Finite Markov Chains II

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Topics in Economic Theory

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Computing Stationary Distributions

We want to solve for a nonzero solution $x = (x_0, x_1, x_2, x_3)$ to

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -4 & 1 & 2 & 1 \\ 4 & -9 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

or

$$\begin{aligned} -4x_0 + 4x_1 \qquad\qquad\qquad &= 0 \\ x_0 - 9x_1 \;\; + x_2 \qquad\quad &= 0 \\ 2x_0 + 2x_1 - 3x_2 + 5x_3 &= 0 \\ x_0 + 3x_1 + 2x_2 - 5x_3 &= 0 \end{aligned}$$

(from Stewart 2009, Section 10.2).

(Scaled) Gaussian Elimination: Reduction

- $$\begin{array}{rr}
\text{Multipliers} & -x_0 + x_1 \qquad\qquad\qquad = 0 \quad \text{(scaled)} \\
1 & x_0 - 9x_1 \;\; + x_2 \qquad\quad = 0 \\
2 & 2x_0 + 2x_1 - 3x_2 + 5x_3 = 0 \\
1 & x_0 + 3x_1 + 2x_2 - 5x_3 = 0
\end{array}$$

- $$\begin{aligned}
-8x_1 \;\; + x_2 \qquad\quad &= 0 \\
4x_1 - 3x_2 + 5x_3 &= 0 \\
4x_1 + 2x_2 - 5x_3 &= 0
\end{aligned}$$

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\begin{array}{rrcl}
\text{Multipliers} & -x_1 + \frac{1}{8}x_2 & = 0 & \text{(scaled)} \\
4 & 4x_1 - 3x_2 + 5x_3 & = 0 & \\
4 & 4x_1 + 2x_2 - 5x_3 & = 0 &
\end{array}
$$

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\begin{aligned}
-\tfrac{5}{2}x_2 + 5x_3 &= 0 \\
\tfrac{5}{2}x_2 - 5x_3 &= 0
\end{aligned}
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$$\begin{array}{rl} \text{Multipliers} & -x_2 + 2x_3 = 0 \quad \text{(scaled)} \\ \frac{5}{2} & \frac{5}{2}x_2 - 5x_3 = 0 \end{array}$$

- (Irrelevant)

$$0 = 0$$

(Scaled) Gaussian Elimination: Backward Substitution

- x_3 is undetermined.
  $\Rightarrow$ Set $x_3 = 1$.
- By $-x_2 + 2x_3 = 0$, we have $x_2 = 2x_3 = 2$.
- By $-x_1 + \frac{1}{8}x_2 = 0$, we have $x_1 = \frac{1}{8}x_2 = \frac{1}{4}$.
- By $-x_0 + x_1 = 0$, we have $x_0 = x_1 = \frac{1}{4}$.

(Scaled) Gaussian Elimination: Normalization

- Normalize x so that $\sum x_i = 1$:

$$x = \left(\tfrac{1}{4}, \tfrac{1}{4}, 2, 1\right) / \left(\tfrac{1}{4} + \tfrac{1}{4} + 2 + 1\right) = \left(\tfrac{1}{14}, \tfrac{1}{14}, \tfrac{4}{7}, \tfrac{4}{7}\right).$$

(Scaled) Gaussian Elimination: Reduction

0. $$\begin{pmatrix} -4 & 1 & 2 & 1 \\ 4 & -9 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1^\circ & 1^* & 2^* & 1^* \\ 1 & -9 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix} \quad \text{(scaling)}$$

$$\Rightarrow \begin{pmatrix} -1^\circ & 1^* & 2^* & 1^* \\ 1 & -8 & 4 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix} \quad \text{(reduction)}$$

(The multipliers and $A[0,0]$ may be left untouched.)

(Scaled) Gaussian Elimination: Reduction

1. $$\begin{pmatrix} * & * & * & * \\ * & -8 & 4 & 4 \\ * & 1 & -3 & 2 \\ * & 0 & 5 & -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} * & * & * & * \\ * & -1^\circ & 4^* & 4^* \\ * & \frac{1}{8} & -3 & 2 \\ * & 0 & 5 & -5 \end{pmatrix} \qquad \text{(scaling)}$$

$$\Rightarrow \begin{pmatrix} * & * & * & * \\ * & -1^\circ & 4^* & 4^* \\ * & \frac{1}{8} & -\frac{5}{2} & \frac{5}{2} \\ * & 0 & 5 & -5 \end{pmatrix} \qquad \text{(reduction)}$$

(The multipliers and $A[1,1]$ may be left untouched.)

(Scaled) Gaussian Elimination: Reduction

2. $$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & -\frac{5}{2} & \frac{5}{2} \\ * & * & 5 & -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & -1^{\circ} & \frac{5}{2}^{*} \\ * & * & 2 & -5 \end{pmatrix} \quad \text{(scaling)}$$

$$\Rightarrow \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & -1^{\circ} & \frac{5}{2}^{*} \\ * & * & 2 & 0 \end{pmatrix} \quad \text{(reduction)}$$

(The multipliers and $A[2,2]$ may be left untouched.)

(Scaled) Gaussian Elimination: Backward Substitution

- We are left with

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} \circ & * & * & * \\ 1 & \circ & * & * \\ 0 & \frac{1}{8} & \circ & * \\ 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

  where $*$ is understood as 0, and $\circ$ as -1.

- Solve backward:

$$\begin{aligned} x_3 &= 1 \\ x_2 &= 2x_3 \\ x_1 &= \tfrac{1}{8}x_2 + 0x_3 \\ x_0 &= 1x_1 + 0x_2 + 0x_3. \end{aligned}$$

Loss of Significance

Consider, for example,

$$P = \begin{pmatrix} 1-(q+\varepsilon) & q & \varepsilon \\ q & 1-(q+\varepsilon) & \varepsilon \\ \varepsilon & \varepsilon & 1-2\varepsilon \end{pmatrix},$$

with $A = P - I$ equal to

$$A = \begin{pmatrix} -(q+\varepsilon) & q & \varepsilon \\ q & -(q+\varepsilon) & \varepsilon \\ \varepsilon & \varepsilon & -2\varepsilon \end{pmatrix},$$

where $0 < q < 1$, and $\varepsilon > 0$ is sufficiently small.

- *Theoretically*, Gaussian elimination leads to

$$\begin{pmatrix} \circ & * & * \\ \frac{q}{q+\varepsilon} & \circ & * \\ \frac{\varepsilon}{q+\varepsilon} & 1 & 0 \end{pmatrix}.$$

- In the 0th step, we have

$$A = \begin{pmatrix} -(q+\varepsilon) & q & \varepsilon \\ \frac{q}{q+\varepsilon} & -(q+\varepsilon) + \frac{q^2}{q+\varepsilon} & \varepsilon + \frac{\varepsilon q}{q+\varepsilon} \\ \frac{\varepsilon}{q+\varepsilon} & \varepsilon + \frac{\varepsilon q}{q+\varepsilon} & -2\varepsilon + \frac{\varepsilon^2}{q+\varepsilon} \end{pmatrix}.$$

- *Numerically*, we do *not* have

$$(q+\varepsilon) - \frac{q^2}{q+\varepsilon} = \varepsilon + \frac{\varepsilon q}{q+\varepsilon},$$

where some degree of significance is lost in the LHS because of subtraction, so that

$$\frac{\varepsilon + \frac{\varepsilon q}{q+\varepsilon}}{(q+\varepsilon) - \frac{q^2}{q+\varepsilon}}$$

can be far from 1.

The Grassmann-Taksar-Heyman (GTH) Algorithm

- Use the property that for each k,

$$A[k,k] = -\sum_{j \neq k} A[k,j]$$

to *avoid subtraction*.

GTH: Reduction

0. $$\begin{pmatrix} -4 & 1 & 2 & 1 \\ 4 & -9 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \circ & \underline{1} & \underline{2} & \underline{1} \\ 1 & -9 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix} \qquad (\text{scale} = 1 + 2 + 1 = 4)$$

$$\Rightarrow \begin{pmatrix} \circ & \underline{1} & \underline{2} & \underline{1} \\ 1 & -8 & 4 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix} \qquad (\text{reduction})$$

(The multipliers and $A[0,0]$ may be left untouched.)

GTH: Reduction

1. $$\begin{pmatrix} * & * & * & * \\ * & -8 & 4 & 4 \\ * & 1 & -3 & 2 \\ * & 0 & 5 & -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} * & * & * & * \\ * & \circ & \underline{4} & \underline{4} \\ * & \frac{1}{8} & -3 & 2 \\ * & 0 & 5 & -5 \end{pmatrix} \qquad \text{(scale} = 4 + 4 = 8\text{)}$$

$$\Rightarrow \begin{pmatrix} * & * & * & * \\ * & \circ & \underline{4} & \underline{4} \\ * & \frac{1}{8} & -\frac{5}{2} & \frac{5}{2} \\ * & 0 & 5 & -5 \end{pmatrix} \qquad \text{(reduction)}$$

(The multipliers and $A[1,1]$ may be left untouched.)

GTH: Reduction

2. $$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & -\frac{5}{2} & \frac{5}{2} \\ * & * & 5 & -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & \circ & \frac{5}{2} \\ * & * & 2 & -5 \end{pmatrix} \quad \text{(scale} = \tfrac{5}{2}\text{)}$$

$$\Rightarrow \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & \circ & \frac{5}{2} \\ * & * & 2 & 0 \end{pmatrix} \quad \text{(reduction)}$$

(The multipliers and $A[2, 2]$ may be left untouched.)