History versus Expectations in Large Population Binary Games

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KFB Dynamics

We discuss another class of forward-looking expectations dynamics, due to Krugman (1991b) and Fukao and Benabou (1993).

- As in perfect foresight dynamics (PFD) due to Matusyama (1991) and Matsui and Matsuyama (1995),
  - there are a continuum of agents, so no single agent has a strategic impact,
  - there are frictions in action revisions, $\Rightarrow$ agents choose actions taking future payoffs into account.
- The way to introduce frictions is different:
  - PFD: action revision opportunities arrive according to independent Poisson processes;
  - KFB dynamics: agents can change actions at any time instant, with adjustment costs which depend on the rate of aggregate adjustment (proportional to $|\dot{x}(t)|$).

- Krugman directly translate the adjustment cost approach (citing Mussa 1978),
  but the contexts are *fundamentally different!*
  - Mussa: a single-agent investment problem, where the adjustment cost depends only on the agent's own choice.
  - Here: a many-agent situation, where the adjustment cost depends on other agents' choice.
- In fact, this is the source of an error in Krugman (1991b), as corrected by Fukao and Benabou (1993).
- Complications, even in defining equilibrium paths.

  (Most papers using this dynamics do not define equilibrium paths...)

  (PFD is much simpler and well behaved.)

- $x(t)$: fraction of agents choosing action 1 (out of two actions).

  Adjustment cost: $|\dot{x}(t)|/\gamma$, $\gamma > 0$.

  When $\dot{x}(t) = 0$, a single agent can instantly change actions infinitely often with no cost.
- Speed of aggregate adjustment depends directly on utilities (i.e., cardinal representation of preferences).
- Still we can formalize the dynamics, define its equilibria, and
- show that this dynamics shares the same stability properties as in PFD, at least in the two-action case
  (in which case a potential function trivially exists).

Stability Results

(Based on Oyama 2009b)

- Even when the static game has multiple equilibria, there is generically a unique equilibrium of the static game that is "stable" in the dynamic game for small frictions.

- A "potential method" as in PFD (due to Hofbauer and Sorger 1999):

  - Equilibrium paths of the dynamic game are translated into
  - solutions of an optimal control problem.

  (More complicated than in PFD, as $x(t)$ can hit the boundary of $[0, 1]$.
  $\Rightarrow$ "optimal control with state-variable constraints")

- Formally, a unique maximizer of the potential function is the unique state that is absorbing and globally accessible whenever the degree of friction is small.

Large Population Binary Games

- There are a continuum of players.
- Each player has two actions, 0 and 1.
- x: fraction of players playing action 1.
  $x = 1$: the state where every player is playing 1;
  $x = 0$: the state where every player is playing 0.
- $f_i(x)$: payoff function for action $i = 0, 1$
  when fraction x of players play 1 (hence $1 - x$ play 0).
  ($f_i \colon [0, 1] \to \mathbb{R}$ is assumed to be Lipschitz continuous.)
- (f_0, f_1) defines a population game.

- Denote

$$f(x) = f_1(x) - f_0(x).$$

Examples

- Economic geography (as in Krugman (1991a)):
  Actions are regions to live in.
- Sector choice and industrialization
  (as in Krugman (1991b), Matsuyama (1991, 1992)):
  Actions are sectors to work for.
- Investment:
  Action 1: to invest, Action 0: not to invest.
- Search in a decentralized market:
  Action 1: to search for trading partner,
  Action 0: not to search.
- Transportation:
  Actions are routes to use.
- Random-matching of a normal form game:
  In this case, $f_i(x)$ is linear in x.

Nash Equilibria

Recall $f(x) = f_1(x) - f_0(x)$. (x: fraction who play action 1)

- $x^* \in [0, 1]$ is a *Nash equilibrium state* of (f_0, f_1) if

$$x^* > 0 \Rightarrow f(x^*) \geq 0, \text{ and } x^* < 1 \Rightarrow f(x^*) \leq 0.$$

- $x^* \in [0, 1]$ is a *strict Nash equilibrium state* of (f_0, f_1) if

$$x^* > 0 \Rightarrow f(x^*) > 0, \text{ and } x^* < 1 \Rightarrow f(x^*) < 0.$$

- **Assumption.** There are finitely many equilibrium states.

  A sufficient condition: f is real analytic (not identically zero).

Potential Function

(Monderer and Shapley 1996, Sandholm 2001)

Recall $f(x) = f_1(x) - f_0(x)$. (x: fraction who play action 1)

Definition.
$F\colon [0,1] \to \mathbb{R}$ is said to be a *potential function* of (f_0, f_1) if

$$\frac{dF}{dx}(x) = f(x). \qquad (*)$$

- Consider the maximization problem:
  Maximize $F(x)$ subject to $x \in [0,1]$.
- Then:
  x^*: solution $\Rightarrow$ x^*: equilibrium state (but not vice versa).

Multiple Equilibria

Recall $f(x) = f_1(x) - f_0(x)$.   (x: fraction who play action 1)

- We consider the case where $f' > 0$ and $f(0) < 0 < f(1)$, so that $x = 0$ and $x = 1$ are both strict equilibrium states.
- In this case, potential function F becomes convex.
- We assume that $F(0) \neq F(1)$, so that F has a *unique* maximizer ($x = 0$ or $x = 1$).

- Note:
  The assumption that $f' > 0$ is made only to simplify the presentation.
  Our main result will hold as long as F has a unique global maximizer x^* and x^* is isolated from other critical points of F.

Modeling Frictions

Future can be important of present decision when

- players incur adjustment costs that depend on others' decision
  $\Rightarrow$ option to wait
  $\cdots$ Krugman (1991, QJE), where cost is given by $|\dot{x}(t)|/\gamma$;

or

- once a player chooses an action,
  he has to stick to that action for some time interval
  $\cdots$ Matsuyama (1991, QJE), Matsui and Matsuyama (1995, JET),
  where action revision opportunities follow a Poisson process.

KFB Dynamics

- A path $x(\cdot)\colon [0,\infty) \to [0,1]$ is said to be *feasible* if continuous and piecewise C^1.
- $(t_1, t_2) \subset [0,\infty)$ is called an *interior interval* of $x(\cdot)$ if $x(t) \in (0,1)$ for all $t \in (t_1, t_2)$.
- $[t_1, t_2] \subset [0,\infty)$ is called a *boundary interval* of $x(\cdot)$ if $x(t) = 0, 1$ for all $t \in [t_1, t_2]$.

- Players can change actions at any time instant with cost $|\dot{x}(t)|/\gamma$ ($\gamma > 0$).

  ($\dot{x}(t) = \lim_{s \searrow t} \dot{x}(s)$ if not differentiable.)

Defining Equilibrium Paths

Given a feasible path $x(\cdot)$,
the value of playing action $i = 0, 1$ satisfies

$$V_i(t) = \sup_{\{t_1,\dots,t_n\}\subset[t,t+\Delta t)} \left\{ \int_t^{t_1} e^{-\theta(s-t)} f_i(x(s))\,ds + \sum_{k=1}^{n} \left(\int_{t_k}^{t_{k+1}} e^{-\theta(s-t)} f_{i_k}(x(s))\,ds - e^{-\theta(t_k-t)} \frac{|\dot{x}(t_k)|}{\gamma} \right) + e^{-\theta\Delta t} V_{i_n}(t+\Delta t) \right\},$$

where $i_k \in \{0,1\} \setminus \{i_{k-1}\}$ $(i_0 = i)$ and $t_{n+1} = t + \Delta t$.

$\theta > 0$: (common) discount rate.

Equilibrium Paths

If $x(\cdot)$ is an equilibrium path, then

- on *interior* intervals,
  indifferent between changing actions and waiting:

$$\dot{x}(t) \leq 0 \Rightarrow V_0(t) - \frac{|\dot{x}(t)|}{\gamma} = V_1(t),$$
$$\dot{x}(t) \geq 0 \Rightarrow V_1(t) - \frac{|\dot{x}(t)|}{\gamma} = V_0(t);$$

- on *boundary* intervals,
  players can change actions with zero cost:

$$V_0(t) = V_1(t).$$

Characterization

$x(\cdot)$ is an equilibrium path from $x^0 \in [0, 1]$ iff $x(0) = x^0$, and $\exists\, q\colon [0, \infty) \to \mathbb{R}$: bounded, continuous and piecewise differentiable such that for all $t \geq 0$,

- if t is in an interior interval, then

$$\begin{aligned} \dot{x}(t) &= \gamma q(t), \\ \dot{q}(t) &= \theta q(t) - f(x(t)), \end{aligned}$$

- if t is in a boundary interval, then

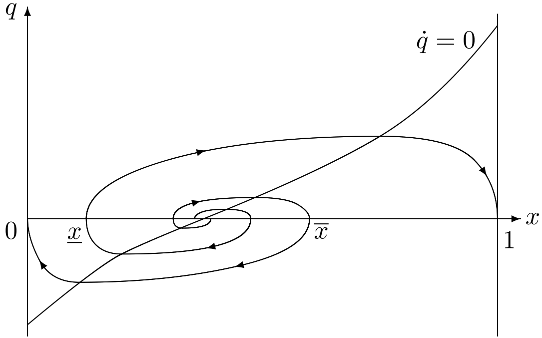
$$q(t) = 0.$$

Here,

$$q(t) = V_1(t) - V_0(t).$$

"Overlap"

$\dot{x}(t) = \gamma q(t),$
$\dot{q}(t) = \theta q(t) - f(x(t)).$



$[\underline{x}, \overline{x}]$ is called the "overlap".
Adjustment cost/discount rate smaller $\Rightarrow$ "overlap" larger.

Stability Concepts

- Equilibrium state $i^* \in \{0, 1\}$ is *absorbing* if
  $\exists$ neighborhood of i^*, $\forall$ equilibrium path converges to i^*.
  (i.e., The overlap does not reach i^*.)
- Equilibrium state $i^* \in \{0, 1\}$ is *globally accessible* if
  $\forall$ initial distribution, $\exists$ equilibrium path that converges to i^*.
  (i.e., The overlap reaches $-i^*$.)

If an absorbing state is also globally accessible,
then it is the unique absorbing state.

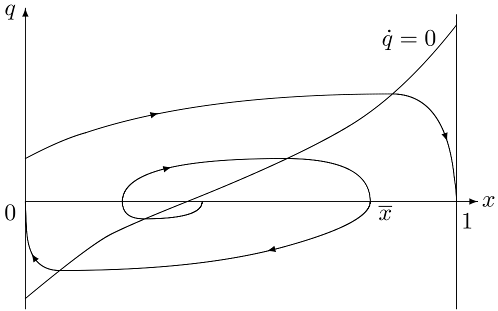
Interested in a state that is absorbing and globally accessible
for small friction $\theta/\sqrt{\gamma}$.

(θ: discount rate; $|\dot{x}(t)|/\gamma$: adjustment cost.)

Main Result

Theorem.
If $\{x^*\} = \max_{x \in [0,1]} F(x)$,
$\Rightarrow x^*$ is absorbing and globally accessible when $\theta/\sqrt{\gamma}$ is small.

F: potential function ($\frac{d}{dx}F(x) = f(x)$).



In the figure, $x = 1$ is absorbing and globally accessible.

Proof Strategy

Follow the proof strategy of Hofbauer and Sorger (1999, JET), who study stability of perfect foresight dynamics due to Matsui and Matsuyama (1995, JET).

- Global accessibility:
  - Consider an associated optimal control problem.
  - Its solution trajectories are equilibrium paths.
  - Its solution trajectories visit the potential maximizer x^*.
  - $+$ Absorption $\Rightarrow$ global accessibility.
- Absorption:
  - The maximized Hamiltonian works as a Lypunov function.

- Notice the *state variable inequality constraint*, $0 \leq x(t) \leq 1$. $x(\cdot)$ may hit the boundary of the state space $[0, 1]$.

Proof of Global Accessibility

Consider the optimal control problem (F: potential function):

$$\text{Max} \quad J(x(\cdot),u(\cdot)) = \int_0^\infty e^{-\theta t}\left(F(x(t)) - \frac{u(t)^2}{2\gamma}\right) dt \tag{5.1a}$$

$$\text{s.t.} \quad \dot{x}(t) = u(t), \tag{5.1b}$$

$$x(t) \geq 0,\ 1 - x(t) \geq 0, \tag{5.1c}$$

$$x(0) = x^0. \tag{5.1d}$$

- **Prop 5.1.** A solution exists for each $x^0 \in [0,1]$.
- **Prop 5.2.** $(x^*(\cdot), u^*(\cdot))$: solution $\Rightarrow$ $x^*(\cdot)$: equilibrium path.
  (The objective function is a "dynamic version of potential function".)
- **Lemma 5.4.** $x^*(\cdot)$ visits neighborhoods of the unique F-max if $\theta/\sqrt{\gamma}$ is small. ("Visit lemma")

Optimality Conditions (1/2)

Necessary conditions for optimality (Hartl et al. (1995, *SIAM Review*)):

$$H(x,u,q) = F(x) - \frac{u^2}{2\gamma} + qu, \tag{5.2}$$

$$L(x,u,q,\nu_0,\nu_1) = H(x,u,q) + \nu_0 x + \nu_1(1-x). \tag{5.3}$$

$\exists\, q(\cdot)$: piecewise absolutely continuous,
$\exists\, \nu_0(\cdot), \nu_1(\cdot)$: piecewise continuous such that

$$H_u(x(t),u(t),q(t)) = -\frac{u(t)}{\gamma} + q(t) = 0, \tag{5.4}$$

$$\begin{aligned} \dot{q}(t) &= \theta q(t) - L_x(x(t),u(t),q(t),\nu_0(t),\nu_1(t)) \\ &= \theta q(t) - f(x(t)) - \nu_0(t) + \nu_1(t), \end{aligned} \tag{5.5}$$

$$\nu_0(t) \geq 0, \quad \nu_0(t)x(t) = 0, \tag{5.6}$$

$$\nu_1(t) \geq 0, \quad \nu_1(t)(1-x(t)) = 0, \tag{5.7}$$

(cont...)

Optimality Conditions (2/2)

Jump conditions for adjoint $q(\cdot)$:
for any time τ in a boundary interval and for any contact time τ,

$$q(\tau^-) = q(\tau^+) + \eta_0(\tau) - \eta_1(\tau), \tag{5.8}$$

$$\eta_0(\tau) \geq 0, \quad \eta_0(\tau)x(\tau) = 0, \tag{5.9}$$

$$\eta_1(\tau) \geq 0, \quad \eta_1(\tau)(1 - x(\tau)) = 0 \tag{5.10}$$

for some $\eta_0(\tau), \eta_1(\tau)$ for each τ.

Show
$q(\tau^-) = q(\tau^+) = 0$ (and hence $q(\cdot)$ is continuous).

"Visit Lemma" 5.4 + Absorption $\Rightarrow$ Global accessibility. (Q.E.D.)

Proof of Absorption

Maximized Hamiltonian:

$$H^*(x,q) = \max_u H(x,u,q) = F(x) + \frac{\gamma}{2}q^2.$$

- **Lemma 5.5.**

$$\frac{d}{dt}H^*(x(t),q(t)) \geq 0.$$

- **Lemma 5.6.** Let $x(\cdot)$ be an equilibrium path from x^0, and $\hat{x} \in [0,1]$ an accumulation point of $x(\cdot)$.
  $\Rightarrow F(\hat{x}) \geq F(x^0)$; and $\hat{x}$ is a critical point of F.

If x^0 is in a neighborhood of the unique max x^* of F
in which x^* is the unique critical point,
$\Rightarrow x(\cdot)$ must converge to x^*. (Q.E.D.)

Comments on Extension to Many-Action Games

- Large population potential games.
- The dynamics:
  Formulation of adjustment costs.
- Idea of proof of global accessibility and absorption.
- Another formulation of the dynamics:
  Introduction of heterogeneity in preferences
  (to prevent the dynamics from hitting the boundary of the state space).

  Cf. Perturbed best response dynamics (Fudenberg and Levine; Hofbauer and Sandholm).

Potential Games

(Monderer and Shapley 1996, Sandholm 2001, 2008, Ui 2008)

$A = \{1, \dots, n\}$: set of actions.

$f_i(x)$: payoff for action $i \in A$,
where
$x \in \Delta(A) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ \sum_{i \in A} x_i = 1\}$.

Definition. $F\colon \bar{\Delta} \to \mathbb{R}$ is said to be a *potential function* of $(f_i)_{i \in A}$ if

$$\frac{\partial F}{\partial x_i}(x) - \frac{\partial F}{\partial x_j}(x) = f_i(x) - f_j(x) \quad \forall\, i, j \in A,\ \forall\, x \in \Delta(A). \ (*)$$

($\bar{\Delta} \subset \mathbb{R}^n$: a full-dimensional subset of $\mathbb{R}^n$ containing $\Delta(A)$.)

- Maximize $F(x)$ subject to $x \in \Delta(A)$.

  x^*: solution $\Rightarrow$ x^*: equilibrium state (but not vice versa).

Examples of Potential Game

- Any population game with two actions.
- Random-matching of a Common interest game/Team game: Games where for any action profile, players get a same payoff.
- New Economic Geography with $\tau_{ji} = \tau_i \ \forall j$: Oyama (2009a).
- Biology: Fisher (1930).
- Transportation economics:
  Beckmann, McGuire, and Winsten (1956).

KFB Dynamics with Many Actions (1/2)

$u_{ji}(t)$: (net) flow from action j to action i, where $u_{ij} = -u_{ji}$, and

$$\dot{x}_i(t) = \sum_{j \neq i} u_{ji}(t).$$

Adjustment cost when changing from j to i: $|u_{ji}(t)|/\gamma$.

KFB Dynamics with Many Actions (2/2)

- The indifference conditions:

$$u_{ji}(t) \geq 0 \Rightarrow V_i(t) - u_{ji}(t)/\gamma = V_j(t),$$
$$u_{ji}(t) \leq 0 \Rightarrow V_j(t) + u_{ji}(t)/\gamma = V_i(t).$$

- Equilibrium dynamics:

$$\dot{x}_i(t) = \gamma \left\{ (n-1)V_i(t) - \textstyle\sum_{j \neq i} V_j(t) \right\},$$
$$\dot{V}_i(t) = \theta V_i(t) - f_i(x(t)),$$

$+$ boundary condition
(if $\dot{x}(t) = 0$ in some time interval, then $V_1(t) = \cdots = V_n(t)$ there.)

Potential Method

Suppose that the game $(f_i)_{i \in A}$ has a potential function F.

- The associated optimal control problem:

$$
\begin{aligned}
\text{Max} \quad & \int_0^\infty e^{-\theta t} \left(F(x(t)) - \frac{1}{2} \sum_i \sum_{j \neq i} \frac{u_{ji}(t)^2}{2\gamma} \right) dt \\
\text{s.t.} \quad & \dot{x}_i(t) = \sum_{j \neq i} u_{ji}(t) \\
& u_{ij}(t) = -u_{ji}(t) \\
& \sum_i x_i(t) = 1 \\
& x_i(t) \geq 0 \\
& x(0) = x^0.
\end{aligned}
$$

The same technique as before should work...

Another Possible Formulation of Dynamics

- Introduce heterogeneity in players w.r.t. their payoffs:
  For a player with "type" $(\alpha_i)_{i \in A} \subset \mathbb{R}^A$, the payoff is given by

  $$u_i(x; \alpha_i) = u_i(x) + \varepsilon \alpha_i. \qquad (\varepsilon > 0, x \in \Delta(A))$$

  α_i is distributed (independently) according to some G_i (with full support).
- For each action i,
  there are some players for whom i is a dominant action.
  $\Rightarrow$ The process $x(t)$ never hits the boundary of $\Delta(A)$.
- What happens when the base game $(u_i)_{i \in A}$ has a potential (and when $\varepsilon \to 0$)?

Concluding Remarks

- Discussed the "Krugman-Fukao-Benabou dynamics".
- It has been shown that there is a unique state that is stable (i.e., globally accessible and absorbing) when the discount rate/adjustment cost is small.
- Stability consideration under this dynamics helps to "select" among multiple equilibria of the underlying static game.

- "Potential method" in potential games: Equilibrium paths of the dynamic game are translated into solutions of a dynamic maximization problem.
- Analog to Hofbauer and Sorger (1999, JET), who considered the "perfect foresight dynamics" due to Matsui and Matsuyama (1995, JET).
- See also: Oyama, Takahashi, and Hofbauer (2008, *Theoretical Economics*), for "monotone method" in supermodular games.

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