# Interim Correlated Rationalizability

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# Paper

- Dekel, E., D. Fudenberg, and S. Morris (2007). "Interim Correlated Rationalizability," Theoretical Economics 2, 15-40.

# Setup

- Players $1, \ldots, I$
- Set of states $\Theta$ (finite)
- Type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^{I}$:
  - $T_i$: set of $i$'s types (countable)
  - $\pi_i \colon T_i \to \Delta(T_{-i} \times \Theta)$: $i$'s belief
- $A_i$: finite set of actions for $i$
- $g_i \colon A \times \Theta \to \mathbb{R}$: payoff function for $i$
- For $\nu_i \in \Delta(T_{-i} \times \Theta \times A_{-i})$,

$$br_i(\nu_i) = \arg\max_{a_i \in A_i} \sum_{\theta, a_{-i}} \sum_{t_{-i}} \nu_i(t_{-i}, \theta, a_{-i}) g_i((a_i, a_{-i}), \theta)$$

# Interim Correlated Rationalizability (ICR)

- $R_{i,0}^{\mathcal{T}}(t_i) = A_i$
- $a_i \in R_{i,k}^{\mathcal{T}}(t_i)$ iff there exists $\nu_i \in \Delta(T_{-i} \times \Theta \times A_{-i})$ such that
  - $\operatorname{marg}_{T_{-i} \times \Theta} \nu_i = \pi_i(t_i)$
  - $\nu_i\big(\big\{(t_{-i}, \theta, a_{-i}) \;\big|\; a_{-i} \in R_{-i,k-1}^{\mathcal{T}}(t_{-i})\big\}\big) = 1$
  - $a_i \in br_i(\nu_i)$

  where $R_{-i,k-1}^{\mathcal{T}}(t_{-i}) = \prod_{j \neq i} R_{j,k-1}^{\mathcal{T}}(t_j)$
- $R_i^{\mathcal{T}}(t_i) = \bigcap_{k=0}^{\infty} R_{i,k}^{\mathcal{T}}(t_i)$

### Definition 1

Action $a_i \in A_i$ is an interim (correlated) rationalizable action for $t_i$ if $a_i \in R_i^{\mathcal{T}}(t_i)$.

# Interim Independent Rationalizability (IIR)

- $IIR^{\mathcal{T}}_{i,0}(t_i) = A_i$
- $a_i \in IIR^{\mathcal{T}}_{i,k}(t_i)$ iff there exist $\nu_i \in \Delta(T_{-i} \times \Theta \times A_{-i})$ and $\sigma_j \colon T_j \to \Delta(A_j)$, $j \neq i$, such that
  - $\nu_i(t_{-i}, \theta, a_{-i}) = \pi(t_i)(t_{-i}, \theta) \prod_{j \neq i} \sigma_j(t_j)(a_j)$
  - $\nu_i\big(\{(t_{-i}, \theta, a_{-i}) \mid a_{-i} \in IIR^{\mathcal{T}}_{-i,k-1}(t_{-i})\}\big) = 1$
  - $a_i \in br_i(\nu_i)$

  where $IIR^{\mathcal{T}}_{-i,k-1}(t_{-i}) = \prod_{j \neq i} IIR^{\mathcal{T}}_{j,k-1}(t_j)$
- $IIR^{\mathcal{T}}_{i}(t_i) = \bigcap_{k=0}^{\infty} IIR^{\mathcal{T}}_{i,k}(t_i)$

## Definition 2

Action $a_i \in A_i$ is an interim independent rationalizable action for $t_i$ if $a_i \in IIR^{\mathcal{T}}_{i}(t_i)$.

# ICR versus IIR

- Example 1
- Example 2

# Independence of Type Spaces

- Given a type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$,
  let $\hat{\pi}_i^*(t_i) \in T^*$ be the belief hierarchy for $t_i \in T_i$.

## Proposition 1

*Consider type spaces $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$ and $\mathcal{T}' = (T_i', \pi_i')_{i=1}^I$.*
*For all $i$, if $\hat{\pi}_i^*(t_i) = \hat{\pi}_i^*(t_i')$, then $R_i^{\mathcal{T}}(t_i) = R_i^{\mathcal{T}'}(t_i')$.*

# Proof

- Let $t_i \in T_i$ and $t'_i \in T'_i$ be such that $\hat{\pi}^1_i(t_i) = \hat{\pi}^1_i(t'_i)$, and take any $\bar{a}_i \in R^{\mathcal{T}}_{i,1}(t_i)$.
- Let $\nu_i \in \Delta(T_{-i} \times \Theta \times A_{-i})$ be such that
  - $\mathrm{marg}_{T_{-i} \times \Theta}\, \nu_i = \pi_i(t_i)$
  - $\bar{a}_i \in br_i(\nu_i)$.
- Define $\sigma'_i \colon \Theta \to \Delta(A_{-i})$ by

$$\sigma'_i(\theta)(a_{-i}) = \frac{\sum_{t_{-i} \in T_{-i}} \nu_i(t_{-i}, \theta, a_{-i})}{\gamma(\theta)},$$

where

$$\gamma(\theta) = \sum_{a_{-i} \in A_{-i}} \sum_{t_{-i} \in T_{-i}} \nu_i(t_{-i}, \theta, a_{-i})$$

$$= \sum_{t_{-i} \in T_{-i}} \pi_i(t_i)(t_{-i}, \theta) = \hat{\pi}^1_i(t_i)(\theta) = \hat{\pi}^1_i(t'_i)(\theta).$$

(Arbitrary if $\gamma(\theta) = 0$.)

- Define $\nu_i' \in \Delta(T_{-i}' \times \Theta \times A_{-i})$ by

$$\nu_i'(t_{-i}', \theta, a_{-i}) = \pi_i'(t_i')(t_{-i}', \theta)\sigma_i'(\theta)(a_{-i}).$$

- Then we have

$$\sum_{a_{-i} \in A_{-i}} \nu_i'(t_{-i}', \theta, a_{-i}) = \pi_i'(t_i')(t_{-i}', \theta),$$

and

$$\begin{aligned}\sum_{t_{-i}' \in T_{-i}'} \nu_i'(t_{-i}', \theta, a_{-i}) &= \hat{\pi}_i^1(t_i')(\theta) \times \frac{\sum_{t_{-i} \in T_{-i}} \nu_i(t_{-i}, \theta, a_{-i})}{\gamma(\theta)} \\ &= \sum_{t_{-i} \in T_{-i}} \nu_i(t_{-i}, \theta, a_{-i}),\end{aligned}$$

so that $\bar{a}_i \in br_i(\nu_i')$.

- Assume that for all $i$, if $\hat{\pi}_i^{k-1}(t_i) = \hat{\pi}_i^{k-1}(t_i')$, then $R_{i,k-1}^{\mathcal{T}}(t_i) = R_{i,k-1}^{\mathcal{T}'}(t_i')$.
- Let $t_i \in T_i$ and $t_i' \in T_i'$ be such that $\hat{\pi}_i^k(t_i) = \hat{\pi}_i^k(t_i')$, and take any $\bar{a}_i \in R_{i,k}^{\mathcal{T}}(t_i)$.
- Let $\nu_i \in \Delta(T_{-i} \times \Theta \times A_{-i})$ be such that
  - $\operatorname{marg}_{T_{-i} \times \Theta} \nu_i = \pi_i(t_i)$
  - $\nu_i\big(\big\{(t_{-i}, \theta, a_{-i}) \bigm| a_{-i} \in R_{-i,k-1}^{\mathcal{T}}(t_{-i})\big\}\big) = 1$
  - $\bar{a}_i \in br_i(\nu_i)$.

- Let $D_{-i}^{k-1} = \{\hat{\pi}_{-i}^{k-1}(t_{-i}) \mid t_{-i} \in T_{-i}\}$.
- Define $\sigma_i'\colon D_{-i}^{k-1} \times \Theta \to \Delta(A_{-i})$ by

$$\sigma_i'(\delta_{-i}, \theta)(a_{-i}) = \frac{\sum_{t_{-i}:\hat{\pi}_{-i}^{k-1}(t_{-i})=\delta_{-i}} \nu_i(t_{-i}, \theta, a_{-i})}{\gamma(\delta_{-i}, \theta)},$$

where

$$\begin{aligned}\gamma(\delta_{-i}, \theta) &= \sum_{a_{-i}\in A_{-i}} \sum_{t_{-i}:\hat{\pi}_{-i}^{k-1}(t_{-i})=\delta_{-i}} \nu_i(t_{-i}, \theta, a_{-i}) \\ &= \sum_{t_{-i}:\hat{\pi}_{-i}^{k-1}(t_{-i})=\delta_{-i}} \pi_i(t_i)(t_{-i}, \theta) \\ &= \sum_{t'_{-i}:\hat{\pi}_{-i}^{k-1}(t'_{-i})=\delta_{-i}} \pi'_i(t'_i)(t'_{-i}, \theta).\end{aligned}$$

(Arbitrary in $R_{i,k-1}^{\mathcal{T}'}(t_i')$ if $\gamma(\delta_{-i}, \theta) = 0$.)

- Define $\nu_i' \in \Delta(T_{-i}' \times \Theta \times A_{-i})$ by

$$\nu_i'(t_{-i}', \theta, a_{-i}) = \pi_i'(t_i')(t_{-i}', \theta)\sigma_i'(\hat{\pi}_{-i}^{k-1}(t_{-i}'), \theta)(a_{-i}),$$

  where $\hat{\pi}_{-i}^{k-1}(t_{-i}') \in D_{-i}^{k-1}$ if $\pi_i'(t_i')(t_{-i}', \theta) > 0$ since $\hat{\pi}_i^k(t_i) = \hat{\pi}_i^k(t_i')$.

- Then we have

$$\begin{aligned}
&\sum_{t_{-i}'} \nu_i'(t_{-i}', \theta, a_{-i}) \\
&= \sum_{\delta_{-i} \in D_{-i}^{k-1}} \sum_{t_{-i}' : \hat{\pi}_{-i}^{k-1}(t_{-i}') = \delta_{-i}} \pi_i'(t_i')(t_{-i}', \theta)\sigma_i'(\delta_{-i}, \theta)(a_{-i}) \\
&= \sum_{\delta_{-i} \in D_{-i}^{k-1}} \gamma(\delta_{-i}, \theta) \frac{\sum_{t_{-i} : \hat{\pi}_{-i}^{k-1}(t_{-i}) = \delta_{-i}} \nu_i(t_{-i}, \theta, a_{-i})}{\gamma(\delta_{-i}, \theta)} \\
&= \sum_{t_{-i}} \nu_i(t_{-i}, \theta, a_{-i}).
\end{aligned}$$

# Example: Email Game

- $\Theta = \{\theta^0, \theta^1\}$
- $T_1 = T_2 = \{0, 1, 2, \ldots\}$
- $\pi_1 \colon T_1 \to \Delta(\Theta \times T_2)$:

$$\pi_1((\theta, t_2)|t_1) = \begin{cases} 1 & \text{if } t_1 = 0,\ t_2 = 0,\ \theta = \theta^0 \\ \frac{1}{2-\varepsilon} & \text{if } t_1 \geq 1,\ t_2 = t_1 - 1,\ \theta = \theta^1 \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_1 \geq 1,\ t_2 = t_1,\ \theta = \theta^1 \\ 0 & \text{otherwise} \end{cases}$$

$\pi_2 \colon T_2 \to \Delta(\Theta \times T_1)$:

$$\pi_2((\theta, t_1)|t_2) = \begin{cases} \frac{1}{2-\varepsilon} & \text{if } t_2 = 0,\ t_1 = 0,\ \theta = \theta^0 \\ \frac{1}{2-\varepsilon} & \text{if } t_2 \geq 1,\ t_1 = t_2,\ \theta = \theta^1 \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_2 \geq 0,\ t_1 = t_2 + 1,\ \theta = \theta^1 \\ 0 & \text{otherwise} \end{cases}$$

- Payoffs:

$\theta = \theta^0$

| | $A_2$ | $B_2$ |
|---|---|---|
| $A_1$ | $0, 4$ | $0, 3$ |
| $B_1$ | $1, 0$ | $1, 2$ |

$\theta = \theta^1$

| | $A_2$ | $B_2$ |
|---|---|---|
| $A_1$ | $4, 4$ | $0, 3$ |
| $B_1$ | $3, 0$ | $2, 2$ |