Refinements and Higher Order Beliefs: A Unified Survey[*]

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Abstract

This paper presents a simple framework that allows us to survey and relate some different strands of the game theory literature. We describe a "canonical" way of adding incomplete information to a complete information game. This framework allows us to give a simple "complete theory" interpretation (Kreps 1990) of standard normal form refinements such as perfection, and to relate refinements both to the "higher order beliefs literature" (Rubinstein 1989; Monderer and Samet 1989; Morris, Rob and Shin 1995; Kajii and Morris 1995) and the "payoff uncertainty approach" (Fudenberg, Kreps and Levine 1988; Dekel and Fudenberg 1990).

1 Introduction

Consider a Nash equilibrium of a complete information game. How robust is that equilibrium? The classical refinements literature (building on Selten 1965, 1975) asks whether that equilibrium may still be played when players "tremble" in their choice of actions. Specifically, does there exist a sequence of full support trembles that justifies the equilibrium play? The normal form refinements literature considers robustness to more complicated action trembles (Myerson 1978) and also robustness to small changes in the game's payoffs (Kohlberg and Mertens 1986). Kreps [1990] and others have argued that the classical refinements literature is flawed because there is no explanation for the trembles. Rather, we should build "complete" theories in which the trembles are endogenously explained: this at least allows us to have an unambiguous interpretation of the refinement. This viewpoint is embodied in two papers, Fudenberg, Kreps and Levine [1988] and Dekel and Fudenberg [1990], that critique existing refinements.

The "higher order beliefs" literature has considered a related question. In analyzing a complete information game, we implicitly assume that payoffs are common knowledge. Which Nash equilibria are robust to a lack of common knowledge? Specifically, which equilibria have the property that they remain equilibria whenever the common knowledge assumption is weakened a little? Rubinstein [1989] demonstrates that even *strict* Nash

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equilibria (which survive all the usual refinements) may not be robust to an (intuitively) small lack of common knowledge. Monderer and Samet [1989] introduce a notion of approximate common knowledge that is sufficient to make strict equilibria robust. Kajii and Morris [1995] ask which Nash equilibria of a complete information game may be equilibrium play of every incomplete information game where payoffs are almost always given by the original complete information game. Many complete information games have no robust equilibria in this sense. This includes games with a unique Nash equilibrium that is strict. But if a game has a unique correlated equilibrium, it is a robust Nash equilibrium; and a many player many action generalization of risk dominance is also sufficient for robustness.

This paper describes a simple framework for relating together all the above results. Our "canonical elaboration" approach, which builds on Fudenberg, Kreps and Levine [1988], works as follows. Fix a complete information game. A canonical elaboration of that complete information game is an incomplete information game with the same players and action sets for each player. Each type of each player is either a "standard" type, whose payoffs are exactly as in the complete information game, or a "committed" type, who has a dominant strategy to play one of the actions of the original game. Importantly, we allow for a large (possibly infinite) number of standard types: although they do not differ in payoffs, they will typically have different beliefs and higher order beliefs about others' payoffs. To see this, consider the following example. There are two actions, a and a'. Player 1 has two possible standard types, t_1 and t_1', and two possible committed types, a and a'. Player 2 has only one possible standard type t_2, and two possible committed types, a and a'. Nature chooses a type profile with assigned probability as shown in the table below.

player 1's type\player 2's type	t_2	a	a'
t_1	$1-2\varepsilon$	0	0
t_1'	ε	ε	0
a	0	0	0
a'	0	0	0

In any case, player 1 is standard with probability 1; thus his payoffs are given by the complete information game. But the two types have different beliefs: if he is of type t_1, he is sure that his opponent is standard. But if he is of type t_1', he will think that player 2 may be committed to play action a with (conditional) probability 1/2. This in turn implies that type t_1 of player 1 is sure that player 2 attaches probability $\frac{\varepsilon}{1-\varepsilon}$ to player 1 attaching probability $\frac{1}{2}$ to player 2 being committed to action a. Thus even though type t_1 is sure that player 2 is not committed to action a, that possibility is relevant to his higher order (specifically, second order) beliefs.

An incomplete information game elaboration of the complete information game is then parameterized completely by a probability distribution over type profiles. We are concerned with sequences of canonical elaborations where, in the limit, types are standard with probability one. For instance, we obtain such a sequence by letting ε go to zero in the example above. We will look at Bayesian-Nash equilibria of the incomplete information game elaboration; hence we will be assuming that players are always *rational* and there are no exogenous trembles. A probability distribution over actions (or *action distribution*) is a *limit equilibrium action distribution* (for some given complete information game and canonical elaboration sequence) if it represents the limit of the action distributions generated by equilibria of the elaboration sequence. Because each player has many standard types, correlation is possible in the limit of the elaborations. Thus any limit equilibrium action distribution is a correlated equilibrium of the original complete information game.

In this framework, we can formalize the generic question we want to ask as follows:

> *Fix a complete information game. Which action distributions are limit equilibrium action distributions of some (all) canonical elaboration sequences satisfying some property in higher order beliefs?*

We can understand each of the above literatures by varying the restrictions we put on elaboration sequences and whether we require robustness to some or all such sequences.

Our results can be summarized in the following table (a more detailed summary is given in the conclusion). We write NE for Nash equilibria and CE for correlated equilibria.

class of elaboration \ type of robustness	Robust to *all* sequences	Justified by *some* full support sequence
No Restrictions	1. not all strict NE 2. risk dominant NE 3. unique CE	undominated CE
Limit Common Knowledge	strict NE	undominated CE
Limit Independence	strict NE	undominated NE
Independence	strictly perfect NE	perfect NE

For the left hand column, we ask which behavior is a limiting equilibrium action distribution for *all* sequences satisfying progressively stronger properties. Since we are requiring robustness to all sequences, we thus weaken the refinement as we impose stronger restrictions. If we impose no extra restrictions on the elaborations (beyond the requirement that standard types are assigned probability one in the limit) then we get the result (in the spirit of Rubinstein 1989) that even strict equilibria may not be robust; a many person many action generalization of risk dominance (Harsanyi and Selten 1988) is a sufficient condition (Kajii and Morris 1995; see also Carlsson and van Damme 1993). A unique correlated equilibrium is also robust (Kajii and Morris 1995).

Now suppose that we add extra requirements to the elaboration sequence. *Limit common knowledge* requires that for all $p < 1$, the probability that there is *common p-belief*[1] that all types are standard also tends to one. All strict equilibria are robust to limit common knowledge elaborations (Monderer and Samet 1989). *Limit independence* requires that the probability distribution over types is independent in the limit. Interestingly enough, Limit independence implies limit common knowledge, so strict equilibria are robust also to limit independent elaborations. *Independence* requires that all elaborations are independent (at the limit and before the limit). Robustness to all independent elaborations is equivalent to *strict perfection* in the sense of Okada [1981]. Strict perfection is equivalent to requiring a singleton strategy profile to be *stable* (Kohlberg and Mertens 1986).

For the right hand column, we ask which behavior is the limiting action equilibrium action distribution for *some* sequence which puts full support (away from the limit) on all committed type profiles and also satisfies the sequence of progressively stronger properties introduced above. Now as we strengthen the restriction on the sequence we *strengthen*

[1] An event is common p-belief if everyone believes it with probability p, everyone believes with probability p that everyone believes it with probability p, and so on (Monderer and Samet 1989).

the refinement. If we put no additional restrictions on the elaboration sequence, the full support requirement implies that no dominated actions are played. An argument of Fudenberg, Kreps and Levine [1988] shows that any undominated correlated equilibrium (a correlated equilibrium where dominated strategies are assigned probability zero) is the limit equilibrium action distribution of some full support elaboration sequence. Thus an action distribution is *justified* by some full support elaboration if and only if it is an undominated correlated equilibrium. Adding the limit common knowledge requirement does not effect this argument. However, requiring limit independence implies that we must have a Nash equilibrium in the limit: thus an action distribution is justified by some full support limit independent elaboration if and only if it is an undominated *Nash* equilibrium. Finally, requiring independence along the sequence makes the canonical elaboration equivalent to the action trembles of Selten [1975]. Thus an action distribution is justified by some full support independent elaboration if and only if it is perfect.

This paper lays out these arguments in essentially this order. We first introduce our notion of canonical elaborations (section 2) and review some techniques and Lemmas from the higher beliefs literature that we will exploit (section 3). We survey results concerning robustness to *all* elaborations in section 4 and review refinements which are justified by some full support elaboration in section 5. Section 6 concludes.

Before we start, let us highlight some limitations of our exercise.

1. The "canonical elaborations" we consider are very simple. In particular, it would make sense to allow "standard types" to have a small amount of uncertainty about their own payoffs; similarly, it would make sense to allow types with very different payoffs who are not necessarily committed to a particular action. We allow neither. We consider simple elaborations because they are sufficient to explain the relation between the literatures we survey; put differently, we believe that the issues raised by considering more complicated elaborations are orthogonal to the main conceptual issues in these literatures. Nonetheless, we do not want to claim they are in any sense the "right" elaborations to consider. Papers such as Fudenberg, Kreps and Levine [1988], Dekel and Fudenberg [1990] and the survey paper of Dekel and Gul [1997] consider different classes of elaborations and discuss their justifications.

2. In analyzing our canonical elaborations, we *assume* that players have common prior beliefs and play according to some (Bayesian) Nash equilibrium. We thus ignore two sets of issues studied in the literature. First, when and how can assumptions on knowledge and higher order knowledge justify equilibrium assumptions (e.g., Aumann [1987], Brandenburger and Dekel [1987] and Aumann and Brandenburger [1995])? Second, which refinements can be justified even before making equilibrium/common prior assumptions (e.g., Börgers [1994], Dekel and Fudenberg [1990] and Gul [1996])?

3. This is a *survey*. We sketch proofs and cite close but not identical proofs in the literature. While our interpretation of results may differ from the literatures we survey, almost all the results are out there even if we do not cite them correctly. We believe that many researchers have thought about "complete theory" issues in refinements but we are not sure of the appropriate citations.

4. In many cases, there are large gaps between the necessary and sufficient conditions we report. In some cases (such as section 4.1) we have tried hard elsewhere to fill these gaps (without success). In other cases (such as sections 4.2 and 4.3) there may be existing results or easy arguments that we have missed.

2 Canonical Elaborations

In this section, a simple framework for studying incomplete information elaborations of complete information games is presented. Our elaborations are a modified version of the elaborations introduced by Fudenberg, Kreps and Levine (1988). Fix a finite set of players, $\mathcal{I} = \{1, ..., I\}$, and their finite action sets, $\{A_i\}_{i \in \mathcal{I}}$. Write $A \equiv A_1 \times .. \times A_I$.

2.1 Complete Information Game

A *complete information game* $\mathbf{g} \equiv \{g_i\}_{i \in \mathcal{I}}$, where each $g_i : A \to \Re$. Write $\mathcal{A}_i \equiv \Delta(A_i)$ for the set of mixed strategies for player i and $\mathcal{A} \equiv \mathcal{A}_1 \times .. \times \mathcal{A}_I$. Write $\alpha \equiv (\alpha_1, .., \alpha_I) \in \mathcal{A}$ for a profile of mixed strategies; write α_{-i} for a profile of mixed strategies excluding player i's. The domain of each g_i is extended to mixed strategies in the usual way. We record some standard complete information game concepts that we will be using.

Definition 1 *Action $a_i \in A_i$ is dominated (in $\mathbf{g}$) if there exists $\alpha_i' \in \mathcal{A}_i$ such that*

$$g_i(\alpha_i', a_{-i}) \geq g_i(a_i, a_{-i}), \text{ for all } a_{-i} \in A_{-i},$$

with strict inequality for at least one a_{-i}.

Definition 2 *Action $a_i \in A_i$ is weakly dominant (in $\mathbf{g}$) if*

$$g_i(a_i, a_{-i}) \geq g_i(a_i', a_{-i}), \text{ for all } a_i' \in A \text{ and } a_{-i} \in A_{-i}.$$

Definition 3 *Mixed strategy profile $\alpha \in \mathcal{A}$ is a Nash equilibrium (of $\mathbf{g}$) if*

$$g_i(\alpha_i, \alpha_{-i}) \geq g_i(\alpha_i', \alpha_{-i}), \text{ for all } i \in \mathcal{I} \text{ and } \alpha_i' \in \mathcal{A}_i.$$

Definition 4 *Action distribution $\mu \in \Delta(A)$ is a correlated equilibrium (of $\mathbf{g}$) if, for all $i \in \mathcal{I}$ and $a_i, a_i' \in A_i$,*

$$\sum_{a_{-i} \in A_{-i}} g_i(a_i, a_{-i}) \mu(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} g_i(a_i', a_{-i}) \mu(a_i, a_{-i}).$$

Definition 5 *Action distribution $\mu \in \Delta(A)$ is a* Nash *equilibrium action distribution (of $\mathbf{g}$) if μ is a correlated equilibrium and $\mu(a) = \prod_{i \in \mathcal{I}} \mu_i(a_i)$, where μ_i is the marginal distribution of μ on A_i. Equivalently, μ is a Nash equilibrium action distribution if $\mu(a) = \prod_{i \in \mathcal{I}} \alpha_i(a_i)$ for some Nash equilibrium α.*

Definition 6 *Action distribution $\mu \in \Delta(A)$ is an undominated correlated equilibrium (of $\mathbf{g}$) if [1] μ is a correlated equilibrium and [2] $\mu(a) = 0$ if a_i is a dominated action.*

Definition 7 *Action distribution $\mu \in \Delta(A)$ is an undominated Nash equilibrium action distribution (of $\mathbf{g}$) if [1] μ is a Nash equilibrium action distribution and [2] $\mu(a) = 0$ if a_i is a dominated action.*

2.2 Incomplete Information Elaborations

In considering elaborations of complete information games, the state space will be *fixed*. Let T_i^* be a countably infinite set of *standard* types of player i. Let A_i be *committed* types of player i. The set of possible types of player i is the union of these two, $T_i = T_i^* \cup A_i$. The state space now consists of all type profiles, $T = T_1 \times .. \times T_I$.

Definition 8 *A canonical elaboration consists of a complete information game* $\mathbf{g}$ *and a probability distribution* $P \in \Delta(T)$.

A canonical elaboration $(\mathbf{g}, P)$ is an incomplete information game under the following interpretation: $P \in \Delta(T)$ gives a probability distribution over types; payoff functions $\mathbf{g}$ give payoffs of standard types; and committed type a_i of player i has a strictly dominant strategy to play a_i.

More formally, let $(\mathbf{g}, P)$ be the incomplete information game with type space T, probability distribution P and incomplete information game payoff functions:

$$u_i(a,t) = \begin{cases} g_i(a), \text{ if } t_i \in T_i^* \\ 1, \text{ if } a_i = t_i \in A_i \\ 0, \text{ if } a_i \neq t_i \text{ and } t_i \in A_i \end{cases}.$$

Thus (for simplicity) we focus on extreme standard and committed types, excluding:

- Standard types with a small amount of uncertainty about their own payoffs;
- Non-standard types without strictly dominant actions.

Now a (mixed) strategy for player i is function $\sigma_i : T_i \to \Delta(A_i)$. Write Σ_i for the set of mixed strategies of player i, and write $\sigma \equiv (\sigma_1, .., \sigma_I) \in \Sigma \equiv \Sigma_1 \times .. \times \Sigma_I$. Write $\sigma_i(a_i|t_i)$ for the probability of action a_i, given type t_i of player i; write $\sigma(a|t) \equiv \prod_{i \in \mathcal{I}} \sigma_i(a_i|t_i)$ and $\sigma_{-i}(a_{-i}|t_{-i}) \equiv \prod_{j \neq i} \sigma_j(a_j|t_j)$.

The standard definition of equilibrium in this setting is:

Definition 9 *Strategy profile* σ *is an equilibrium if, for all* $i \in \mathcal{I}$ *and* $\sigma_i' \in \Sigma_i$,

$$\sum_{t \in T} \sum_{a \in A} u_i(a,t)\, \sigma_i(a|t)\, P(t) \geq \sum_{t \in T} \sum_{a \in A} u_i(a,t)\, \sigma_i'(a_i|t_i)\, \sigma_{-i}(a_{-i}|t_{-i})\, P(t).$$

This is equivalent to:

Definition 10 *Strategy profile* σ *is an equilibrium if, for all* $i \in \mathcal{I}$, $t_i \in T_i$ *with* $P(t_i) > 0$, *and* $a_i \in A_i$ *with* $\sigma_i(a_i|t_i) > 0$,

$$a_i \in \underset{a_i' \in A_i}{\arg\max} \sum_{t_{-i} \in T_{-i}} \sum_{a_{-i} \in A_{-i}} u_i((a_i', a_{-i}), t)\, \sigma_{-i}(a_{-i}|t_{-i})\, P(t_{-i}|t_i),$$

where $P(t_{-i}|t_i)$ *is the conditional probability of type profile* t_{-i} *given* t_i, *i.e.,* $P(t_{-i}|t_i) = \frac{P(t_i, t_{-i})}{\sum_{t_{-i}' \in T_{-i}'} P(t_i, t_{-i}')}$.

Substituting for the particular form of u_i, we have:

Definition 11 *Strategy profile σ is an equilibrium if, for all $i \in \mathcal{I}$ and $t_i \in T_i$ with $P(t_i) > 0$, [1] $\sigma_i(a_i|t_i) = 1$, if $t_i = a_i$; and*

$$\text{[2] } a_i \in \underset{a_i' \in A_i}{\arg\max} \sum_{t_{-i} \in T_{-i}} \sum_{a_{-i} \in A_{-i}} g_i(a_i', a_{-i}) \sigma_{-i}(a_{-i}|t_i) P(t_{-i}|t_i),$$

if $t_i \in T_i^*$ and $\sigma_i(a_i|t_i) > 0$.

We are interested in *average* behavior in equilibria of the canonical elaborations:

Definition 12 *Action distribution $\mu \in \Delta(A)$ is an equilibrium action distribution [EAD] of $(\mathbf{g}, P)$ if there is an equilibrium σ of $(\mathbf{g}, P)$ with $\mu(a) = \sum_{t \in T} \sigma(a|t) P(t)$ for all $a \in A$.*

2.3 Elaboration Sequences

Definition 13 *Sequence $\{P^k\}_{k=1}^{\infty}$, each $P^k \in \Delta(A)$, is an elaboration sequence if $P^k \to P^{\infty}$ pointwise and $P^k[T^*] \to 1$.*

Definition 14 *Action distribution $\mu \in \Delta(A)$ is a limiting equilibrium action distribution [LEAD] of $[\mathbf{g}, \{P^k\}_{k=1}^{\infty}]$ if there exist equilibrium action distributions $\{\mu^k\}_{k=1}^{\infty}$ of $[\mathbf{g}, \{P^k\}_{k=1}^{\infty}]$ with $\mu^k \to \mu$. If $\mu(a^*) = 1$, a^* is a limit equilibrium of $[\mathbf{g}, \{P^k\}_{k=1}^{\infty}]$.*

Lemma 1 *If μ is an LEAD of $[\mathbf{g}, \{P^k\}_{k=1}^{\infty}]$, then μ is a correlated equilibrium of $\mathbf{g}$.*

Proof. (Sketch: Corollary 3.5 of Kajii and Morris [1995] can be used to give a complete proof). If μ^k is a EAD of $(\mathbf{g}, P^k)$ for large k, it can be shown to be an approximate correlated equilibrium of $\mathbf{g}$. The limit of a convergent sequence is thus a correlated equilibrium. ■

The generic question that we will be asking in this paper is the following:

> *Fix $\mathbf{g}$. Which $\mu \in \Delta(A)$ are limiting equilibrium action distributions of $[\mathbf{g}, \{P^k\}_{k=1}^{\infty}]$ for some (all) elaboration sequence(s) $\{P^k\}_{k=1}^{\infty}$ satisfying property X?*

3 Preliminaries

We introduce some techniques from the higher order beliefs literature that will be extensively used in later sections. Thus we first introduce the idea of **p**-evident, or almost public, events (Monderer and Samet 1989). Then we introduce the **p**-dominant equilibria (Morris, Rob and Shin 1995). Finally, we relate these together to provide "the basic Lemma," on which much of the higher order beliefs literature builds.

3.1 Almost Public Events

An event is almost public if whenever it is true, everyone believes that it is true with high probability. We will be interested only in events that are the products of sets of types of each player.

Definition 15 *$S \subseteq T$ is a simple event if $S = \times_{i \in \mathcal{I}} S_i$, where each $S_i \subseteq T_i$.*

Given simple event S, write $S_{-i} = S_1 \times .. \times S_{i-1} \times S_{i+1} \times .. \times S_I$. If $P(t_i) > 0$, define

$$P(S_{-i}|t_i) \equiv \frac{\sum\limits_{t_{-i} \in S_{-i}} P(t_i, t_{-i})}{\sum\limits_{t_{-i} \in T_{-i}} P(t_i, t_{-i})}.$$

Fix a vector $\mathbf{p} = (p_1, ..., p_I)$, each $p_i \in [0, 1]$.

Definition 16 *A simple event S is $\mathbf{p}$-evident (under P) if whenever it is true, each individual i believes it with probability at least p_i. Formally, S is $\mathbf{p}$-evident if for all $i \in \mathcal{I}$ and $t_i \in S_i$ such that $P(t_i) > 0$, $P(S_{-i}|t_i) \geq p_i$.*

3.2 p-Dominant Equilibria

An action profile a^* is a $\mathbf{p}$-dominant equilibrium (of $\mathbf{g}$) if, for each player i, action a_i^* is a best response whenever he assigns probability at least p_i to other players following action profile a^*:

Definition 17 *Action Profile a^* is a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ if, for all $i \in \mathcal{I}$, $a_i \in A_i$ and $\lambda \in \Delta(A_{-i})$ with $\lambda\left(a_{-i}^*\right) \geq p_i$,*

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i})\, g_i(a_i^*, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i})\, g_i(a_i, a_{-i}).$$

The following are immediate by definition.

- a^* is a $(1, .., 1)$-dominant equilibrium if and only if a^* is a Nash equilibrium.
- a^* is a $(0, .., 0)$-dominant equilibrium if and only if each a_i^* is a weakly dominant action.

If a^* is a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$, then a^* is a $\mathbf{p}'$-dominant for any probability vector $\mathbf{p}' \geq \mathbf{p}$. So if a^* is a Nash equilibrium (thus $(1, .., 1)$-dominant), we will be interested in the smallest $\mathbf{p}$ for which a^* is a $\mathbf{p}$-dominant equilibrium.

3.3 Higher Order Beliefs and the Equilibria of Incomplete Information Games

Lemma 2 *(The Basic Lemma). Let a^* be a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$. Let S be a $\mathbf{p}$-evident subset of T^* (under P). Then $(\mathbf{g}, P)$ has an equilibrium with $\sigma_i\left(a_i^* | \, t_i\right) = 1$ for all $t_i \in S_i$.*

Proof. (Sketch: see Kajii and Morris 1995, Lemma 5.2, for full proof). Consider the modified game where we *require* $\sigma_i(a_i^*|t_i) = 1$ for all $t_i \in S_i$. Let σ^* be any equilibrium of the modified game. By construction, σ^* is an equilibrium of the original game. ■

For any simple event S, define

$$\begin{aligned} B_*^{\mathbf{p},k}(S) &\equiv \left\{ t \in T : \begin{array}{l} \text{for all } i \text{ with } P^k(t_i) > 0, \\ \text{(i) } t_i \in S_i; \text{ (ii) } P^k(S_{-i}|t_i) \geq p_i \end{array} \right\} \\ \text{and } C^{\mathbf{p},k}(S) &\equiv \bigcap_{n \geq 1} \left[B^{\mathbf{p},k}\right]^n (S). \end{aligned}$$

Lemma 3 *(The* **p***-Belief Lemma). For any simple event S, $C^{\mathbf{p},k}(S)$ is* **p***-evident (under P^k) and contains all* **p***-evident events contained in S, i.e., $E \subseteq B_*^{\mathbf{p},k}(E)$ and $E \subseteq S \Rightarrow E \subseteq C^{\mathbf{p},k}(S)$.*

This is a special case of the characterization of common p-belief in Monderer and Samet [1989].

Corollary 1 *Let a^* be a* **p***-dominant equilibrium of* **g***. Then $(\mathbf{g}, P^k)$ has an equilibrium σ with $\sigma_i(a_i^* | t_i) = 1$ for all $i \in \mathcal{I}$ if $t \in C^{\mathbf{p},k}(T^*)$.*

This follows immediately from the **p**-Belief Lemma (Lemma 3) and the Basic Lemma (Lemma 2).

4 Robustness to *all* Elaborations

One way of examining the robustness of equilibria of **g** is to ask if behavior close to those equilibria is possible equilibrium behavior in *all* nearby elaborations. In this section, we consider this question under progressively stronger restrictions on the elaboration sequences. This implies progressively weaker solution concepts. For a given restriction X on the canonical elaborations, we will say:

Definition 18 *Action distribution $\mu \in \Delta(A)$ is robust to (type X) elaborations if μ is a LEAD of $[\mathbf{g}, \{P^k\}_{k=1}^{\infty}]$ for every (type X) elaboration sequence $\{P^k\}_{k=1}^{\infty}$.*

4.1 No Restrictions

Consider first the case with no additional restrictions on the elaborations. A variation of an example due to Rubinstein [1989] shows that it is possible to have a^* a strict Nash equilibrium of **g** and an elaboration sequence $\{P^k\}_{k=1}^{\infty}$ such that every limiting equilibrium action distribution of $[\mathbf{g}, \{P^k\}_{k=1}^{\infty}]$ has $\mu(a^*) = 0$. This implies that not all strict Nash equilibria are robust to canonical elaborations.

Example 1 *The complete information game* **g** *is described by the following matrix.*

	A	B
A	$1,1$	$-2,0$
B	$0,-2$	$0,0$

(1)

The elaboration sequence $P^k \to P^{\infty}$, where P^{∞} is given by the rule: $P^{\infty}(t_1, t_2) = \alpha(1-\alpha)^{2n}$ if $t_1 = t_2 = n$; $= \alpha(1-\alpha)^{2n+1}$ if $t_1 = n = t_2 - 1$; $= 0$ otherwise, where $0 < \alpha < 1/2$. This is illustrated in the following table:

	A	B	0	1	$\cdot$	$k-1$	k	$k+1$	$\cdot$
A	0	0	0	0	$\cdot$	0	0	0	$\cdot$
B	0	0	0	0	$\cdot$	0	0	0	$\cdot$
0	0	0	α	0	$\cdot$	0	0	0	$\cdot$
1	0	0	$\alpha(1-\alpha)$	$\alpha(1-\alpha)^2$	$\cdot$	0	0	0	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$k-1$	0	0	0	0	$\cdot$	$\alpha(1-\alpha)^{2k-4}$	0	0	$\cdot$
k	0	0	0	0	$\cdot$	$\alpha(1-\alpha)^{2k-3}$	$\alpha(1-\alpha)^{2k-2}$	0	$\cdot$
$k+1$	0	0	0	0	$\cdot$	0	$\alpha(1-\alpha)^{2k-1}$	$\alpha(1-\alpha)^{2k}$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

P^k is identical to P^∞ except that $P^k(k-1,k) = P^k(k,k) = 0$ and $P^k(B,k-1) = \alpha(1-\alpha)^{2k-3}$ and $P^k(B,k) = \alpha(1-\alpha)^{2k-2}$. P^k is illustrated in the following table:

	A	B	0	1	$\cdot$	$k-1$	k	$k+1$	$\cdot$
A	0	0	0	0	$\cdot$	0	0	0	$\cdot$
B	0	0	0	0	$\cdot$	$\alpha(1-\alpha)^{2k-1}$	$\alpha(1-\alpha)^{2k}$	0	$\cdot$
0	0	0	α	0	$\cdot$	0	0	0	$\cdot$
1	0	0	$\alpha(1-\alpha)$	$\alpha(1-\alpha)^2$	$\cdot$	0	0	0	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$k-1$	0	0	0	0	$\cdot$	$\alpha(1-\alpha)^{2k-2}$	0	0	$\cdot$
k	0	0	0	0	$\cdot$	0	0	0	$\cdot$
$k+1$	0	0	0	0	$\cdot$	0	$\alpha(1-\alpha)^{2k+1}$	$\alpha(1-\alpha)^{2k+2}$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

- *The Electronic Mail Interpretation.* Suppose players communicate with each other as follows. Player 2 sends a message to player 1, which gets lost with probability α. If the message is lost, communication stops. If player 1 receives the message, he sends a message to player 2, which also gets lost with probability α. The communication process continues like this. A player's type corresponds to the number of messages he has received. The game $(\mathbf{g}, P^\infty)$ corresponds to a situation where players communicate according to this protocol but payoffs are always given by matrix (1), independent of types. The game $(\mathbf{g}, P^k)$ corresponds to the situation where player 1, if he receives exactly k messages, has a dominant strategy to play B. All other types of player 1, and all types of player 2, have payoffs given by matrix (1).

One equilibrium of $(\mathbf{g}, P^\infty)$ has all standard types playing A (another has all standard types playing B). But the essentially[2] unique equilibrium of $(\mathbf{g}, P^k)$ has action B played by all standard types. To see why, observe that committed type B of player 1 plays B by assumption. Now in any equilibrium, standard types $k-1$ and k of player 2 assign probabilities at least $\frac{1-\alpha}{2-\alpha}$ and $\frac{1}{2-\alpha}$ respectively to player 1 choosing B. Both these probabilities are strictly more than $\frac{1}{3}$ (since $\alpha < \frac{1}{2}$). Given the payoff matrix (1), this ensures that action B is the unique best response. But now a similar argument ensures that B must be played by types $k-1$ and $k+1$ of player 1. The argument iterates to ensure the result. Thus the *unique* LEAD of $\left[\mathbf{g}, \left\{P^k\right\}_{k=1}^{\infty}\right]$ has $\mu((B,B)) = 1$.

Kajii and Morris [1995] show more:

Proposition 1 *[1] There exist an open set of games for which no $\mu \in \Delta(A)$ is robust to elaborations. This includes games with a unique Nash equilibrium that is strict. [2] If $\mathbf{g}$ has a unique correlated equilibrium μ^*, then μ^* is robust to elaborations. [3] If a^* is a $\mathbf{p}$-dominant equilibrium (of $\mathbf{g}$), with $\sum_{i \in \mathcal{I}} p_i \leq 1$, then a^* is robust to elaborations.*[3]

Proof. [1] is proved by a more complicated version of the above example (see KM example 3.1). [2] follows from the fact that all LEADs are correlated equilibria (Lemma 1). [3] is proved by combining Corollary 1 with the following Proposition. ■

[2]Throughout the paper an equilibrium is said to be essentially unique if it is unique up to zero probability types.

[3]The notion of "robustness to incomplete information" considered in Kajii and Morris [1995] is slightly different to that described here. First, we allowed a richer class of non-standard types. This makes no difference to this result. Second, we implicitly required robustness to elaboration sequences without well-defined limits. This makes a small difference, effecting only equilibria which are $\mathbf{p}$-dominant with $\sum_{i \in \mathcal{I}} p_i = 1$ but not $\mathbf{p}$-dominant with $\sum_{i \in \mathcal{I}} p_i < 1$ (see section 9.3 of Kajii and Morris 1995).

Proposition 2 *If $P^k[S] \to 1$ as $k \to \infty$, then $P^k\left[C^{\mathbf{p},k}(S)\right] \to 1$ for all $\mathbf{p}$ with $\sum_{i\in\mathcal{I}} p_i \leq 1$.*[4]

In the case of two player two action games with two strict equilibria, the requirement that a^* is a $\mathbf{p}$-dominant equilibrium with $\sum_{i\in\mathcal{I}} p_i \leq 1$ is equivalent to the requirement that a^* is risk dominant in the sense of Harsanyi and Selten [1988]. But except for non generic cases, a two player two action game has (1) a unique strict pure strategy equilibrium; (2) a unique mixed strategy equilibrium; or (3) two pure strategy equilibria, in which case only one is risk dominant. The unique equilibrium in cases (1) and (2) is also a unique correlated equilibrium. So for such generic games, Proposition 1 gives a complete characterization of robustness.

Corollary 2 *In generic two player two action games, there exists a unique equilibrium robust to elaborations.*

Carlsson and van Damme [1993a] have given a closely related justification for the risk dominant equilibrium. Suppose that players of a two player two action game observe a continuous signal of the payoffs (represented by a vector in $\Re^8$) with a small amount of symmetric noise. This generates an incomplete information game. They show that as the noise goes to zero (specifically, the support of the noise), the unique equilibrium of the incomplete information game has the risk dominant equilibrium played everywhere.

It is useful to understand the difference between the Carlsson and van Damme result and the robustness question discussed here. There are some inessential differences: Carlsson and van Damme have a continuum of types and do not have the ex ante probability of any particular payoff vector tending to one in the limit. But the crucial difference is that while they demonstrate an (intuitive and general) class of elaboration sequences for which only the risk dominant equilibrium is a limit equilibrium, they do not show the non-existence of some different class of elaboration sequences for which the risk dominant equilibrium is not a limit equilibrium. This is the harder part of showing robustness.

Morris [1997] describes another class of games where it is possible to provide a complete characterization of robustness. A "symmetric binary action co-ordination" (SBAC) game is a game where players have a choice of two actions; each player's payoff depends only on his own action and the number of his opponents choosing each action; and there are strategic complementarities, i.e., an action becomes a better response as more players choose that action. In an N-player SBAC, an action is said to be a uniform best response if it is best response to a conjecture putting uniform probability on the number of opponents choosing each action. In a generic SBAC, exactly one action is the uniform best response. The unique robust equilibrium of a SBAC is the action profile where all players choose the uniform best response.[5]

4.2 Limit Common Knowledge Elaborations

What restriction on the elaborations is required to ensure the robustness of at least *strict* Nash equilibria? Monderer and Samet [1989] identified one. Observe that for every

[4] This result is tight: if $\sum_{i\in\mathcal{I}} p_i > 1$, it is possible to construct an elaboration sequence $\left\{P^k\right\}_{k=1}^{\infty}$ with $P^k\left[C^{\mathbf{p},k}(T^*)\right] = 0$ for all k.

[5] This result again follows the work of Carlsson and van Damme (see Carlsson and van Damme [1993b] and also Kim [1996]) who showed that the uniform best response must be played in the limit equilibrium of a particular elaboration sequence with uniform noise.

elaboration sequence $\left\{P^k\right\}_{k=1}^{\infty}$ with $P^k \to P^{\infty}$, $P^{\infty}\left[C^{\mathbf{1},\infty}(T^*)\right] = 1$. Thus we might be interested in the following property:

Definition 19 *Elaboration sequence $\left\{P^k\right\}_{k=1}^{\infty}$ satisfies limit common knowledge if $P^k\left[C^{\mathbf{p},k}(T^*)\right] \to 1$ for all $\mathbf{p} < \mathbf{1}$.*

We will show that a slight generalization of strict equilibrium is a sufficient condition for robustness to limit common knowledge elaborations.

Definition 20 *Pure strategy profile a^* is a semi-strict equilibrium of $\mathbf{g}$ if, for each i and $a_i \in A_i$, either $g_i\left(a_i^*, a_{-i}^*\right) > g_i\left(a_i, a_{-i}^*\right)$ or $g_i\left(a_i^*, a_{-i}\right) \geq g_i\left(a_i, a_{-i}\right)$ for all $a_{-i} \in A_{-i}$.*

Proposition 3 *If a^* is a semi-strict equilibrium of $\mathbf{g}$, then a^* is robust to limit common knowledge elaborations.*

The Proposition is an immediate consequence of two results showing [1] that a^* is semi-strict if and only if it is $\mathbf{p}$-dominant for some $\mathbf{p} < \mathbf{1}$ (Corollary 3 below); and [2] that if a^* is a $\mathbf{p}$-dominant equilibrium for some $\mathbf{p} < \mathbf{1}$, then a^* is robust to limit common knowledge elaborations (Lemma 5 below).

The sufficient condition of Proposition 3 is not necessary: part 2 of Proposition 1 shows that if a two player two action game has a unique, strictly mixed, Nash equilibrium, then that equilibrium is robust to all (including limit common knowledge) elaborations, although it certainly does not satisfy the sufficient condition of Proposition 3.

The following Lemma characterizes for which $\mathbf{p}$ an action profile is $\mathbf{p}$-dominant. This Lemma can then be used to give an exact characterization of when an action profile is $\mathbf{p}$-dominant for some $\mathbf{p} < \mathbf{1}$. Fix game $\mathbf{g}$ and Nash equilibrium a^*. Write

$$A_i^+ = \left\{a_i \in A_i : g_i\left(a_i^*, a_{-i}\right) < g_i\left(a_i, a_{-i}\right) \text{ for some } a_{-i} \in A_{-i}\right\}.$$

If A_i^+ is empty, set $\overline{p}_i = 0$. If A_i^+ is non-empty, then for each $a_i \in A_i^+$, choose $f_i\left(a_i\right) \in \underset{a_{-i} \in A_{-i}}{\arg\max}\left(g_i\left(a_i, a_{-i}\right) - g_i\left(a_i^*, a_{-i}\right)\right)$ and set

$$\overline{p}_i = \max_{a_i \in A_i^+} \left\{ \frac{g_i\left(a_i, f_i\left(a_i\right)\right) - g_i\left(a_i^*, f_i\left(a_i\right)\right)}{\left(g_i\left(a_i^*, a_{-i}^*\right) - g_i\left(a_i, a_{-i}^*\right)\right) + \left(g_i\left(a_i, f_i\left(a_i\right)\right) - g_i\left(a_i^*, f_i\left(a_i\right)\right)\right)} \right\}. \tag{2}$$

Lemma 4 *For any $\mathbf{p} \in [0,1]^I$, a^* is a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ if and only if $p_i \geq \overline{p}_i$ for all $i \in \mathcal{I}$.*

Proof. (if) Suppose $p_i \geq \overline{p}_i$ for all $i \in \mathcal{I}$. Now by construction of $\overline{\mathbf{p}}=\left\{\overline{p}_i\right\}_{i \in \mathcal{I}}$,

$$p_i\left(g_i\left(a_i^*, a_{-i}^*\right) - g_i\left(a_i, a_{-i}^*\right)\right) + \left(1 - p_i\right)\left(g_i\left(a_i^*, a_{-i}\right) - g_i\left(a_i, a_{-i}\right)\right) \geq 0 \tag{3}$$

for all $a_i \in A_i$ and $a_{-i} \in A_{-i}$. So for any $a_i \in A_i$ and $\lambda \in \Delta\left(A_{-i}\right)$ with $1 > \lambda\left(a_{-i}^*\right) \geq p_i \geq \overline{p}_i$, we have:

$$\begin{aligned}
& \sum_{a_{-i} \in A_{-i}} \lambda\left(a_{-i}\right)\left(g_i\left(a_i^*, a_{-i}\right) - g_i\left(a_i, a_{-i}\right)\right) \\
\geq & \left(\begin{array}{l} \lambda\left(a_{-i}^*\right)\left(g_i\left(a_i^*, a_{-i}^*\right) - g_i\left(a_i, a_{-i}^*\right)\right) \\ +\left(1 - \lambda\left(a_{-i}^*\right)\right) \sum\limits_{a_{-i} \neq a_{-i}^*} \left(\frac{\lambda\left(a_{-i}\right)}{\left(1 - \lambda\left(a_{-i}^*\right)\right)}\right)\left(g_i\left(a_i^*, a_{-i}\right) - g_i\left(a_i, a_{-i}\right)\right) \end{array} \right) \\
= & \sum_{a_{-i} \neq a_{-i}^*} \left(\frac{\lambda\left(a_{-i}\right)}{\left(1 - \lambda\left(a_{-i}^*\right)\right)}\right) \left(\begin{array}{l} \lambda\left(a_{-i}^*\right)\left(g_i\left(a_i^*, a_{-i}^*\right) - g_i\left(a_i, a_{-i}^*\right)\right) \\ +\left(1 - \lambda\left(a_{-i}^*\right)\right)\left(g_i\left(a_i^*, a_{-i}\right) - g_i\left(a_i, a_{-i}\right)\right) \end{array} \right) \\
\geq & \ 0, \text{ by (3).}
\end{aligned}$$

This proves that a^* is $\mathbf{p}$-dominant.

(Only if) If $\overline{p}_i > 0$, then there exists $\widehat{a}_i \in A_i$ and $\widehat{a}_{-i} \in A_{-i}$ such that for all $p_i < \overline{p}_i$,

$$p_i \left(g_i \left(a_i^*, a_{-i}^*\right) - g_i \left(\widehat{a}_i, a_{-i}^*\right)\right) + (1 - p_i) \left(g_i \left(a_i^*, \widehat{a}_{-i}\right) - g_i \left(\widehat{a}_i, \widehat{a}_{-i}\right)\right) < 0.$$

But if $p_i < \overline{p}_i$, $\overline{p}_i > 0$; now consider $\lambda \in \Delta(A_{-i})$ with $\lambda\left(a_{-i}^*\right) = p_i$ and $\lambda\left(\widehat{a}_{-i}\right) = 1 - p_i$;

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) \left(g_i\left(a_i^*, a_{-i}\right) - g_i\left(\widehat{a}_i, a_{-i}\right)\right) = \left\{ \begin{array}{c} p_i \left(g_i \left(a_i^*, a_{-i}^*\right) - g_i \left(\widehat{a}_i, a_{-i}^*\right)\right) \\ + (1 - p_i) \left(g_i \left(a_i^*, \widehat{a}_{-i}\right) - g_i \left(\widehat{a}_i, \widehat{a}_{-i}\right)\right) \end{array} \right\} < 0$$

so a_i^* is not a best response to λ. Thus a^* is not $\mathbf{p}$-dominant for any $\mathbf{p}$ with $p_i < \overline{p}_i$. ■

We can use Lemma 4 to give an exact characterization of when an action profile is $\mathbf{p}$-dominant for some $\mathbf{p} < \mathbf{1}$.

Corollary 3 *Pure strategy profile a^* is a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$, for some $\mathbf{p} < \mathbf{1}$, if and only if it is semi-strict.*

Proof. By construction, $\overline{\mathbf{p}} < \mathbf{1}$ if and only if $g_i\left(a_i^*, a_{-i}^*\right) > g_i\left(a_i, a_{-i}^*\right)$ for all $a_i \in A_i^+$. But this is exactly the definition of semi-strict equilibrium, since a_i is not an element of A_i^+ exactly if $g_i\left(a_i^*, a_{-i}\right) \geq g_i\left(a_i, a_{-i}\right)$ for all $a_{-i} \in A_{-i}$. ■

Lemma 5 *If a^* is a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$, for some $\mathbf{p} < \mathbf{1}$, then a^* is robust to limit common knowledge elaborations.*

Proof. Follows immediately from Corollary 1. ■

The converse is false; see example 2. So limit common knowledge does not fully characterize semi-strict equilibria. However, the following result holds: if $\left\{P^k\right\}_{k=1}^{\infty}$ is an elaboration sequence that does not satisfy limit common knowledge, then there exists a complete information game $\mathbf{g}$ with a strict equilibrium a^* such that every LEAD μ of $\left[\mathbf{g}, \left\{P^k\right\}_{k=1}^{\infty}\right]$ has $\mu(a^*) = 0$. See the Appendix. This result is closely related to the approximate lower hemicontinuity results of Monderer and Samet [1996] and Kajii and Morris [1997].

4.3 Limit Independent Elaborations

Definition 21 *Probability distribution $P \in \Delta(T)$ is independent if $P(t) = \prod_{i \in \mathcal{I}} P_i(t_i)$, for some $P_i \in \Delta(T_i)$.*

Definition 22 *Elaboration sequence $\left\{P^k\right\}_{k=1}^{\infty}$, with $P^k \to P^{\infty}$, satisfies limit independence if P^{∞} is independent.*

Lemma 6 *If sequence $\left\{P^k\right\}_{k=1}^{\infty}$ satisfies limit independence, then it satisfies limit common knowledge.*

Proof. First note that since the state space is countable, $P^k \to P^{\infty}$ pointwise implies that convergence is uniform in states. Let $\left\{P^k\right\}_{k=1}^{\infty}$ be an elaboration sequence satisfying limit independence. Write

$$\mathbf{A}_i^k(\varepsilon) = \left\{ t_i \in T_i^* : \begin{array}{l} P^k(t_i) > 0 \text{ and} \\ \left|P^k\left(S_{-i} \middle| t_i\right) - P^{\infty}\left(S_{-i} \middle| t_i\right)\right| \leq \varepsilon, \text{ for all } S_{-i} \subseteq T_{-i} \end{array} \right\}.$$

Since $P^k[T^*] \to 1$ and P^k converges uniformly, $P^k\left(\mathbf{A}_i^k(\varepsilon)\right) \to 1$ as $k \to \infty$. Fix ε, and choose $K(\varepsilon)$ such that $P^\infty\left(\bigcap_{i\in\mathcal{I}} \mathbf{A}_i^k(\varepsilon)\right) \geq 1-\varepsilon$ for all $k \geq K(\varepsilon)$. Now for all $k \geq K(\varepsilon)$ and $t_i \in \mathbf{A}_i^k(\varepsilon)$ with $P^k(t_i) > 0$, $\left|P^k\left(\mathbf{A}_{-i}^k(\varepsilon)\middle|\, t_i\right) - P^\infty\left(\mathbf{A}_{-i}^k(\varepsilon)\middle|\, t_i\right)\right| \leq \varepsilon$ and $P^\infty\left(\mathbf{A}_{-i}^k(\varepsilon)\right) \geq 1-\varepsilon$. By independence, $P^\infty\left(\mathbf{A}_{-i}^k(\varepsilon)\middle|\, t_i\right) = P^\infty\left(\mathbf{A}_{-i}^k(\varepsilon)\right)$. Thus $P^k\left(\mathbf{A}_{-i}^k(\varepsilon)\middle|\, t_i\right) \geq 1-2\varepsilon$. So for all $k \geq K(\varepsilon)$,

$$\bigcap_{i\in\mathcal{I}} \mathbf{A}_i^k(\varepsilon) \subseteq B_*^{(1-2\varepsilon,..,1-2\varepsilon),k}\left(\bigcap_{i\in\mathcal{I}} \mathbf{A}_i^k(\varepsilon)\right),$$

so by Lemma 3, $\bigcap_{i\in\mathcal{I}} \mathbf{A}_i^k(\varepsilon) \subseteq C^{(1-2\varepsilon,..,1-2\varepsilon),k}(T^*)$ and $\left\{P^k\right\}_{k=1}^\infty$ satisfies limit common knowledge. ■

Proposition 4 *If a^* is a semi-strict equilibrium of $\mathbf{g}$, then a^* is robust to limit independent elaborations.*

Proof. Follows immediately from Proposition 3 and Lemma 6. ■

4.4 Independent Elaborations

Definition 23 *Elaboration sequence $\left\{P^k\right\}_{k=1}^\infty$ satisfies independence if each P^k is independent.*

- If $\left\{P^k\right\}_{k=1}^\infty$ satisfies independence, then it satisfies limit independence.

Okada [1981] introduced the following definition of *strictly perfect* equilibria. Fix $\widetilde{\varepsilon} \equiv \{\widetilde{\varepsilon}_i\}_{i\in\mathcal{I}}$, each $\widetilde{\varepsilon}_i : A_i \to \Re_{++}$. Let $\mathbf{g}(\widetilde{\varepsilon})$ be the complete information game where payoffs are given by $\mathbf{g}$, but each player i is constrained to chose action a_i with probability at least $\varepsilon_i(a_i)$. Strategy profile α is an equilibrium of $\mathbf{g}(\widetilde{\varepsilon})$ if and only if $\alpha_i(a_i) = \widetilde{\varepsilon}_i(a_i)$ holds whenever $g_i(a_i, \alpha_{-i}) < g_i(a_i', \alpha_{-i})$ for some $a_i' \in A_i$.

Definition 24 *Strategy profile $\alpha \in \mathcal{A}$ is a strictly perfect equilibrium (of $\mathbf{g}$) if for every sequence of functions $\left\{\widetilde{\varepsilon}^k\right\}_{k=1}^\infty$, with $\widetilde{\varepsilon}_i^k(a_i) \to 0$ for every $i \in \mathcal{I}$ and $a_i \in A_i$, there exists a sequence $\left\{\alpha^k\right\}_{k=1}^\infty$, each α^k an equilibrium of $\mathbf{g}(\widetilde{\varepsilon}^k)$, with $\alpha^k \to \alpha$.*

Definition 25 *Action distribution μ is a strictly perfect equilibrium action distribution if $\mu(a) = \prod_{i\in\mathcal{I}} \alpha_i(a_i)$ for some strictly perfect equilibrium α.*

Proposition 5 *Action distribution $\mu \in \Delta(A)$ is robust to independent elaborations if and only if it is strictly perfect.*

Proof. Suppose μ is robust to independent elaborations. Fix *any* $\left\{\widetilde{\varepsilon}^k\right\}_{k=1}^\infty$, with $\widetilde{\varepsilon}^k \to 0$. Consider any independent elaboration sequence $\left\{P^k\right\}_{k=1}^\infty$ with $P^k = \prod_{i\in\mathcal{I}} P_i^k$ and $P_i^k(a_i) = \widetilde{\varepsilon}_i(a_i)$. There exists a sequence of equilibrium strategies $\left\{\sigma^k\right\}_{k=1}^\infty$ which induce action distributions $\mu^k \to \mu$ with the property $\mu^k = \prod_{i\in\mathcal{I}} \mu_i^k$, where μ_i^k is the distribution on A_i induced by σ_i^k. Let $\alpha_i^k(a_i) = P_i^k(a_i) + \sum_{t_i\in T_i^*} P_i^k(t_i)\,\sigma_i^k(a_i\,|t_i)$. Note that $\sigma_i^k(a_i\,|t_i) > 0$

for some $t_i \in T_i^*$ implies that a_i is a best response to α_{-i}^k. Thus α_i^k is best response to α_{-i}^k in the constrained game $\mathbf{g}(\widehat{\varepsilon}^k)$. Since $P^k(T^*) \to 1$ and $\prod_{i \in \mathcal{I}} \alpha_i^k \to \mu$, μ is a strictly perfect equilibrium action distribution.

Conversely, suppose $\mu = \prod_{i \in \mathcal{I}} \alpha_i$ is a strictly perfect Nash equilibrium action distribution. Consider any independent elaboration sequence $\left\{P^k\right\}_{k=1}^{\infty}$. Let $\widehat{\varepsilon}_i^k(a_i) = P_i^k(a_i)$. Let $\left\{\alpha^k\right\}_{k=1}^{\infty}$ be any sequence of equilibria of $\left\{\mathbf{g}\left(\widehat{\varepsilon}^k\right)\right\}_{k=1}^{\infty}$with $\alpha^k \to \alpha$. Let $\sigma_i^k(a_i|t_i^*) = \frac{\alpha_i^k(a_i) - P_i^k(a_i)}{\sum_{t_i \in T_i^*} P_i^k(t_i)}$ for all $t_i^* \in T_i^*$. Clearly σ^k is an equilibrium of $\left(\mathbf{g}, P^k\right)$. ■

4.5 Tightening Results: Examples

The only tight result in this section was the equivalence of strict perfection and robustness to all independent elaborations. In all the other results, we have a gap between known necessary and known sufficient conditions. Here we report three examples highlighting what tight results might and might not be feasible.

Example 2 *Player* 1 *chooses the row, player* 2 *chooses the column:*

	L	M	R
U	$0,0$	$1,0$	$-1,0$
D	$0,0$	$0,0$	$0,0$

.

Strategy profile (U, L) is a Nash equilibrium. It is not semi-strict, since $g_1(U, L) = g_1(D, L)$ and $g_1(U, R) < g_1(D, R)$. So it is not $\mathbf{p}$-dominant for any $\mathbf{p} < \mathbf{1}$ (by Corollary 3). Thus it does not satisfy the sufficient condition that we identified for robustness to all limit common knowledge elaborations. Nonetheless, it *is* robust not only to all limit common knowledge elaborations, but to *all* elaborations. To see why, consider *any* elaboration sequence $\left\{P^k\right\}_{k=1}^{\infty}$. Let $\gamma^k = \sqrt{1 - P^k\left[T_2^* | T_1^*\right]} \to 0$ as $k \to \infty$ and

$$\widehat{T}_1^k = \left\{t_1 \in T_1^* : P^k(t_1) = 0 \text{ or } \gamma^k P^k\left(T_2^* | t_1\right) + P^k\left(M | t_1\right) \geq P^k\left(R | t_1\right)\right\}.$$

The following strategy profile for standard types gives an equilibrium of the game $\left(\mathbf{g}, P^k\right)$:

$$\sigma_1^k\left(U | t_1^*\right) = 1 - \sigma_1^k\left(D | t_1\right) = \begin{cases} 1, \text{ if } t_1 \in \widehat{T}_1^k \\ 0, \text{ if } t_1 \in T_1^* \setminus \widehat{T}_1^k \end{cases}$$

$$\text{and } \sigma_2^k\left(a_2 | t_2\right) = \begin{cases} 1 - \gamma^k, \text{ if } a_2 = L \\ \gamma^k, \text{ if } a_2 = M \\ 0, \text{ if } a_2 = R \end{cases} \quad \text{for all } t_2 \in T_2^*.$$

Now $t_1 \notin \widehat{T}_1^k \Rightarrow \gamma^k P^k\left(T_2^* | t_1\right) + P^k\left(M | t_1\right) < P^k\left(R | t_1\right) \Rightarrow \gamma^k P^k\left(T_2^* | t_1\right) < 1 - P^k\left(T_2^* | t_1\right) \Rightarrow P^k\left(T_2^* | t_1\right) < \frac{1}{1+\gamma^k}$. Thus for each k,

$$\begin{aligned} 1 - \left(\gamma^k\right)^2 &= P^k\left[T_2^* | T_1^*\right] \\ &\leq P^k\left[\widehat{T}_1^k \middle| T_1^*\right] + \left(1 - P^k\left[\widehat{T}_1^k \middle| T_1^*\right]\right)\left(\frac{1}{1 + \gamma^k}\right). \end{aligned}$$

Re-arranging gives

$$\begin{aligned}
P^k\left[\widehat{T}_1^k \middle| T_1^*\right] &\geq \frac{\left(1+\gamma^k\right)\left(1-\left(\gamma^k\right)^2\right)-1}{\gamma^k} \\
&= 1-\gamma^k-\left(\gamma^k\right)^2 \\
&\to 1 \text{ as } k\to\infty.
\end{aligned}$$

Thus if μ^k is the EAD corresponding to strategy profile σ^k,

$$\mu^k\left((U,L)\right) \geq P^k\left[T^*\right] P^k\left[\widehat{T}_1^k \middle| T_1^*\right]\left(1-\gamma^k\right) \to 1.$$

Example 3 *Player* 1 *chooses the row, player* 2 *chooses the column, and player* 3 *chooses the matrix:*

	L	R
L	$0,1,1$	$1,0,1$
R	$0,1,1$	$0,0,1$

.

	L	R
L	$1,1,0$	$-1,0,0$
R	$0,1,0$	$0,0,0$

(L,L,L) is strictly perfect but not robust to limit independent elaborations (and thus not robust to limit common knowledge elaborations).

- (L,L,L) is strictly perfect: consider any $\widetilde{\varepsilon}^k \to 0$. If α^k is an equilibrium of $\mathbf{g}\left(\widetilde{\varepsilon}^k\right)$, then $\alpha_2^k(R) = \widetilde{\varepsilon}_2^k(R)$ and $\alpha_3^k(R) = \widetilde{\varepsilon}_3^k(R)$. Thus player 1's payoff to action L is $\widetilde{\varepsilon}_2^k(R)\left(1-\widetilde{\varepsilon}_3^k(R)\right) + \widetilde{\varepsilon}_3^k(R)\left(1-\widetilde{\varepsilon}_2^k(R)\right) - \widetilde{\varepsilon}_2^k(R)\,\widetilde{\varepsilon}_3^k(R)$. As $k\to\infty$, this expression becomes positive. Since player 1's payoff to R is 0, we have $\alpha_1^k(R) = \widetilde{\varepsilon}_1^k(R)$ in the unique equilibrium of $\mathbf{g}\left(\widetilde{\varepsilon}^k\right)$ for k sufficient large. Thus $\alpha_i^k(R)\to 0$ for all i and (L,L,L) is strictly perfect.
- (L,L,L) is not robust to limit independent elaborations: consider elaboration sequence $\left\{P^k\right\}_{k=1}^{\infty}$, with $P^k(t^*) = 1-\varepsilon^k$ and $P^k\left(t_1^*,R,R\right) = \varepsilon^k$, for some $t^*\in T^*$ and real numbers $\varepsilon^k\to 0$. Every EAD of $\left(\mathbf{g},P^k\right)$ has $\mu^k(R,L,L) = 1-\varepsilon^k$ and $\mu^k(L,L,L) = \varepsilon^k$. Thus (L,L,L) is not robust to limit independent elaborations.

Example 4 *Player* 1 *chooses the row, player* 2 *chooses the column, and player* 3 *chooses the matrix:*

	L	R
L	$-1,-1,-1$	$-1,1,1$
R	$1,1,-1$	$1,-1,1$

.

	L	R
L	$1,-1,1$	$1,1,-1$
R	$-1,1,1$	$-1,-1,-1$

The game has the following "matching pennies" interpretation. Each player chooses L or R. Each player gets -1 if he matches the choice of the player preceding him in the cycle $1\to 2\to 3\to 1...$, 1 otherwise.

This game has a unique Nash equilibrium (all randomize 50/50). This equilibrium satisfies all standard refinements. In particular, it is regular, and thus strictly perfect and essential (van Damme 1991). But we will show that it is not robust to limit common knowledge elaborations.

Let $T_1^* = \{t_1,t_1',...\}$, $T_2^* = \{t_2,t_2',..\}$ and $T_3^* = \{t_3,..\}$. Let P^k satisfy $P^k(t_1,t_2,L) = \varepsilon^k$, $P^k(t_1,t_2,t_3) = \frac{2}{3}$ and $P^k\left(t_1',t_2',t_3\right) = \frac{1}{3}-\varepsilon^k$ with $P^k(t) = 0$ for all other t. Thus (ignoring zero probability types) we have:

	t_3	L
t_1, t_2	$\frac{2}{3}$	ε^k
t_1', t_2'	$\frac{1}{3} - \varepsilon^k$	0

where $\varepsilon^k \to 0$ as $k \to \infty$.

The game $(\mathbf{g}, P^k)$ has an essentially unique Bayesian Nash equilibrium. This equilibrium has $\sigma_1^k(L|t_1) = \frac{1}{2}$, $\sigma_1^k(L|t_1') = 1$, $\sigma_2^k(L|t_2) = \frac{3}{4}(1-\varepsilon^k)$, $\sigma_2^k(L|t_2') = 0$ and $\sigma_3^k(L|t_3) = \frac{1}{2} - \frac{3}{4}\varepsilon^k$. To see why, first note that we would get a contradiction if type t_1, t_2, or t_3 of player 1,2 and 3 respectively played a pure strategy. But if type t_2 of player 2 plays a mixed strategy, we must have $\sigma_1^k(L|t_1) = \frac{1}{2}$. But if type t_1 of player 1 plays a mixed strategy, he must assess the probability that 3 plays L to be $\frac{1}{2}$. Thus $\frac{3\varepsilon^k}{2+3\varepsilon^k}(1) + \frac{2}{2+3\varepsilon^k}\left(\sigma_3^k(L|t_3)\right) = \frac{1}{2}$; so $\sigma_3^k(L|t_3) = \frac{1}{2} - \frac{3}{4}\varepsilon^k$. But $\sigma_3^k(L|t_3) < \frac{1}{2} \Rightarrow \sigma_1^k(L|t_1') = 1 \Rightarrow \sigma_2^k(L|t_2') = 0$. Finally, if type t_3 of 3 plays a mixed strategy, he must assess the probability that 2 plays L to be $\frac{1}{2}$; thus $\frac{2}{3-3\varepsilon^k}\left(\sigma_2^k(L|t_2)\right) + \frac{1-3\varepsilon^k}{3-3\varepsilon^k}(0) = \frac{1}{2}$; so $\sigma_2^k(L|t_2) = \frac{3}{4} - \frac{3}{4}\varepsilon^k$.

What happens as $k \to \infty$? Type t_1 of player 1 plays L with probability $\frac{1}{2}$, type t_1' of player 1 always plays L, type t_2 of player 2 plays L with probability $\frac{3}{4}$, type t_2' of player 2 always plays R, while type t_3 of player 3 plays L with probability $\frac{1}{2}$. Writing μ^k for the equilibrium action distribution generated by the strategies above, we have $\mu^k \to \mu$ where $\mu(L,L,L) = \frac{1}{8}$, $\mu(L,L,R) = \frac{1}{8}$, $\mu(L,R,L) = \frac{5}{24}$, $\mu(L,R,R) = \frac{5}{24}$, $\mu(R,L,L) = \frac{1}{8}$, $\mu(R,L,R) = \frac{1}{8}$, $\mu(R,R,L) = \frac{1}{24}$ and $\mu(R,R,R) = \frac{1}{24}$. This is a correlated equilibrium of $\mathbf{g}$.[6]

5 Justified by Full Support Elaborations

Another way of examining the robustness of equilibria of $\mathbf{g}$ is to ask if behavior close to those equilibria is possible equilibrium behavior in *some* elaboration where all committed types are possible. In this section, we consider this question under the same progressively stronger restrictions on the elaboration sequences considered in the previous section. But now the solution concepts become progressively stronger.

Definition 26 *Probability distribution $P \in \Delta(T)$ has full support if, for all $t_i \in T_i^*$ with $P(t_i) > 0$, $P(a_{-i}|t_i) > 0$ for all $a_{-i} \in A_{-i}$.*

Definition 27 *Elaboration sequence $\{P^k\}_{k=1}^{\infty}$ has full support if each P^k has full support.*

For a given property X of canonical elaborations, we will say:

Definition 28 *Action distribution $\mu \in \Delta(A)$ is justified by some full support (type X) elaboration if μ is a LEAD of $\left[\mathbf{g}, \{P^k\}_{k=1}^{\infty}\right]$ for some full support (type X) elaboration sequence $\{P^k\}_{k=1}^{\infty}$.*

[6] If the game $\mathbf{g}$ is perturbed, but we hold fixed the elaboration sequence $\{P^k\}_{k=1}^{\infty}$, one can verify that $\left[\mathbf{g}, \{P^k\}_{k=1}^{\infty}\right]$ has a unique LEAD equal to the unique Nash equilibrium action distribution of the perturbed game $\mathbf{g}$. However, it is presumably possible to choose $\{P^k\}_{k=1}^{\infty}$ (as a function of the perturbed $\mathbf{g}$) in order to show that the Nash equilibria of the perturbed games are not robust either.

5.1 No Restrictions

We will use the following well known characterization of when an action is dominated.

Lemma 7 *Action $a_i \in A_i$ is not dominated (in* $\mathbf{g}$*) if and only if there exists a full support probability distribution $\phi \in \Delta_{++}(A_{-i})$ such that a_i is a best response to ϕ, i.e. for all $a_i' \in A_i$,*

$$\sum_{a_{-i}\in A_{-i}} \phi(a_{-i})\, g_i(a_i, a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} \phi(a_{-i})\, g_i(a_i', a_{-i}).$$

Proof. See Pearce [1984], appendix B. ■

Proposition 6 *Action distribution μ is justified by some full support elaboration if and only if it is an undominated correlated equilibrium.*

The following proof is close to the arguments of Fudenberg, Kreps and Levine [1988].
Proof. Suppose μ is a LEAD of some full support elaboration sequence $\left[\mathbf{g}, \left\{P^k\right\}_{k=1}^{\infty}\right]$. By Lemma 1, μ is a correlated equilibrium of $\mathbf{g}$. If a_i is dominated and distribution P^k has full support, then Lemma 7 implies $\mu^k(a_i) = 0$ in any EAD of $\left(\mathbf{g}, P^k\right)$. Thus $\mu(a_i) = 0$ in every LEAD of $\left[\mathbf{g}, \left\{P^k\right\}_{k=1}^{\infty}\right]$.

Now suppose μ is an undominated correlated equilibrium of $\mathbf{g}$. Let $A_i^+ = \{a_i \in A_i : \mu(a_i) > 0\}$. By Lemma 7, for each $a_i \in A_i^+$, we can define $\phi_i(.|a_i) \in \Delta_{++}(A_{-i})$ such that a_i is a best response to $\phi_i(.|a_i)$. Now let $N = \sum\limits_{i\in\mathcal{I}} \#A_i^+$, fix $\varepsilon^k \to 0$, let $f_i : A_i^+ \to T_i^*$ be onto and write $f(a) = (f_1(a_1), ..., f_I(a_I))$. Thus for each action a_i, we identify a unique *standard* type, $f_i(a_i)$. Now set $P^k(f(a)) = \left(1 - N\varepsilon^k\right)\mu(a)$ for all $a \in A$, $P^k\left(f_i(a_i), a_{-i}'\right) = \varepsilon^k \phi_i\left(a_{-i}'\middle| a_i\right)$ for all $a_i \in A_i^+$ and $a_{-i}' \in A_{-i}$, and $P^k(t) = 0$ for all other t. Sequence $\left\{P^k\right\}_{k=1}^{\infty}$ has full support. Consider the strategy profile where each standard type $f_j(a_j)$ chooses action a_j. Under this strategy profile, type $f_i(a_i)$ attaches probability

$$\left(\frac{\left(1 - N\varepsilon^k\right)\mu(a_i)}{\left(1 - N\varepsilon^k\right)\mu(a_i) + \varepsilon^k}\right)\mu\left(a_{-i}'\middle| a_i\right) + \left(\frac{\varepsilon^k}{\left(1 - N\varepsilon^k\right)\mu(a_i) + \varepsilon^k}\right)\phi_i\left(a_{-i}'\middle| a_i\right)$$

to his opponents playing a_{-i}'. This gives a convex combination of distributions with a_i a best response to both. Thus this strategy profile is an equilibrium of $\left(\mathbf{g}, P^k\right)$. But if μ^k is the action distribution generated by this strategy profile, $\mu^k \to \mu$. ■

5.2 Limit Common Knowledge Elaborations

Proposition 7 *Action distribution μ is justified by some full support limit common knowledge elaboration if and only if it is an undominated correlated equilibrium.*

Proof. Exactly the same argument as that for Proposition 6 works, since the elaboration sequences constructed in the proof of that Proposition satisfied limit common knowledge. ■

Börgers [1994] showed how one round of deletion of weakly dominated strategies follows from approximate common knowledge of rationality.

5.3 Limit Independent Elaborations

Proposition 8 *Action distribution μ is justified by some full support limit independent elaboration if and only if it is an undominated Nash equilibrium.*

Proof. Again, the same argument as that for Proposition 6 works, except that now independence in the limit ensures that μ is a Nash equilibrium. ■

A version of this result appears in Fudenberg, Kreps and Levine [1988] (they assumed a unique standard type, which implies limit independence). Dekel and Fudenberg [1990] clarified the role of limit independent elaborations in deleting (one round of) weakly dominated strategies.

5.4 Independent Elaborations

Consider the perturbed game $\mathbf{g}(\widetilde{\varepsilon})$ introduced in section 4.4. The following notion is due to Selten [1965, 1975].

Definition 29 *Strategy profile $\alpha \in \mathcal{A}$ is a perfect equilibrium if there exist a sequence of functions $\widetilde{\varepsilon}^k$, each $\widetilde{\varepsilon}_i^k(a_i) \to 0$ as $k \to \infty$, and $\left\{\alpha^k\right\}_{k=1}^{\infty}$, with $\alpha^k \to \alpha$, such that each α^k is an equilibrium of $\mathbf{g}\left(\widetilde{\varepsilon}^k\right)$.*

An equivalent characterization is the following: strategy profile $\alpha \in \mathcal{A}$ is a perfect Nash equilibrium if there exists real numbers $\varepsilon^k \to 0$ and $\alpha^k \to \alpha$, such that $\alpha_i^k(a_i) < \varepsilon^k$ holds whenever $g_i(a_i, \alpha_{-i}^k) < g_i(a_i', \alpha_{-i}^k)$ for some $a_i' \in A_i$.

Definition 30 *Action distribution μ is a perfect equilibrium action distribution if $\mu(a) = \prod_{i \in \mathcal{I}} \alpha_i(a_i)$ for some perfect equilibrium α.*

Proposition 9 *Action distribution μ is justified by some full support independent elaboration if and only if it is a perfect equilibrium action distribution.*

Proof. Suppose μ is justified by some full support independent elaboration. Then there exists independent elaboration sequence $P^k = \prod_{i \in \mathcal{I}} P_i^k$, a sequence of equilibrium strategy profiles of $\left(\mathbf{g}, P^k\right)$, $\left\{\sigma^k\right\}_{k=1}^{\infty}$, which induce action distributions $\mu^k \to \mu$. Clearly $\mu^k = \prod_{i \in \mathcal{I}} \mu_i^k$ where μ_i^k is the distribution on A_i induced by σ_i^k. Let $\alpha_i^k(a_i) = P_i^k(a_i) + \sum_{t_i \in T_i^*} P_i^k(t_i)\sigma_i^k(a_i | t_i)$. Clearly α_i^k has full support. Also note that $\sigma_i^k(a_i | t_i) > 0$ implies that a_i is a best response to α_{-i}^k. Thus if a_i is not a best response to α_{-i}^k, then $\alpha_i^k(a_i) \leq 1 - P_i^k(T_i^*) \leq 1 - P^k(T^*)$. Thus α^k is a full support strategy profile which assigns probability less than $1 - P^k(T^*)$ to actions that are not best responses. Since $P^k(T^*) \to 1$ and $\prod_{i \in \mathcal{I}} \alpha_i^k \to \mu$, μ is a perfect equilibrium action distribution. The converse is straightforward. ■

It is well known that all perfect equilibria are undominated Nash equilibria. The following example shows that, with more than two players, they need not be equivalent (see van Damme 1991, page 29).

Example 5 *Player* 1 *chooses the row, player* 2 *chooses the column, and player* 3 *chooses the matrix:*

	L	R
L	$1,1,1$	$1,0,1$
R	$1,1,1$	$0,0,1$

	L	R
L	$1,1,0$	$0,0,0$
R	$0,1,0$	$1,0,0$

(L, L, L) is the unique perfect equilibrium, but (R, L, L) is an undominated Nash equilibrium.

6 Discussion

6.1 Summary

We conclude by summarizing the results surveyed in the paper in a little more detail. In each case, we either report the best sufficient conditions (sc) and necessary conditions (nc) that we know, or we give an exact characterization (*) where available:

class of elaboration \ type of robustness	Robust to *all* sequences	Justified by *some* full support sequence
No Restrictions	sc: unique CE sc: **p**-dom. with $\sum_{i \in \mathcal{I}} p_i \leq 1$ nc: strictly perfect NE note: not all strict NE	*: undominated CE
Limit Common Knowledge	sc: unique CE sc: semi-strict NE nc: strictly perfect NE note: not all essential NE	*: undominated CE
Limit Independence	sc: unique CE sc: semi-strict NE nc: strictly perfect NE	*: undominated NE
Independence	*: strictly perfect NE	*: perfect NE

What should we conclude from this table? The right hand column indicates which outcomes might be reasonable equilibrium outcomes for a game $\mathbf{g}$, assuming that there is some possibility that the analyst's model $\mathbf{g}$ is not quite complete. The left hand column indicates which equilibrium outcomes of game $\mathbf{g}$ are equilibrium outcomes, *whatever* the fine details that have been missed from the analysis.

In each case, what restrictions on the elaborations are relevant? In Kajii and Morris [1995], we argued that if you want to avoid assuming (full or approximate) common knowledge of payoffs on the part of the players, then no extra restrictions are justified. But the limit common knowledge case clearly provides an important benchmark. Fudenberg, Kreps and Levine [1988] argue for limit independence in analyzing the right hand column on the grounds that we should allow for the possibility of correlation *conditional* on the game not being as expected. The refinements literature has focussed on independent elaborations for no obvious reason.

6.2 Two More Restrictions

Two more natural restrictions to place on elaboration sequences $\{P^k\}_{k=1}^{\infty}$ are:

Definition 31 *Elaboration sequence* $\left\{P^k\right\}_{k=1}^{\infty}$ *has finite support if* $\left\{t : P^k(t) > 0 \text{ for some } k\right\}$ *is finite.*

Definition 32 *Elaboration sequence* $\left\{P^k\right\}_{k=1}^{\infty}$ *has a unique standard type if* $\left\{t_i \in T_i^* : P_i^k(t_i) > 0 \text{ for some } k\right\}$ *is a singleton for all* i *and* k.

Fudenberg and Tirole [1991, Theorem 14.5] showed that if $\left\{P^k\right\}_{k=1}^{\infty}$ has finite support, it satisfies limit common knowledge. It may still fail to satisfy limit independence. If $\left\{P^k\right\}_{k=1}^{\infty}$ has a unique standard type, then it satisfies limit independence. It may fail to satisfy independence.

Fudenberg and Tirole [1991, Theorem 14.6] show that if an equilibrium is *essential* [Wu and Jiang 1962], it is robust to all elaboration sequences with a unique standard type. For a generic choice of normal form payoffs, all equilibria are essential. It seems likely that essential equilibria are robust to limit independent elaborations, but the unique standard type result doesn't show this. Example 4 shows that not all essential equilibria are robust to limit common knowledge elaborations (or even finite support elaborations).

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Appendix

In this appendix, we show that if sequence $\left\{P^k\right\}_{k=1}^{\infty}$ does not satisfy limit common knowledge, then we can find a game $\mathbf{g}$ with strict Nash equilibrium a^* such that a^* is not a LEAD of $\left[\mathbf{g}, \left\{P^k\right\}_{k=1}^{\infty}\right]$.

More formally, we have:

Definition 33 *Elaboration sequence $\left\{P^k\right\}_{k=1}^{\infty}$ satisfies payoff continuity if, for any strict Nash equilibrium a^* of any complete information game $\mathbf{g}$, a^* is a limit equilibrium of $\left[\mathbf{g}, \left\{P^k\right\}_{k=1}^{\infty}\right]$.*

Proposition 10 *$\left\{P^k\right\}_{k=1}^{\infty}$ satisfies payoff continuity if and only if $\left\{P^k\right\}_{k=1}^{\infty}$ satisfies limit common knowledge.*

Proof. "if" follows from Lemma 5. For "only if," fix $a \in A$ and let $E_i^{\mathbf{p},k}(a)$ be the projection of $C^{\mathbf{p},k}\left(\underset{i\in\mathcal{I}}{\times}\left(T_i^* \cup \{a_i\}\right)\right)$ on T_i, i.e., $C^{\mathbf{p},k}\left(\underset{i\in\mathcal{I}}{\times}\left(T_i^* \cup \{a_i\}\right)\right) = \underset{i\in\mathcal{I}}{\times} E_i^{\mathbf{p},k}(a)$, each $E_i^{\mathbf{p},k}(a) \subseteq T_i$. Now we have

$$P^k\left[E_{-i}^{\mathbf{p},k}(a)\middle| t_i\right] \geq p_i, \text{ for all } t_i \in E_i^{\mathbf{p},k}(a) \text{ with } P^k(t_i) > 0. \tag{4}$$

Since $E_j^{\mathbf{p},k}(a) \subseteq T_j^* \cup \{a_j\}$ for all $j \in \mathcal{I}$,

$$P^k\left[\underset{j\neq i}{\times}\left(T_j^* \cup \{a_j\}\right)\middle| t_i\right] \geq p_i, \text{ for all } t_i \in E_i^{\mathbf{p},k}(a) \text{ with } P^k(t_i) > 0. \tag{5}$$

Write $F_i^{\mathbf{p},k} = \underset{a\in A}{\cap} E_i^{\mathbf{p},k}(a)$, $F_{-i}^{\mathbf{p},k} = \underset{j\neq i}{\times} F_j^{\mathbf{p},k}$ and $N = \#A$. By (4),

$$P^k\left[F_{-i}^{\mathbf{p},k}\middle| t_i\right] \geq 1 - N(1-p_i), \text{ for all } t_i \in F_i^{\mathbf{p},k} \text{ with } P^k(t_i) > 0. \tag{6}$$

By (5),

$$P^k\left[\underset{j\neq i}{\times}\left(T_j^* \cup \{a_j\}\right)\middle| t_i\right] \geq p_i \geq 1 - N(1-p_i), \text{ for all } t_i \in F_i^{\mathbf{p},k} \text{ with } P^k(t_i) > 0. \tag{7}$$

Thus

$$\underset{a\in A}{\cap}\left(C^{\mathbf{p},k}\left(\underset{i\in\mathcal{I}}{\times}\left(T_i^* \cup \{a_i\}\right)\right)\right) = \underset{i\in\mathcal{I}}{\cap} F_i^{\mathbf{p},k} \subseteq C^{\mathbf{q},k}(T^*) \tag{8}$$

whenever $q_i = 1 - N(1-p_i)$ [equivalently, whenever $p_i = 1 - \frac{1}{N}(1-q_i)$]. Now suppose that P^k does not satisfy limit common knowledge. That is, there exists $\mathbf{q} < \mathbf{1}$ and $\varepsilon > 0$ such that $P^k\left[C^{\mathbf{q},k}(T^*)\right] \leq 1 - \varepsilon$ for infinitely many k. Then there exists $a^* \in A$, $\mathbf{p} < \mathbf{1}$ and $\delta > 0$ such that $P^k\left[C^{\mathbf{p},k}\left(\underset{i\in\mathcal{I}}{\times}\left(T_i^* \cup \{a_i^*\}\right)\right)\right] \leq 1 - \delta$ for infinitely many k. Let $\eta < 1 - p_i$ for all i. Now consider the following game:

$$g_i(a) = \begin{cases} \eta, \text{ if } a = a^* \\ 0, \text{ if } a_i = a_i^* \text{ and } a_{-i} \neq a_{-i}^* \\ 0, \text{ if } a_i \neq a_i^* \text{ and } a_{-i} = a_{-i}^* \\ 1, \text{ if } a_i \neq a_i^* \text{ and } a_{-i} \neq a_{-i}^* \end{cases}.$$

Action profile a^* is never played by type profiles not in $C^{\mathbf{p},k}\left(\underset{i\in\mathcal{I}}{\times}\left(T_i^* \cup \{a_i^*\}\right)\right)$. Thus $\mu^k(a^*) < 1 - \delta$ if μ^k is an equilibrium action distribution of $\left(\mathbf{g}, P^k\right)$; so $\mu(a^*) < 1 - \delta$ if μ is a LEAD of $\left[\mathbf{g}, \left\{P^k\right\}_{k=1}^{\infty}\right]$. ■