta))$ to $[a_1]$ is a rationalizable foresight path. From Lemma A.1, no RF-stable set contains $(0,1/2,1/2)$. ▮

Proof of Claim 3.3. It can be verified that for any given δ and for any $x^0 \in \Delta$, there exists a perfect foresight path that starts from x^0 and converges to either one of $[a_1]$ and $(0,1/2,1/2)$. Combining Lemmata A.1–A.4 with this observation completes the proof of the Claim. ▮

A.2 Proof of Proposition 3.3

We introduce a Banach space X, the set of bounded functions $f\colon [0,\infty) \to \mathbb{R}^n$ with the norm

$$\|f\|_r = \sup_{t\ge 0} e^{-rt}|f(t)|$$

for $r > 0$.

Lemma A.5. *$\Phi^0 \subset X$ is compact.*

Proof. Observe first that due to the Ascoli-Arzelà theorem,

$$K = \{\phi\colon [0,\infty) \to \Delta \subset \mathbb{R}^n \mid \phi \text{ is Lipschitz with constant } \lambda\}$$

is a compact subset of X. Thus, it is sufficient to show that Φ^0, which is a subset of K, is closed.

Take a sequence $\{\phi^m\}$ such that $\phi^m \in \Phi^0$ for all m, and assume $\phi^m \to \phi$. Suppose that there exist i and t such that

$$\dot{\phi}_i(t) < -\lambda\phi_i(t).$$

Then, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$, $(\phi_i(t+\varepsilon) - \phi_i(t))/\varepsilon < -\lambda\phi_i(t)$. It follows that for a sufficiently large m, $(\phi_i^m(t+\varepsilon) - \phi_i^m(t))/\varepsilon < -\lambda\phi_i^m(t)$, or

$$\phi_i^m(t+\varepsilon) < \phi_i^m(t)(1-\lambda\varepsilon). \tag{A.6}$$

On the other hand, since $\dot{\phi}_i^m(s) \geq -\lambda\phi_i^m(s)$ holds for any s and any m, we have

$$\begin{aligned} \phi_i^m(t+\varepsilon) &\geq \phi_i^m(t)\, e^{-\lambda\varepsilon} \\ &> \phi_i^m(t)(1-\lambda\varepsilon), \end{aligned}$$

which contradicts (A.6). ▮

Lemma A.6. *For all k, Φ^k is closed.*

Proof. First, by Lemma A.5, Φ^0 is closed. Suppose next that Φ^{k-1} is closed. Let $\{\phi^m\}$ be such that $\phi^m \in \Phi^k$ for all m, and assume $\phi^m \to \phi$. Take any i and t such that $\dot{\phi}_i(t) > -\lambda\phi_i(t)$. Observe that for any $\varepsilon > 0$, there exists M such that for all $m \geq M$,

$$\dot{\phi}_i^m(t^m) > -\lambda\phi_i^m(t^m)$$

holds for some $t^m \in (t-\varepsilon, t+\varepsilon)$. Take a sequence $\{\varepsilon^\ell\}$ such that $\varepsilon^\ell > 0$ and $\varepsilon^\ell \to 0$. Then we can take a subsequence $\{\phi^{m_\ell}\}$ of $\{\phi^m\}$ such that $\dot{\phi}_i^{m_\ell}(t^\ell) > -\lambda\phi_i^{m_\ell}(t^\ell)$ holds for some $t^\ell \in (t-\varepsilon^\ell, t+\varepsilon^\ell)$. For each ϕ^{m_ℓ}, since it is contained in Φ^k, there exists $\psi^\ell \in \Phi^{k-1}$ such that $\psi^\ell(t^\ell) = \phi^{m_\ell}(t^\ell)$ and $a_i \in BR(\psi^\ell)(t^\ell)$. Since Φ^0 is compact and, by the hypothesis, Φ^{k-1} is closed, a subsequence (again denoted by) ψ^ℓ converges to some $\psi \in \Phi^{k-1}$, which satisfies $\psi(t) = \phi(t)$. Moreover, since the payoff $V(\cdot)(\cdot)$ is continuous, and hence, $BR(\cdot)(\cdot)$ is upper semi-continuous, we have $a_i \in BR(\psi)(t)$, so that $\phi \in \Phi^k$. ▮

Proof of Proposition 3.3. We first show that Φ^* has the best response property. Take any $\phi \in \Phi^*$, and any i and t such that $\dot{\phi}_i(t) > -\lambda\phi_i(t)$. Since $\phi \in \Phi^k$ for all k, we can take a sequence $\{\psi^k\}$ with $\psi^k \in \Phi^k\,(\subset \Phi^0)$ such that $\psi^k(t) = \phi(t)$ and $a_i \in BR(\psi^k)(t)$. Since Φ^0 is compact due to Lemma A.5, a subsequence (again denoted by) ψ^k converges to some $\psi \in \Phi^0$ with $\psi(t) = \phi(t)$. For each k, $\{\psi^{k'}\}_{k' \geq k}$ is contained in Φ^k, so that the limit ψ is in Φ^k since Φ^k is closed due to Lemma A.6. Therefore, $\psi \in \Phi^*\ (= \bigcap_{k=0}^{\infty} \Phi^k)$. Moreover, due to the upper semi-continuity of BR, we have $a_i \in BR(\psi)(t)$.

Second, it is straightforward to observe that if Φ' has the best response property, then $\Phi' \subset \Phi^k$ for all k, and hence, $\Phi' \subset \Phi^*$. ▮

A.3 Proof of Proposition 3.5

Proof. In order to show that $\bar{H}^*$ is the unique RF-stable set, it suffices to verify that if δ is sufficiently close to -1, then for any two action distributions in $\bar{H}^*$, from each of the two distributions there exists a rationalizable foresight path that converges to the other. It is easy to see that no distribution outside $\bar{H}^*$ is contained in any RF-stable set.

Take any $[a_i] \in \bar{H}^*$. Then, we can take an action distribution $z^i \in \bar{H}^*$ such that $\overline{BR}(z^i) = \{a_i\}$, due to the genericity of the payoffs. Take any $w \in \Delta$, and consider the linear path ψ^i from w to z^i given by $\psi^i(t) = (1 - e^{-\lambda t})z^i + e^{-\lambda t}w$, $t \geq 0$. As δ goes to -1, $V_j(\psi^i)$ converges to $\sum_k z^i_k u_{jk}$. Therefore, by way of the choice of z^i, $v_i(w|\delta) = V_i(\psi^i) - \max_{j \neq i} V_j(\psi^i)$ converges to a positive number as δ goes to -1. Since the set of such functions $\{v_i(\cdot|\delta)\}_{i,\delta}$ is equicontinuous and each function is defined on a compact set, there exists $\bar{\delta} > -1$ such that for all $\delta \in (-1, \bar{\delta})$, $v_i(w|\delta) > 0$ holds for all $[a_i] \in \bar{H}^*$ and all $w \in \Delta$. Take such a δ.

Take any two $x, y \in \bar{H}^*$. Consider the linear path from x to y: $\phi(t) = (1-e^{-\lambda t})y + e^{-\lambda t}x$. We show that any such path is a rationalizable foresight path. For each t and each i with $y_i > 0$, let $\psi^{i,t}$ be the path given by $\psi^{i,t}(\tau) = (1 - e^{-\lambda\tau})z^i + e^{-\lambda\tau}\phi(t)$, where z^i satisfies $\{a_i\} = \overline{BR}(z^i)$. By way of the choice of δ, $a_i \in BR(\psi^{i,t})(t)$. Thus, ϕ is in Φ^1. Since ϕ is the linear path from x to y where x and y are arbitrarily chosen, every such path is in Φ^1. Repeating this procedure, we establish that every linear path connecting two distributions in $\bar{H}^*$ is in Φ^k for all k, and hence, in Φ^*, which implies that the unique RF-stable set is identical with $\bar{H}^*$. ∎

A.4 Proof of Proposition 4.3

Proof. (a) We would like to show that $H^*(\cdot)$ is given by

$$H^*(z) = \begin{cases} \{[a_1]\} & \text{if } z_1 < \alpha^\star, \\ \Delta & \text{if } \alpha^\star \leq z_1 \leq 1 - \beta^\star, \\ \{[a_2]\} & \text{if } z_1 > 1 - \beta^\star. \end{cases}$$

For this purpose, it suffices to show that

$$H^k(z) = \begin{cases} \{[a_1]\} & \text{if } z_1 < \alpha^k, \\ \Delta & \text{if } \alpha^k \leq z_1 \leq 1 - \beta^k, \\ \{[a_2]\} & \text{if } z_1 > 1 - \beta^k \end{cases} \tag{A.7}$$

holds for all $k = 0, 1, 2, \ldots$, with sequences $\{\alpha^k\}_{k=0}^\infty$ and $\{\beta^k\}_{k=0}^\infty$ such that $0 = \alpha^0 \leq \alpha^1 \leq \cdots < \hat{x}_1$, $0 = \beta^0 \leq \beta^1 \leq \cdots < \hat{x}_2$, and

$$\lim_{k\to\infty} \alpha^k = \alpha^\star, \qquad \lim_{k\to\infty} \beta^k = \beta^\star.$$

Let $H^0(\cdot) \equiv \Delta$, and $\alpha^0 = 0$ and $\beta^0 = 0$. Suppose that $H^k(\cdot)$ be given by (A.7) and that $\alpha^k < \hat{x}_1$ and $\beta^k < \hat{x}_2$. We show that H^{k+1} is given by

$$H^{k+1}(z) = \begin{cases} \{[a_1]\} & \text{if } z_1 < \alpha^{k+1}, \\ \Delta & \text{if } \alpha^{k+1} \leq z_1 \leq 1 - \beta^{k+1}, \\ \{[a_2]\} & \text{if } z_1 > 1 - \beta^{k+1} \end{cases} \tag{A.8}$$

where $\alpha^k \leq \alpha^{k+1} < \hat{x}_1$ and $\beta^k \leq \beta^{k+1} < \hat{x}_2$.

We consider those z's with $z_1 \in [0, \hat{x}_1]$. Take any such $z = (z_1, z_2)$. Consider first the path $\psi \in \Psi^k(z)$ given by

$$\psi(t) = \begin{cases} [a_1] - ([a_1] - z)\, e^{-\lambda t} & \text{if } t < T_1, \\ \hat{x} & \text{if } t \geq T_1, \end{cases}$$

where T_1 satisfies $z_2\, e^{-\lambda T_1} = \hat{x}_2$. Since $V_1(\psi) \geq V_2(\psi)$ holds, $[a_1]$ is always in $H^{k+1}(z)$.

We then check if there exists a path to which a_2 is a best response. The best scenario for a_2 to be a best response is expressed by the path $\psi' \in \Psi^k(z)$ such that

$$\psi'(t) = \begin{cases} [a_1] - ([a_1] - z)\, e^{-\lambda t} & \text{if } t < T_2, \\ (1 - \beta^k, \beta^k) & \text{if } t \geq T_2, \end{cases}$$

where $T_2 \in (0, \infty]$ is given by

$$(1 - z_1)\, e^{-\lambda T_2} = \beta^k.$$

The expected discounted payoffs along this path are calculated as:

$$\begin{aligned} V_1(\psi') &= b \left[(\lambda + \theta) \int_0^{T_2} e^{-(\lambda+\theta)s}\, z_2\, e^{-\lambda s}\, ds + (\lambda + \theta) \int_{T_2}^{\infty} e^{-(\lambda+\theta)s}\, \beta^k\, ds \right] \\ &= b \left[\frac{1+\delta}{2+\delta} (1 - z_1) + \frac{1}{2+\delta} \frac{(\beta^k)^{2+\delta}}{(1 - z_1)^{1+\delta}} \right]; \\ V_2(\psi') &= c \left[(\lambda + \theta) \int_0^{T_2} e^{-(\lambda+\theta)s} \{1 - (1 - z_1)\, e^{-\lambda s}\}\, ds \right. \\ &\qquad\qquad \left. + (\lambda + \theta) \int_{T_2}^{\infty} e^{-(\lambda+\theta)s} (1 - \beta^k)\, ds \right] \\ &= c \left[1 - \frac{1+\delta}{2+\delta} (1 - z_1) - \frac{1}{2+\delta} \frac{(\beta^k)^{2+\delta}}{(1 - z_1)^{1+\delta}} \right]. \end{aligned}$$

It follows that $V_1(\psi') \leq V_2(\psi')$ if and only if $z_1 \geq \alpha^{k+1}$, where $\alpha^{k+1} \in [0, \hat{x}_1)$ is given by

$$\frac{1+\delta}{2+\delta} (1 - \alpha^{k+1}) + \frac{1}{2+\delta} \frac{(\beta^k)^{2+\delta}}{(1 - \alpha^{k+1})^{1+\delta}} = \frac{c}{b+c} = \hat{x}_2,$$

or

$$P(\alpha^{k+1}, \beta^k) = 0,$$

if such an $\alpha^{k+1} \in (0, \hat{x}_1)$ exists; otherwise, $\alpha^{k+1} = 0$: in such a case, $V_1(\psi') \leq V_2(\psi')$ always holds.

A similar argument for those z's with $z_1 \in (\hat{x}_1, 1]$ enables us to conclude that the other threshold is given by $\beta^{k+1} \in [0, \hat{x}_2)$ that solves

$$Q(\alpha^k, \beta^{k+1}) = 0.$$

It can be verified that such a β^{k+1} always exists in $(0, \hat{x}_2)$. We have thus proved that $H^{k+1}(\cdot)$ is given by (A.8).

Repeating this procedure, we can verify by their construction that $\{\alpha^k\}$ and $\{\beta^k\}$ are both nondecreasing sequences and converge to $\alpha^\star$ and $\beta^\star$ as $k \to \infty$, respectively.

(b) (i) We only show that $\alpha^\star(\delta)$ is increasing in $\delta \in (\delta^\star, \infty)$. A dual argument shows that $\beta^\star(\delta)$ is increasing in $\delta \in (\delta^\star, \infty)$. For $\delta \in (0, \delta^\star]$ (in the case where $\delta^\star > 0$), $\beta^\star(\delta) = 1 - \hat{x}_1(2+\delta)/(1+\delta)$, which is increasing in δ.

For $(\alpha, \beta) \in [0, \hat{x}_1] \times [0, \hat{x}_2]$ and $\delta \in [0, \infty)$, define

$$p(\alpha, \beta, \delta) = (1-\alpha)\left\{\frac{1+\delta}{2+\delta} + \frac{1}{2+\delta}\left(\frac{\beta}{1-\alpha}\right)^{2+\delta} - \frac{\hat{x}_2}{1-\alpha}\right\},$$

$$q(\alpha, \beta, \delta) = (1-\beta)\left\{\frac{1+\delta}{2+\delta} + \frac{1}{2+\delta}\left(\frac{\alpha}{1-\beta}\right)^{2+\delta} - \frac{\hat{x}_1}{1-\beta}\right\}.$$

For $\alpha \in [0, \hat{x}_1)$ and $\delta \in (\delta^\star, \infty)$, let $\check{\beta}(\alpha, \delta)$ and $\hat{\beta}(\alpha, \delta)$ satisfy $p(\alpha, \check{\beta}(\alpha, \delta), \delta) = 0$ and $q(\alpha, \hat{\beta}(\alpha, \delta), \delta) = 0$, respectively. With δ being fixed, the loci of $\beta = \check{\beta}(\alpha, \delta)$ and $\beta = \hat{\beta}(\alpha, \delta)$ are those of $P(\alpha, \beta) = 0$ and $Q(\alpha, \beta) = 0$ (as depicted in Fig. 3(a)), respectively. Then define

$$f(\alpha, \delta) = \hat{\beta}(\alpha, \delta) - \check{\beta}(\alpha, \delta).$$

Recall that $\alpha^\star(\delta)$ satisfies $f(\alpha^\star(\delta), \delta) = 0$.

By observing that $\check{\beta}$ and $\hat{\beta}$ are concave and convex in α, respectively, and hence f is convex in α and that $\lim_{\alpha \to \hat{x}_1} f(\alpha, \delta) = 0$, $(\partial f/\partial \alpha)(0, \delta) < 0$, and $\lim_{\alpha \to \hat{x}_1} (\partial f/\partial \alpha)(\alpha, \delta) = \infty$, we have

$$\frac{\partial f}{\partial \alpha}(\alpha^\star(\delta), \delta) < 0,$$

as depicted in Fig. 4. Note here that $f(0, \delta) > 0$ if and only if $\delta > \delta^\star$.

On the other hand,

$$\frac{\partial p}{\partial \delta}(\alpha, \beta, \delta) = \frac{1-\alpha}{(2+\delta)^2}\left[1 - \left(\frac{\beta}{1-\alpha}\right)^{2+\delta}\left\{1 - (2+\delta)\log\left(\frac{\beta}{1-\alpha}\right)\right\}\right] > 0$$

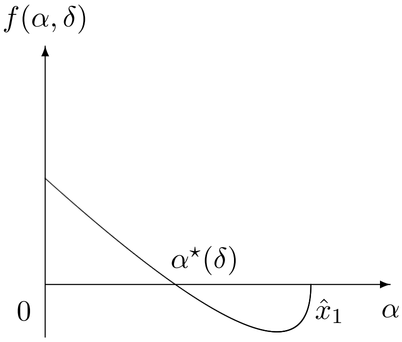


Figure 4: Graph of $f(\alpha,\delta)$

for all $(\alpha,\beta) \in (0,\hat{x}_1) \times (0,\hat{x}_2)$ and $\delta \in (\delta^\star,\infty)$, since

$$\left(\frac{\beta}{1-\alpha}\right)^{-(2+\delta)} > 1-(2+\delta)\log\left(\frac{\beta}{1-\alpha}\right).$$

Similarly, we have $(\partial q/\partial\delta)(\alpha,\beta,\delta) > 0$ for all $(\alpha,\beta) \in (0,\hat{x}_1) \times (0,\hat{x}_2)$ and $\delta \in (\delta^\star,\infty)$. We also have

$$\begin{aligned}
\frac{\partial p}{\partial \beta}(\alpha,\beta,\delta) &= \left(\frac{\beta}{1-\alpha}\right)^{1+\delta} > 0,\\
\frac{\partial q}{\partial \beta}(\alpha,\beta,\delta) &= -\frac{1+\delta}{2+\delta}\left\{1-\left(\frac{\alpha}{1-\beta}\right)^{2+\delta}\right\} < 0
\end{aligned}$$

for all $(\alpha,\beta) \in (0,\hat{x}_1) \times (0,\hat{x}_2)$ and $\delta \in (\delta^\star,\infty)$. Therefore,

$$\begin{aligned}
\frac{\partial f}{\partial \delta}(\alpha,\delta) &= \frac{\partial \hat{\beta}}{\partial \delta}(\alpha,\delta) - \frac{\partial \check{\beta}}{\partial \delta}(\alpha,\delta)\\
&= -\frac{(\partial q/\partial\delta)(\alpha,\hat{\beta}(\alpha,\delta),\delta)}{(\partial q/\partial\beta)(\alpha,\hat{\beta}(\alpha,\delta),\delta)} + \frac{(\partial p/\partial\delta)(\alpha,\check{\beta}(\alpha,\delta),\delta)}{(\partial p/\partial\beta)(\alpha,\check{\beta}(\alpha,\delta),\delta)} > 0.
\end{aligned}$$

It follows that

$$\frac{d\,\alpha^\star(\delta)}{d\,\delta} = -\frac{(\partial f/\partial\delta)(\alpha^\star(\delta),\delta)}{(\partial f/\partial\alpha)(\alpha^\star(\delta),\delta)} > 0. \tag{A.9}$$

(ii) Due to $f(0,\delta^\star)=0$, the continuity of f, and (A.9), we have

$$\lim_{\delta\to\delta^\star} \alpha^\star(\delta) = 0.$$

If $\delta^\star > 0$, then for $\delta \in (0,\delta^\star]$, $1-\beta^\star(\delta) = \hat{x}_1(2+\delta)/(1+\delta)$, so that $\lim_{\delta\to 0}(1-\beta^\star(\delta)) = 2\hat{x}_1$. If $\delta^\star = 0$ (i.e., $\hat{x}_1 = 1/2$), then $\beta^\star(\delta) = \alpha^\star(\delta)$, and thus $\lim_{\delta\to 0}(1-\beta^\star(\delta)) = 1$ $(= 2\hat{x}_1)$.

(iii) Note from the proof of (a) that $\alpha^1 < \alpha^\star(\delta) < \hat{x}_1$ where $\alpha^1 = \hat{x}_1 - \hat{x}_2/(1+\delta)$ for large δ. Since $\hat{x}_1 - \hat{x}_2/(1+\delta) \to \hat{x}_1$ as $\delta \to \infty$, we have

$$\lim_{\delta\to\infty} \alpha^\star(\delta) = \hat{x}_1.$$

Analogously, we have $\lim_{\delta\to\infty} \beta^\star(\delta) = \hat{x}_2$, or $\lim_{\delta\to\infty}(1-\beta^\star(\delta)) = 1 - \hat{x}_2 = \hat{x}_1$. ▌

References

Alós-Ferrer, C. (1999). "Dynamical Systems with a Continuum of Randomly Matched Agents," *Journal of Economic Theory* **86**, 245-267.

Aubin, J.-P. and A. Cellina (1984). *Differential Inclusions*, Spinger-Verlag, Berlin.

Bernheim, B. D. (1984). "Rationalizable Strategic Behavior," *Econometrica* **52**, 1007-1028.

Boylan, R. T. (1992). "Laws of Large Numbers for Dynamical Systems with Randomly Matched Individuals," *Journal of Economic Theory* **57**, 473-504.

Feldman, M. and C. Gilles (1985). "An Expository Note on Individual Risk without Aggregate Uncertainty," *Journal of Economic Theory* **35**, 26-32.

Gilboa, I. and A. Matsui (1991). "Social Stability and Equilibrium," *Econometrica* **59**, 859-867.

Gilboa, I. and A. Matsui (1992). "A Model of Random Matching," *Journal of Mathematical Economics* **21**, 185-197.

Hofbauer, J. and G. Sorger (1999). "Perfect Foresight and Equilibrium Selection in Symmetric Potential Games," *Journal of Economic Theory* **85**, 1-23.

Hofbauer, J. and G. Sorger (2002). "A Differential Game Approach to Evolutionary Equilibrium Selection," *International Game Theory Review* **4**, 17-31.

Judd, K. L. (1985). "The Law of Large Numbers with a Continuum of IID Random Variables," *Journal of Economic Theory* **35**, 19-25.

Kajii, A. and S. Morris (1997). "The Robustness of Equilibria to Incomplete Information," *Econometrica* **65**, 1283-1309.

Kaneda, M. (1995). "Industrialization under Perfect Foresight: A World Economy with a Continuum of Countries," *Journal of Economic Theory* **66**, 437-462.

Krugman, P. (1991). "History versus Expectations," *Quarterly Journal of Economics* **106**, 651-667.

Matsui, A. (1992). "Best Response Dynamics and Socially Stable Strategies," *Journal of Economic Theory* **57**, 343-362.

Matsui, A. and K. Matsuyama (1995). "An Approach to Equilibrium Selection," *Journal of Economic Theory* **65**, 415-434.

Matsuyama, K. (1991). "Increasing Returns, Industrialization, and Indeterminacy of Equilibrium," *Quarterly Journal of Economics* **106**, 617-650.

Matsuyama, K. (1992). "The Market Size, Entrepreneurship, and the Big Push," *Journal of the Japanese and International Economics* **6**, 347-364.

Morris, S., R. Rob, and H. S. Shin (1995). "p-Dominance and Belief Potential," *Econometrica* **63**, 145-157.

Oyama, D. (2002). "p-Dominance and Equilibrium Selection under Perfect Foresight Dynamics," *Journal of Economic Theory* **107**, 288-310.

Oyama, D., S. Takahashi, and J. Hofbauer (2003). "Monotone Methods for Equilibrium Selection under Perfect Foresight Dynamics," mimeo.

Pearce, D. G. (1984). "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica* **52**, 1029-1050.