Analysis in Metric Space III

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Topics in Economic Theory

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Compactness

Let (S, ρ) be a metric space.

Definition 1

- $K \subset S$ is *precompact* if every sequence contained in K has a subsequence that converges to a point of S.
- $K \subset S$ is *compact* if every sequence contained in K has a subsequence that converges to a point of K.

Proposition 1 (Exercises 3.2.15, 3.2.17)

1. *A subset of a compact set is precompact.*
2. *The closure of a precompact set is compact.*

Proof

- Let K be a precompact set.

  Consider any sequence $(x_n) \subset \operatorname{cl} K$.
  We want to show that (x_n) has a convergent subsequence whose limit is in $\operatorname{cl} K$.
- For each $n \in \mathbb{N}$, let $y_n \in K$ be such that $\rho(x_n, y_n) < 1/n$.
- Since K is precompact, there are a subsequence $(y_{n(k)})$ and a point $y \in S$ such that $y_{n(k)} \to y$.

- We claim $x_{n(k)} \to y$.

  Fix any $\varepsilon > 0$.

  Let $N \in \mathbb{N}$ be such that $\rho(y_{n(k)}, y) < \varepsilon/2$ for all $k \geq N$.

  We can assume that N is large enough that $1/n(k) < \varepsilon/2$ for all $k \geq N$.

- Then
  $\rho(x_{n(k)}, y) \leq \rho(x_{n(k)}, y_{n(k)}) + \rho(y_{n(k)}, y) < 1/n(k) + \varepsilon/2 < \varepsilon$.

- Since $\operatorname{cl} K$ is closed, we have $y \in \operatorname{cl} K$.

Let (S, ρ) be a metric space.

Definition 2

- An *open cover* of $K \subset S$ is a collection $\{G_\alpha\}_{\alpha \in A}$ of open sets such that $K \subset \bigcup_{\alpha \in A} G_\alpha$.

Proposition 2 (Theorem 3.2.10)

$K \subset S$ is compact if and only if for any open cover $\{G_\alpha\}_{\alpha \in A}$ of K, there exists a finite subset $A' \subset A$ such that $\{G_\alpha\}_{\alpha \in A'}$ is an open cover of K.

Proposition 3 (Exercise 3.2.15)

A closed subset of a compact set is compact.

Proof 1

- Let F be a closed subset of a compact set K.

  Let $\{G_\alpha\}_{\alpha \in A}$ be any open cover of F.
  We want to find a finite subcover.

- By the closedness of F, $S \setminus F$ is an open set.

  Therefore, $\{\{G_\alpha\}_{\alpha \in A}, S \setminus F\}$ is an open cover of K.

- By the compactness of K, there are finitely many $\alpha_1, \ldots, \alpha_n$ such that $K \subset \bigcup_{i=1}^n G_{\alpha_i} \cup (S \setminus F)$.

- Since $F \cap (S \setminus F) = \emptyset$, we have $F \subset \bigcup_{i=1}^n G_{\alpha_i}$.

Proof 2

- Let F be a closed subset of a compact set K in a metric space.
- Let (x_n) be any sequence in F.
- Since $(x_n) \subset K$ and K is compact,
  (x_n) has a convergent subsequence (x_{n_k}) and its limit is in K.
- By the closedness of F, the limit must be in F.

Proposition 4 (Exercise 3.2.12)

Every compact subset of a metric space is bounded.

Proof 1

- Let K be a compact subset of a metric space (S, ρ).

  Fix any $\varepsilon > 0$. $\{B(\varepsilon; x)\}_{x \in K}$ is an open cover of K.

- By the compactness of K, there are finitely many points $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n B(\varepsilon; x_i)$.

- Let $\delta = \max_{i=1,\ldots,n} \rho(x_i, x_1) < \infty$, and let $m \in \mathbb{N}$ be such that $\delta + \varepsilon < m$.

- We claim $K \subset B(m; x_1)$.

  Indeed, take any $x \in K$. Let x_i, $i = 1, \ldots, k$ be such that $x \in B(\varepsilon; x_i)$.

  Then we have $\rho(x, x_1) \leq \rho(x_i, x_1) + \rho(x, x_i) \leq \delta + \varepsilon < m$.

Proof 2

- Let K be a subset of a metric space (S, ρ).

  Suppose that K is not bounded.
  We want to show that K is not compact.

- Fix any $x_0 \in S$.

  Since K is not bounded,
  for each $n \in \mathbb{N}$ we can take an $x_n \in K \cap (S \setminus B(n; x_0))$.

  We want to show that the sequence $(x_n) \subset K$ has no convergent subsequence.

- Take any $x \in S$. Let $N \in \mathbb{N}$ be such that $\rho(x, x_0) < N$.

- Then for all $n \geq N + 1$, we have
  $\rho(x_n, x) \geq \rho(x_n, x_0) - \rho(x, x_0) > (N + 1) - N = 1$.

Proposition 5 (Exercise 3.2.13)

Every compact subset of a metric space is closed.

Proof 1

- Let K be a compact subset of a metric space (S, ρ).
  We want to show that $S \setminus K$ is open.

- Take any $x \in S \setminus K$.

  For each $y \in K$, let U_y and V_y be open sets such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$.

  (For example, let $U_y = B(\frac{1}{2}\rho(x, y); x)$ and $V_y = B(\frac{1}{2}\rho(x, y); y)$.)

- Since $\{V_y\}_{y \in K}$ is an open cover of the compact set K, there are finitely many $y_1, \ldots, y_n \in K$ such that $K \subset \bigcup_{i=1}^n V_{y_i}$.

- Let $U = \bigcap_{i=1}^n U_{y_i}$, which is open.
  Then $x \in U$ and $U \cap K = \emptyset$.

Proof 2

- Let K be a compact subset of a metric space (S, ρ).

  Let $(x_n) \subset K$, and suppose that $x_n \to x$.
  We want to show that $x \in K$.

- By the compactness of K, (x_n) has a convergent subsequence and its limit x' is in K.

- But (x_n) itself is convergent, so that we must have $x' = x$.

  $\therefore x \in K$.

Remark

- Thus, a compact subset of a metric space is closed and bounded.
- But the converse is not true in general.
- For example, consider $\ell_\infty = \{x \colon \mathbb{N} \to \mathbb{R} \mid \|x\|_\infty < \infty\}$, endowed with the norm $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

  The unit closed ball around 0 $(= (0, 0, 0, \ldots))$, $D(1; 0) \subset \ell_\infty$, which is closed and bounded, is not precompact (nor compact).
- To see this, consider the sequence $(1, 0, 0, \ldots), (0, 1, 0, \ldots), \ldots$.

  None of its subsequences is convergent.
- But a closed and bounded subset of $\mathbb{R}^k$ is compact.

Proposition 6 (Theorem A.2.10)

Any bounded sequence in $(\mathbb{R}, |\cdot|)$ *has a convergent subsequence.*

Proof

- Let $(x_n) \subset \mathbb{R}$ be a bounded sequence, and suppose that $(x_n) \subset [a_1, b_1]$.
- At least one of $\{n \in \mathbb{N} \mid x_n \in [a_1, (a_1 + b_1)/2]\}$ and $\{n \in \mathbb{N} \mid x_n \in [(a_1 + b_1)/2, b_1]\}$ is an infinite set.

  Let $[a_2, b_2] = [a_1, (a_1 + b_1)/2]$ if the former is infinite, and $[a_2, b_2] = [(a_1 + b_1)/2, b_1]$ otherwise.

  Repeat this procedure, to have a sequence $I_m = [a_m, b_m]$.
- Construct a subsequence as follows:

  Pick any $x_{n(1)} \in I_1$, and for $m = 2, 3, \ldots$, let $n(m) > n(m-1)$ be such that $x_{n(m)} \in I_m$.
- $(x_{n(m)})$ is Cauchy (by the Archimedean Axiom), and so it is convergent by Axiom A.2.4 (Completeness of $\mathbb{R}$).

Proposition 7

1. *A subset of* $(\mathbb{R}^k, \|\cdot\|_\infty)$ *is precompact if and only if it is bounded.*
2. *A subset of* $(\mathbb{R}^k, \|\cdot\|_\infty)$ *is compact if and only if it is closed and bounded.*

Proof

- Let $K \subset \mathbb{R}^k$ be a bounded set, and take any $(x_n) \subset K$, which is bounded.
- $(x_n^1) \subset \mathbb{R}$ is bounded and has a convergent subsequence $(x^1_{f_1(n)})$.

  $(x^2_{f_1(n)}) \subset \mathbb{R}$ is bounded and has a convergent subsequence $(x^2_{f_2(n)})$.

  Repeat this procedure, which terminates in finitely many steps.
- The resulting subsequence is convergent.

Optimization, Equivalence

Proposition 8 (Theorem 3.2.20)

Let S and T be metric spaces.

If $f\colon S \to T$ is continuous and $K \subset S$ is compact, then $f(K)$ is compact.

Proof 1

- $\{G_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$.

  We want to find a finite subcover.
- By the continuity of f, $\{f^{-1}(G_\alpha)\}_{\alpha \in A}$ is an open cover of K.
- By the compactness of K, there are $\alpha_1, \dots, \alpha_n \in A$ such that $K \subset \bigcup_{i=1}^n f^{-1}(G_{\alpha_i})$.
- Therefore we have $f(K) \subset \bigcup_{i=1}^n G_{\alpha_i}$.

  $\because$ If $y \in f(K)$, then there are $x \in K$ and i such that $f(x) = y$ and $x \in f^{-1}(G_{\alpha_i})$, which implies that $y \in G_{\alpha_i}$ for some i.

Proof 2

- Let (y_n) be any sequence in $f(K)$.

  We want to find a convergent subsequence whose limit is in $f(K)$.

- Let $(x_n) \subset K$ be such that $f(x_n) = y_n$ for each $n \in \mathbb{N}$.

- Since K is compact, (x_n) has a convergent subsequence (x_{n_k}) whose limit, which we denote x, is in K.

- Consider the subsequence (y_{n_k}).

  By the continuity of f, we have $y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$.

Proposition 9 (Theorem 3.2.22)

Let S be a metric space.

If $K \subset S$, $K \neq \emptyset$, is compact and $f\colon K \to \mathbb{R}$ is continuous, then there exist $x^*, x^{**} \in K$ such that $f(x^*) \geq f(x)$ for all $x \in K$ and $f(x^{**}) \leq f(x)$ for all $x \in K$.

Proof

- By the previous theorem, $f(K)$ is compact, and hence is bounded and closed.
- Therefore $\sup f(K)$ and $\inf f(K)$ exist, and $\sup f(K), \inf f(K) \in f(K)$.
- By the definition of $f(K)$, there are x^* and x^{**} such that $f(x^*) = \sup f(K)$ and $f(x^{**}) = \inf f(K)$.

Let (S, ρ) be a metric space.

Definition 3

Let S be a nonempty set, and ρ and ρ' be metrics on S.

ρ and ρ' are *equivalent* if there exist c, c' such that

$$\rho(x, y) \leq c\rho'(x, y) \text{ and } \rho'(x, y) \leq c'\rho(x, y)$$

for all $x, y \in S$.

Two norms on a vector space are equivalent if the induced metrics are equivalent.

- The equivalence defined above is an equivalence relation.
- If ρ and ρ' are equivalent, then for any $\{x_n\} \subset S$, $\{x_n\}$ is ρ-convergent if and only if it is ρ'-convergent, ...

Proposition 10 (Theorem 3.2.30)

All norms on $\mathbb{R}^k$ are equivalent.

Proof

- Let $\|\cdot\|$ be any norm on $\mathbb{R}^k$.

  We show that $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent.

- First, let $c = k \max_j \|e^j\|$, where e^j is the jth unit vector in $\mathbb{R}^k$.

  Then we have, for any $x \in \mathbb{R}^k$,
  $\|x\| = \left\| \sum_{j=1}^k x^j e^j \right\| \le \sum_{j=1}^k |x^j| \|e^j\| \le c\|x\|_\infty$.

- Second, $x \mapsto \|x\|$ is continuous in $(\mathbb{R}^k, \|\cdot\|_\infty)$,
  since $|\|x_n\| - \|x\|| \le \|x_n - x\| \le c\|x_n - x\|_\infty \to 0$
  as $x_n \to x$ in $\|\cdot\|_\infty$.

- Now consider the set $E = \{x \in \mathbb{R}^k \mid \|x\|_\infty = 1\}$, which is bounded and closed in $(\mathbb{R}^k, \|\cdot\|_\infty)$, and hence is compact.
- Therefore, by Theorem 3.2.22 $x \mapsto \|x\|$ has a minimizer x^* on E.

  Note that $x^* \neq 0$ so that $\|x^*\| \neq 0$.
- Then for all $x \in \mathbb{R}^k$, we have

$$\|x\| = \left\| \frac{x}{\|x\|_\infty} \right\| \|x\|_\infty \geq \|x^*\| \|x\|_\infty.$$

  Letting $c' = 1/\|x^*\|$, we have $c'\|x\| \geq \|x\|_\infty$.

Fixed Points

Let (S, ρ) be a metric space.

Definition 4

- $T\colon S \to S$ is *nonexpansive* on S if
$$\rho(Tx, Ty) \leq \rho(x, y) \quad \forall\, x, y \in S.$$
- $T\colon S \to S$ is *contracting* on S if
$$\rho(Tx, Ty) < \rho(x, y) \quad \forall\, x, y \in S \text{ with } x \neq y.$$
- $T\colon S \to S$ is *uniformly contracting* on S with modulus λ if $0 \leq \lambda < 1$ and
$$\rho(Tx, Ty) \leq \lambda \rho(x, y) \quad \forall\, x, y \in S.$$

Proposition 11 (Theorem 3.2.36)

Let (S, ρ) be a complete metric space.

- *If $T \colon S \to S$ is uniformly contracting on S with modulus λ, then T has a unique fixed point $x^* \in S$.*
- *Moreover, for any $x \in S$, $\rho(T^n x, x^*) \leq \lambda^n \rho(x, x^*)$ for all $n \in \mathbb{N}$, and hence $T^n x \to x^*$ as $n \to \infty$.*

Proposition 12 (Theorem 3.2.38)

Let (S, ρ) *be a compact metric space.*

- *If* $T\colon S \to S$ *is contracting on* S, *then* T *has a unique fixed point* $x^* \in S$.
- *Moreover, for any* $x \in S$, $T^n x \to x^*$ *as* $n \to \infty$.