Analysis in Metric Space II

Daisuke Oyama

Topics in Economic Theory

October 8, 2014

Open Sets, Closed Sets

Let (S, ρ) be a metric space. Let $B(r; x) = \{y \in S \mid \rho(y, x) < r\}$.

Definition 1

For $E \subset S$,

- The *closure* of E, denoted $\operatorname{cl} E$, is defined by

$$\operatorname{cl} E = \{x \in S \mid B(\varepsilon; x) \cap E \neq \emptyset \text{ for all } \varepsilon > 0\}.$$

- The *interior* of E, denoted $\operatorname{int} E$, is defined by

$$\operatorname{int} E = \{x \in S \mid B(\varepsilon; x) \subset E \text{ for some } \varepsilon > 0\}.$$

- Observe that by definition, $E \subset \operatorname{cl} E$ and $\operatorname{int} E \subset E$.
- $x \in S$ is said to *adhere* to E if $x \in \operatorname{cl} E$.
- $x \in S$ is said to be *interior* to E if $x \in \operatorname{int} E$.

Definition 2

- $E \subset S$ is *closed* if E contains all points that adhere to E, i.e., if $\operatorname{cl} E \subset E$.
- $E \subset S$ is *open* if all points in E are interior to E, i.e., if $E \subset \operatorname{int} E$.

- Since $E \subset \operatorname{cl} E$ by definition, it is equivalent to define: $E \subset S$ is closed if $\operatorname{cl} E = E$.
- Since $\operatorname{int} E \subset E$ by definition, it is equivalent to define: $E \subset S$ is open if $\operatorname{int} E = E$.

Proposition 1

$\operatorname{cl} E = S \setminus \operatorname{int}(S \setminus E)$.

Proposition 2 (Theorem 3.1.18)

E is closed if and only if $S \setminus E$ is open.

Proposition 3 (Exercise 3.1.19)

For any $x \in S$ *and any* $r \geq 0$,
$B(r;x) = \{y \in S \mid \rho(y,x) < r\}$ *is open.*

Proposition 4

For any $x \in S$ *and any* $r \geq 0$,
$D(r;x) = \{y \in S \mid \rho(y,x) \leq r\}$ *is closed.*

Proposition 5

1. $\operatorname{cl} \emptyset = \emptyset$.
2. $E \subset \operatorname{cl} E$.
3. 3.1 *If* $E \subset F$, *then* $\operatorname{cl} E \subset \operatorname{cl} F$.

   3.2 $\operatorname{cl}(E \cup F) = \operatorname{cl} E \cup \operatorname{cl} F$.
4. $\operatorname{cl}(\operatorname{cl} E) \subset \operatorname{cl} E$.

Proposition 6 (Exercise 3.1.30)

1. $\operatorname{cl} E$ *is closed.*
2. *If* F *is closed and* $F \supset E$, *then* $\operatorname{cl} E \subset F$.
3. $\operatorname{cl} E$ *is equal to the intersection of all closed sets containing* E.

Proposition 7 (Exercises 3.1.20, 27, 28)

1. $\emptyset$ *and* S *are closed.*
2. *If* $\{F_\alpha\}_{\alpha \in A}$ *is a collection of closed sets, then* $\bigcap_{\alpha \in A} F_\alpha$ *is also closed.*
3. *If* A *is finite and* $\{F_\alpha\}_{\alpha \in A}$ *is a collection of closed sets, then* $\bigcup_{\alpha \in A} F_\alpha$ *is also closed.*

Definition 3

For $A \subset \mathbb{R}$, $A \neq \emptyset$,

1. $s = \sup A$ if
   1.1 $a \leq s \ \forall\, a \in A$, and
   1.2 if $a \leq u \ \forall\, a \in A$, then $s \leq u$;
2. $i = \inf A$ if
   2.1 $i \leq a \ \forall\, a \in A$, and
   2.2 if $\ell \leq a \ \forall\, a \in A$, then $\ell \leq i$.

If A is not bounded above, we define $\sup A = \infty$.
If A is not bounded below, we define $\inf A = -\infty$.

(Sometimes it is convenient to define $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.)

Lemma 8

$a \leq u \; \forall \, a \in A \Rightarrow s \leq u$
$\iff \; \forall \, \varepsilon > 0 : \; \exists \, a \in A : \; s - \varepsilon < a.$

Proposition 9 (Theorem 3.1.22)

If $F \subset \mathbb{R}$*,* $F \neq \emptyset$*, is bounded and closed, then* $\sup F \in F$*.*

Let (S, ρ) be a metric space.

Proposition 10 (Exercise 3.1.16)

$\operatorname{cl} F = \{x \in S \mid x = \lim_{n\to\infty} x_n$ *for some* $(x_n) \subset F\}$.

Proposition 11 (Theorem 3.1.17)

$F \subset S$ *is closed if and only if*
$\lim_{n\to\infty} x_n \in F$ *for any convergent sequence* $(x_n) \subset F$.

Let S and T be metric spaces.

Proposition 12 (Exercise 3.1.16)

$f\colon S \to T$ is continuous if and only if $f^{-1}(G)$ is open in S for any open set $G \subset T$.

($f^{-1}(G) = \{x \in S \mid f(x) \in G\}$.)

Completeness

Definition 4

A sequence (x_n) in a metric space (S, ρ) is a *Cauchy* sequence if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\rho(x_m, x_n) < \varepsilon \text{ for all } m, n \geq N.$$

A sequence in a normed vector space $(V, \|\cdot\|)$ is a Cauchy sequence if it is a Cauchy sequence in the metric space induced by $(V, \|\cdot\|)$.

Definition 5

A subset A of a metric space (or a normed vector space) is *complete* if every Cauchy sequence in A converges to some point in A.

(A complete normed vector space is called a *Banach space*.)

Proposition 13 (Theorem 3.2.3)

Let S be a complete metric space.

$A \subset S$ is complete if and only if it is closed.

Axiom 1

$(\mathbb{R}, |\cdot|)$ is complete.

$(bU, \|\cdot\|_\infty)$ denotes the set of bounded functions $f\colon U \to \mathbb{R}$ endowed with the norm $\|\cdot\|_\infty$ defined by $\sup_{x\in U}|f(x)|$.

Proposition 14 (Theorem 3.2.6)

Let U be any nonempty set.

$(bU, \|\cdot\|_\infty)$ *is complete (i.e., it is a Banach space).*

For a metric space U,
let $(bcU, \|\cdot\|_\infty)$ be the set of bounded continuous functions $f\colon U \to \mathbb{R}$ endowed with the norm $\|\cdot\|_\infty$ defined by $\sup_{x \in U} |f(x)|$.

Proposition 15 (Theorem 3.2.9)

Let U be a metric space.

$(bcU, \|\cdot\|_\infty)$ *is complete (i.e., it is a Banach space).*