Rationalizability and Correlated Equilibria

Daisuke Oyama

Topics in Economic Theory

September 11, 2015

Paper

- Brandenburger, A. and E. Dekel (1987). "Rationalizability and Correlated Equilibria," Econometrica 55, 1391-1402.

Normal Form Games

- Players $1, \ldots, I$
- A_i: finite set of actions for i
- $g_i \colon A \to \mathbb{R}$: payoff function for i

We identify the normal form game with $\mathbf{g} = (g_i)$.

Notation

- $\Delta(S)$: the set of probability distributions over S
- g_i is extended to $\Delta(A_{-i})$ by

$$g_i(a_i, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i, a_{-i}) \qquad (\pi_i \in \Delta(A_{-i})).$$

- The set of i's best responses to $\pi_i \in \Delta(A_{-i})$:

$$br_i(\pi_i) = \{a_i \in A_i \mid g_i(a_i, \pi_i) \geq g_i(a_i', \pi_i) \ \forall\, a_i' \in A_i\}.$$

(Correlated) Rationalizability

- $R_i^0 = A_i$
- $R_i^k = \{a_i \in A_i \mid a_i \in br_i(\pi_i) \; \exists \, \pi_i \in \Delta(R_{-i}^{k-1})\}$
  where $R_{-i}^{k-1} = \prod_{j \neq i} R_j^{k-1}$
- $R_i^\infty = \bigcap_{k=0}^\infty R_i^k$
  (note $R_i^0 \supset R_i^1 \supset R_i^2 \supset \cdots$)

Definition 1

Action $a_i \in A_i$ is a (correlated) rationalizable action if $a_i \in R_i^\infty$.

Independent Rationalizability

- $IR_i^0 = A_i$
- $IR_i^k = \{a_i \in A_i \mid a_i \in br_i(x_{-i}) \; \exists \, x_{-i} \in \prod_{j \neq i} \Delta(IR_j^{k-1})\}$
- $IR_i^\infty = \bigcap_{k=0}^{\infty} IR_i^k$

Definition 2

Action $a_i \in A_i$ is an independent rationalizable action if $a_i \in IR_i^\infty$.

- Trivially, for $I = 2$, $IR_i^\infty = R_i^\infty$, and in general, $IR_i^\infty \subset R_i^\infty$.
- For $I \geq 3$, in some cases, $IR_i^\infty \neq R_i^\infty$.

Iterative Elimination of Strictly Dominated Actions

- Action $a_i \in A_i$ is strictly dominated against $A'_{-i} \subset A_{-i}$ if there exists $x_i \in \Delta(A_i)$ such that $g_i(x_i, a_{-i}) > g_i(a_i, a_{-i})$ for all $a_{-i} \in A'_{-i}$.
- $U_i^0 = A_i$
- $U_i^k = \{a_i \in A_i \mid a_i \text{ is not strictly dominated against } U_{-i}^{k-1}\}$
- $U_i^\infty = \bigcap_{k=0}^\infty U_i^k$

Definition 3

Action $a_i \in A_i$ is an iteratively undominated action if $a_i \in U_i^\infty$.

Equivalence

Proposition 1

$R^{\infty} = U^{\infty}$.

Use the following.

Theorem 1

Let $Z \in \mathbb{R}^{M \times N}$. The following conditions are equivalent:

1. *For any $y \in \mathbb{R}^N_+$, if $Zy \leq 0$, then $y = 0$.*
2. *There exists $x \in \mathbb{R}^M_+$ such that $x'Z \gg 0$.*

(Fix i and a_i, and let $Z_{mn} = g_i(m, n) - g_i(a_i, n)$.)

Alternatively, use the following.

Theorem 2

Let $K \subset \mathbb{R}^N$ *be a convex set.*
If $K \cap \mathbb{R}^N_{++} = \emptyset$*, then there exists* $p \in \mathbb{R}^N_+$*,* $p \neq 0$*, such that* $p \cdot z \leq 0$ *for all* $z \in K$*.*

(Fix i and a_i, and let $K = \{g_i(x_i, \cdot) - g_i(a_i, \cdot) \in \mathbb{R}^{|A_{-i}|} \mid x_i \in \Delta(A_i)\}$).

Best Response Sets

For $D \subset \Delta(A_{-i})$, we write
$br_i(D) = \{a_i \in A_i \mid a_i \in br_i(\pi_i)\ \exists\, \pi_i \in D\}$.

Proposition 2

1. *$br_i(\Delta(R^\infty_{-i})) \subset R^\infty_i$ for all i.*
2. *$R^\infty_i \subset br_i(\Delta(R^\infty_{-i}))$ for all i.*
3. *If $A'_i \subset br_i(\Delta(A'_{-i}))$ for all i, then $A'_i \subset R^\infty_i$ for all i.*

Information Structure

- Ω: (finite) set of states
- $P_i \in \Delta(\Omega)$: i's prior belief
- $\mathcal{Q}_i$: i's information partition of Ω

  $Q_i(\omega) \in \mathcal{Q}_i$: the partition cell that contains ω

A strategy of i is a $\mathcal{Q}_i$-measurable function $f_i \colon \Omega \to A_i$.

Correlated Equilibrium I

Definition 4

$(\Omega, (P_i), (\mathcal{Q}_i), (f_i))$ is a correlated equilibrium of $\mathbf{g}$ if for all i, j, $P_i = P_j$ (denoted P), and for all i,

$$\sum_{\omega \in \Omega} P(\omega)\big(g_i(f(\omega)) - g_i(f_i'(\omega), f_{-i}(\omega))\big) \geq 0$$

for all strategy f_i'.

Correlated Equilibrium II

Definition 5

An action distribution $\mu \in \Delta(A)$ is a correlated equilibrium of g if for all i and all a_i,

$$\mu(a_i) > 0 \Rightarrow a_i \in br_i(\mu(\cdot|a_i)).$$

- $\mu(a_i) = \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})$.
- If $\mu(a_i) > 0$, $\mu(a_{-i}|a_i) = \mu(a_i, a_{-i})/\mu(a_i)$.

Equivalence

The action distribution $\mu \in \Delta(A)$ induced by $(\Omega, P, (\mathcal{Q}_i), (f_i))$ is defined by $\mu(a) = P(\{\omega \mid f(\omega) = a\})$.

Proposition 3

The two definitions are equivalent in the following sense:

- *If $(\Omega, P, (\mathcal{Q}_i), (f_i))$ is a correlated equilibrium (I), then its induced action distribution is a correlated equilibrium (II).*
- *If μ is a correlated equilibrium (II), then there exists some correlated equilibrium (I) that induces μ.*

Nash Equilibrium

Proposition 4

A profile of mixed actions $(x_i)_{i=1}^{I} \in \prod_{i=1}^{I} \Delta(A_i)$ *is a Nash equilibrium if and only if the action distribution* $\mu \in \Delta(A)$ *given by* $\mu(a) = \prod_{i=1}^{I} x_i(a_i)$ *is a correlated equilibrium.*

Subjective Correlated Equilibrium I

Definition 6

$(\Omega, (P_i), (\mathcal{Q}_i), (f_i))$ is a subjective correlated equilibrium of $\mathbf{g}$ if for all i,

$$\sum_{\omega \in \Omega} P_i(\omega)\big(g_i(f(\omega)) - g_i(f_i'(\omega), f_{-i}(\omega))\big) \geq 0$$

for all strategy f_i'.

Subjective Correlated Equilibrium II

Definition 7

A profile of action distributions $(\mu_i)_{i=1}^I \in (\Delta(A))^I$ is a subjective correlated equilibrium of $\mathbf{g}$ if for all i and all a_i,

$$\mu_i(a_i) > 0 \Rightarrow a_i \in br_i(\mu_i(\cdot|a_i)).$$

- $\mu_i(a_i) = \sum_{a_{-i} \in A_{-i}} \mu_i(a_i, a_{-i})$.
- If $\mu(a_i) > 0$, $\mu_i(a_{-i}|a_i) = \mu_i(a_i, a_{-i})/\mu_i(a_i)$.

Equivalence

Proposition 5

The two definitions are equivalent in the following sense:

- *If* $(\Omega, (P_i), (\mathcal{Q}_i), (f_i))$ *is a subjective correlated equilibrium (I), then its induced profile of action distributions is a subjective correlated equilibrium (II).*
- *If* (μ_i) *is a subjective correlated equilibrium (II), then there exists some subjective correlated equilibrium (I) that induces* (μ_i)*.*

A Posteriori Equilibrium I

Definition 8

$(\Omega, (P_i), (\mathcal{Q}_i), (f_i))$ is an a posteriori equilibrium of $\mathbf{g}$ if for all i and all ω,

$$\sum_{\omega' \in \Omega} P_i(\omega' | Q_i(\omega)) \big(g_i(f_i(\omega), f_{-i}(\omega')) - g_i(a'_i, f_{-i}(\omega')) \big) \geq 0$$

for all a'_i, where $P(\cdot | Q_i(\omega)) \in \Delta(Q_i(\omega))$ is defined even when $P(Q_i(\omega)) = 0$.

Rationalizability and A Posteriori Equilibrium

Proposition 6

1. *For any a posteriori equilibrium* $(\Omega, (P_i), (\mathcal{Q}_i), (f_i))$, $f_i(\omega) \in R_i^\infty$ *for any* i *and any* $\omega \in \Omega$.
2. *There exist an a posteriori equilibrium* $(\Omega, (P_i), (\mathcal{Q}_i), (f_i))$ *such that if* $a \in R^\infty$, *then* $f(\omega) = a$ *for some* $\omega \in \Omega$.

Proof

1.

- Fix any a posteriori equilibrium $(\Omega, (P_i), (\mathcal{Q}_i), (f_i))$.

  We want to show that for all k, $f_i(\omega) \in R_i^k$ for all i and all ω.
- Trivially $f_i(\omega) \in R_i^0 = A_i$ for all i and all ω.
- Suppose $f_i(\omega) \in R_i^{k-1}$ for all i and all ω.

  Take any i and $\omega \in \Omega$.

  Define $\pi_i \in \Delta(A_{-i})$ by
  $\pi_i(a_{-i}) = P_i(\{\omega' \mid f_{-i}(\omega') = a_{-i}\} | Q_i(\omega))$.
- By the induction hypothesis, $\pi_i(R_{-i}^{k-1}) = 1$.
- By the definition of a posteriori equilibrium, $f_i(\omega) \in br_i(\pi_i)$.

  Therefore, $f_i(\omega) \in R_i^k$.

2.

- Recall that R^∞ has the best response property, so that for each $a_i \in R_i^\infty$, there is some $\pi_i^{a_i} \in \Delta(R_{-i}^\infty)$ such that $a_i \in br_i(\pi_i^{a_i})$.
- Construct $(\Omega, (P_i), (\mathcal{Q}_i), (f_i))$ as follows:

  $\Omega = R^\infty$, $P_i(a) = \pi_i^{a_i}/|R_i^\infty|$, $Q_i(a) = \{a' \in A \mid a'_i = a_i\}$, $f_i(a) = a_i$.

A Posteriori Equilibrium II

Definition 9

A profile of action distributions $(\mu_i)_{i=1}^I \in (\Delta(A))^I$ is an a posteriori equilibrium of $\mathbf{g}$ if it is a subject correlated equilibrium and $\mu_i(R^\infty) = 1$ for all i.

(Oyama and Tercieux 2010, Definition 2.2)