Hierarchies of Beliefs and Common Knowledge

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Topics in Economic Theory

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Paper

- Brandenburger, A. and E. Dekel (1993). “Hierarchies of Beliefs and Common Knowledge,” Journal of Economic Theory 59, 189-198.

Setup

- Two agents (for simplicity)
- States of uncertainty Θ (fixed):

  Polish space, i.e., metric space that is

  - complete
  - separable (having a countable dense subset)

  (Needed to apply Kolmogorov's Extension Theorem)

Preliminaries

- For any metric space Z, let $\Delta(Z)$ be the space of probability measures on the Borel field (σ-algebra) of Z.

  Endow $\Delta(Z)$ with the weak topology: $\mu^n \to \mu$ iff $\int f \, d\mu^n \to \int f \, d\mu$ for any bounded continuous function f on Z.

  (This topology is metrizable.)

- If Z is Polish, then so is $\Delta(Z)$.

- A countable product of Polish spaces is Polish in the product topology.

Hierarchies of Beliefs

- Define iteratively

$$
\begin{aligned}
X^0 &= \Theta, \\
X^1 &= X^0 \times \Delta(X^0) = \Theta \times \Delta(\Theta), \\
X^2 &= X^1 \times \Delta(X^1) = \Theta \times \Delta(\Theta) \times \Delta(\Theta \times \Delta(\Theta)), \\
&\vdots \\
X^k &= X^{k-1} \times \Delta(X^{k-1}), \\
&\vdots
\end{aligned}
$$

- Let $T^0 = \prod_{k=0}^{\infty} \Delta(X^k)$.
- A type of agent i: $t_i = (\delta_i^1, \delta_i^2, \delta_i^3, \ldots) \in T^0 = \prod_{k=0}^{\infty} \Delta(X^k)$:
  - $\delta_i^1 \in \Delta(X^0) = \Delta(\Theta)$: i's first order belief
  - $\delta_i^2 \in \Delta(X^1) = \Delta(\Theta \times \Delta(\Theta))$: i's second order belief, i.e., joint belief over Θ and the space of j's first order beliefs
  - $\ldots$

Coherency (Consistency)

- For example, if $(\delta_i^1, \delta_i^2, \ldots)$ is a type of i, where $\delta_i^1 \in \Delta(\Theta)$ and $\delta_i^2 \in \Delta(\Theta \times \Delta(\Theta))$, then it should hold that

$$\delta_i^2(E \times \Delta(\Theta)) = \delta_i^1(E) \text{ for all } E \in \mathcal{B}(\Theta),$$

or $\operatorname{marg}_{X^0} \delta_i^2 = \delta_i^1$.

Definition 1

A type $(\delta^1, \delta^2, \ldots) \in T^0$ is *coherent* if for all $k \geq 2$, $\operatorname{marg}_{X^{k-2}} \delta_k = \delta_{k-1}$.

Let $T^1 \subset T^0$ denote the set of all coherent types.

Homeomorphism between T^1 and $\Delta(\Theta \times T^0)$

- For a sequence of probability measures $(\delta^1, \delta^2, \delta^3, \ldots)$, where $\delta^1 \in \Delta(Z^0), \delta^2 \in \Delta(Z^0 \times Z^1), \delta^3 \in \Delta(Z^0 \times Z^1 \times Z^2), \ldots,$ that is coherent (consistent), there is a unique measure $\delta \in \Delta(Z^0 \times Z^1 \times Z^2 \times \cdots)$ such that $\operatorname{marg}_{Z^0 \times \cdots \times Z^{k-1}} \delta = \delta^k$, where $Z^0 = X^0$, $Z^1 = \Delta(X^0)$, $Z^2 = \Delta(X^1)$, $\ldots$.

  $\cdots$ Kolmogorov's Extension Theorem

- Let $f \colon T^1 \to \Delta(\Theta \times T^0)$ denote this mapping from t to δ.

- f is a homeomorphism,
  i.e., it is one-to-one and onto, and f and f^{-1} are continuous.
  - For $\delta \in \Delta(X^0 \times \Delta(X^0) \times \Delta(X^1) \times \cdots)$,
    $f^{-1}(\delta)^k = \operatorname{marg}_{X^k} \delta$.
  - For $t = (\delta^1, \delta^2, \delta^3, \ldots) \in T^1 \subset \Delta(X^0) \times \Delta(X^1) \times \cdots$,
    $\operatorname{marg}_{X^k} f(t) = \delta^k$.

Proposition 1

T^1 is homeomorphic to $\Delta(\Theta \times T^0)$ by f.

Common Certainty of Coherency

- $T^2 = \{t \in T^1 \mid f(t)(\Theta \times T^1) = 1\}$

  $(= \{t \in T^1 \mid \delta^k(\Theta \times \mathrm{proj}_{\Delta(X^0)\times\cdots\times\Delta(X^{k-1})} T^1) = 1 \ \forall\, k \geq 2\})$
- $T^3 = \{t \in T^1 \mid f(t)(\Theta \times T^2) = 1\}$
- $\cdots$
- $T^* = \bigcap_{k=1}^{\infty} T^k$

  $\cdots$ set of types that satisfy common certainty of coherency

Homeomorphism between T^* and $\Delta(\Theta \times T^*)$

- It holds that $T^* = \{t \in T^1 \mid f(t)(\Theta \times T^*) = 1\}$.
- Thus $f(T^*) = \{\delta \in \Delta(\Theta \times T^0) \mid \delta(\Theta \times T^*) = 1\}$ (since f is one-to-one).
- Identify (by the obvious homeomorphism) $\{\delta \in \Delta(\Theta \times T^0) \mid \delta(\Theta \times T^*) = 1\}$ with $\Delta(\Theta \times T^*)$.
- Then the restriction $g\colon T^* \to \Delta(\Theta \times T^*)$ of f to T^*, defined by $g(t)(E) = f(t)(E)$ for all measurable $E \subset \Theta \times T^*$, is a homeomorphism.

Proposition 2

T^* is homeomorphic to $\Delta(\Theta \times T^*)$ by g.

General Type Spaces

- A *type space* is $((T_1, \pi_1), (T_2, \pi_2))$ where
  - T_i is a polish space, and
  - $\pi_i \colon T_i \to \Delta(\Theta \times T_{-i})$ is continuous.
- $((T^*, g), (T^*, g))$ is a particular type space.
- Explicit type: Given $t_i \in T_i$:
  - $\hat{\pi}_i^1(t_i) \in \Delta(X^0)$ defined by $\hat{\pi}_i^1(t_i)(E^0) = \pi(t_i)(E^0 \times T_{-i})$ for measurable $E^0 \subset \Delta(X^0)$ (where $X^0 = \Theta$);
  - $\hat{\pi}_i^2(t_i) \in \Delta(X^1)$ defined by $\hat{\pi}_i^2(t_i)(E^1) = \pi_i(t_i)(\{(\theta, t_{-i}) \mid (\theta, \hat{\pi}_{-i}^1(t_{-i})) \in E^1\})$ for measurable $E^1 \subset \Delta(X^1)$ (where $X^1 = \Theta \times \Delta(\Theta)$);
  - $\cdots$.

  $\Rightarrow \hat{\pi}_i^*(t_i) = (\hat{\pi}_i^1(t_i), \hat{\pi}_i^2(t_i), \ldots) \in T^0$.
- By construction $\hat{\pi}_i^*(t_i) \in T^1$.

- For all k,

$$\begin{aligned}&\hat{\pi}_i^k(t_i)(\Theta \times \mathrm{proj}_{\Delta(X^0)\times\Delta(X^{k-1})} T^1)\\ &= \pi_i(t_i)(\{(\theta, t_{-i}) \mid (\hat{\pi}_{-i}^1, \ldots, \hat{\pi}_{-i}^{k-1}) \in \mathrm{proj}_{\Delta(X^0)\times\cdots\times\Delta(X^{k-1})} T^1\})\\ &= \pi_i(t_i)(\Theta \times T_{-i}) = 1.\end{aligned}$$

- $\ldots$

- Hence $\hat{\pi}_i^*(t_i) \in T^*$.

- Let $\hat{T}_i = \hat{\pi}_i^*(T_i) \subset T^*$.
  Then $g(\hat{\pi}_i^*(t_i))(\Theta \times \hat{T}_{-i}) = 1$ for all $t_i \in T_i$.

Proposition 3

Any type space $((T_1, \pi_1), (T_2, \pi_2))$ *can be embedded in* $((T^*, g), (T^*, g))$ *as a belief-closed subspace.*

Example: Email Game Type Space

- $\Theta = \{\theta^0, \theta^1\}$
- $T_1 = T_2 = \{0, 1, 2, \ldots\}$
- $\pi_1 \colon T_1 \to \Delta(\Theta \times T_2)$:

$$\pi_1((\theta, t_2)|t_1) = \begin{cases} 1 & \text{if } t_1 = 0,\ t_2 = 0,\ \theta = \theta^0 \\ \frac{1}{2-\varepsilon} & \text{if } t_1 \geq 1,\ t_2 = t_1 - 1,\ \theta = \theta^1 \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_1 \geq 1,\ t_2 = t_1,\ \theta = \theta^1 \\ 0 & \text{otherwise} \end{cases}$$

$\pi_2 \colon T_2 \to \Delta(\Theta \times T_1)$:

$$\pi_2((\theta, t_1)|t_2) = \begin{cases} \frac{1}{2-\varepsilon} & \text{if } t_2 = 0,\ t_1 = 0,\ \theta = \theta^0 \\ \frac{1}{2-\varepsilon} & \text{if } t_2 \geq 1,\ t_1 = t_2,\ \theta = \theta^1 \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_2 \geq 0,\ t_1 = t_2 + 1,\ \theta = \theta^1 \\ 0 & \text{otherwise} \end{cases}$$