Approximating Common Knowledge with Common Beliefs

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Topics in Economic Theory

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Papers

- Monderer, D. and D. Samet (1989). "Approximating Common Knowledge with Common Beliefs," Games and Economic Behavior 1, 170-190.
- Kajii, A. and S. Morris (1997b). "Refinements and Higher Order Beliefs: A Unified Survey."

Type Spaces

- Type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$:
  - T_i: set of i's types (countable)
  - $\pi_i : T_i \to \Delta(T_{-i})$: i's belief
- $T = \prod_{i=1}^I T_i$, $T_{-i} = \prod_{j \neq i} T_j$
- If there is a common prior $P \in \Delta(T)$ with $P(t_i) = P(\{t_i\} \times T_{-i}) > 0$ for all i and t_i,

  $$\pi_i(t_i)(E_{-i}) = \frac{P(\{t_i\} \times E_{-i})}{P(t_i)}$$

  for $E_{-i} \subset T_{-i}$.
- An event $E \subset T$ is simple if $E = \prod_{i=1}^I E_i$ for some $E_i \subset T_i$, $i = 1, \ldots, I$.

  Let $\mathcal{S} \subset 2^T$ denote the set of simple events.

p-Belief Operator

- $B_i^p \colon \mathcal{S} \to 2^{T_i}$:

$$B_i^p(E) = \{t_i \in T_i \mid t_i \in E_i \text{ and } \pi_i(t_i)(E_{-i}) \geq p\}.$$

Proposition 1

1. $B_i^p(E) \subset E_i$.
2. *If* $E \subset F$, *then* $B_i^p(E) \subset B_i^p(F)$.
3. *If* $E^0 \supset E^1 \supset \cdots$, *then* $B_i^p(\bigcap_{k=0}^{\infty} E^k) = \bigcap_{k=0}^{\infty} B_i^p(E^k)$.

(3. If $E^0 \supset E^1 \supset \cdots$, then $\pi_i(t_i)(\bigcap_{k=0}^{\infty} E_{-i}^k) = \lim_{k\to\infty} \pi_i(t_i)(E_{-i}^k)$.)

Common $\mathbf{p}$-Belief (Iteration)

- For $\mathbf{p} \in [0,1]^I$,

$$B_*^{\mathbf{p}}(E) = \prod_{i=1}^{I} B_i^{p_i}(E),$$

$$C^{\mathbf{p}}(E) = \bigcap_{k=1}^{\infty} (B_*^{\mathbf{p}})^k(E).$$

Definition 1

$E \in \mathcal{S}$ is *common* $\mathbf{p}$*-belief at* $t \in T$ if $t \in C^{\mathbf{p}}(E)$.

Common $\mathbf{p}$-Belief (Fixed Point)

Definition 2

$E \in \mathcal{S}$ is $\mathbf{p}$-*evident* if

$$E \subset B_*^{\mathbf{p}}(E).$$

(Equivalent to the condition with "$E = B_*^{\mathbf{p}}(E)$".)

Definition 3

$E \in \mathcal{S}$ is *common* $\mathbf{p}$-*belief at* $t \in T$ if there exists a $\mathbf{p}$-evident event F such that

$$t \in F \subset B_*^{\mathbf{p}}(E).$$

(Equivalent to the condition with "$t \in F \subset E$".)

Equivalence

Proposition 2

$C^{\mathbf{p}}(E)$ is $\mathbf{p}$-evident, i.e., $C^{\mathbf{p}}(E) \subset B_*^{\mathbf{p}}(C^{\mathbf{p}}(E))$.

Proof.

$C^{\mathbf{p}}(E) = \bigcap_{k=1}^{\infty} B_*^{\mathbf{p}}((B_*^{\mathbf{p}})^{k-1}(E)) = B_*^{\mathbf{p}}(\bigcap_{k=1}^{\infty}(B_*^{\mathbf{p}})^{k-1}(E))$. □

Proposition 3

$C^{\mathbf{p}}(E)$ is the largest $\mathbf{p}$-evident event in E, i.e.,
if $F \subset E$ and $F \subset B_*^{\mathbf{p}}(F)$, then $F \subset C^{\mathbf{p}}(E)$.

Proof.

First, $F \subset B_*^{\mathbf{p}}(F) \subset B_*^{\mathbf{p}}(E)$.

Suppose $F \subset (B_*^{\mathbf{p}})^n(E)$. Then
$F \subset B_*^{\mathbf{p}}(F) \subset B_*^{\mathbf{p}}((B_*^{\mathbf{p}})^n(E)) = (B_*^{\mathbf{p}})^{n+1}(E)$. □

Equivalence

Proposition 4

The two definitions are equivalent, i.e.,

$$t \in C^{\mathbf{p}}(E) \iff \exists\, F : F \subset B_*^{\mathbf{p}}(F) \text{ and } t \in F \subset B_*^{\mathbf{p}}(E).$$

Proof.

- "Only if":

  $C^{\mathbf{p}}(E)$ is $\mathbf{p}$-evident by Proposition 2, and $C^{\mathbf{p}}(E) \subset B_*^{\mathbf{p}}(C^{\mathbf{p}}(E))$.

- "If":

  $F \subset C^{\mathbf{p}}(E)$ by Proposition 3.

Example: Email Game

- $T_1 = T_2 = \{0, 1, 2, \dots\}$
- $\pi_1 : T_1 \to \Delta(\mathrm{T}_2)$:

$$\pi_1(t_2|t_1) = \begin{cases} 1 & \text{if } t_1 = 0,\ t_2 = 0 \\ \frac{1}{2-\varepsilon} & \text{if } t_1 \geq 1,\ t_2 = t_1 - 1 \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_1 \geq 1,\ t_2 = t_1 \\ 0 & \text{otherwise} \end{cases}$$

$\pi_2 : T_2 \to \Delta(T_1)$:

$$\pi_2(t_1|t_2) = \begin{cases} \frac{1}{2-\varepsilon} & \text{if } t_2 = 0,\ t_1 = 0 \\ \frac{1}{2-\varepsilon} & \text{if } t_2 \geq 1,\ t_1 = t_2 \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_2 \geq 0,\ t_1 = t_2 + 1 \\ 0 & \text{otherwise} \end{cases}$$

- Let $E_1 = T_1 \setminus \{0\}$ and $E_2 = T_2$, and $p_i \geq \frac{1}{2}$.

Connection to Games 1

- Type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$
- Players $1, \dots, I$
- Binary actions $A_i = \{0, 1\}$
- $F \in \mathcal{S}$ is identified with the (pure) strategy profile σ such that $\sigma_i(t_i) = 1$ if and only if $t_i \in F_i$.
- Fix $E \in \mathcal{S}$.
- Incomplete information game $\mathbf{u}^{\mathbf{p}}$:

  If $t_i \in E_i$: for all t_{-i} with $\pi_i(t_i)(t_{-i}) > 0$,

$$u_i^{p_i}(1, a_{-i}, t_i, t_{-i}) = \begin{cases} 1 - p_i & \text{if } a_{-i} = \mathbf{1}_{-i}, \\ -p_i & \text{otherwise,} \end{cases}$$

$$u_i^{p_i}(0, a_{-i}, t_i, t_{-i}) = 0.$$

  If $t_i \notin E_i$: 0 is a dominant action.

- $B_i^{p_i}(E_i \times F_{-i})$ is the (largest) best response to F_{-i} (play 1 if indifferent).
- $1 \in R_i(t_i)$ if and only if $t_i \in C_i^{\mathbf{p}}(E)$.
- F is an equilibrium if and only if $F \subset E$ and F is $\mathbf{p}$-evident.
- $C^{\mathbf{p}}(E)$ is the largest equilibrium.

Connection to Games 2

- Players $1, \ldots, I$
- Actions A_i (finite)
- Complete information game $\mathbf{g}$, $g_i \colon A \to \mathbb{R}$
- $a^* \in A$ is a $\mathbf{p}$-*dominant equilibrium* of $\mathbf{g}$ if

  $$a_i^* \in br_i(\lambda_i)$$

  for any $\lambda_i \in \Delta(A_{-i})$ such that $\lambda_i(a_{-i}^*) \geq p_i$.
- Incomplete information game $\mathbf{u}$, $u_i \colon A \times T \to \mathbb{R}$
- Let

  $$T_i^{g_i} = \{t_i \in T_i \mid u_i(a, t_i, t_{-i}) = g_i(a) \text{ for all } a \in A \text{ and } \text{for all } t_{-i} \in T_{-i} \text{ with } \pi_i(t_i)(t_{-i}) > 0\},$$

  and $T^{\mathbf{g}} = \prod_{i=1}^{I} T_i^{g_i}$.

Lemma 5

Suppose that a^* is a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$. Then $\mathbf{u}$ has an equilibrium σ such that $\sigma(t)(a^*) = 1$ for all $t \in C^{\mathbf{p}}(T^{\mathbf{g}})$.

Proof

- For each i, let $F_i = B_i^{p_i}(C^{\mathbf{p}}(T^{\mathbf{g}}))$ $(\subset T_i^{g_i})$.

  Then $C^{\mathbf{p}}(T^{\mathbf{g}}) \subset F$ (in fact $C^{\mathbf{p}}(T^{\mathbf{g}}) = F$).

- Consider the modified game $\mathbf{u}'$ where each player i must play a_i^* if $t_i \in F_i$.

  Let σ^* be any equilibrium of $\mathbf{u}'$.

  We want to show that σ^* is also an equilibrium of $\mathbf{u}$.

- For $t_i \in T_i \setminus F_i$,
  $\sigma_i^*(t_i)$ is a best response to σ_{-i}^* by construction.

- Suppose $t_i \in F_i$.

  Then by definition, $\pi_i(t_i)(C^{\mathbf{p}}(T^{\mathbf{g}})) \geq p_i$, and hence i assigns probability at least p_i to the others playing a_{-i}^*.

  Therefore, $\sigma_i^*(t_i) = a_i^*$ is a best response to σ_{-i}^*.

Proposition 6

Suppose that a^* is a strict equilibrium of $\mathbf{g}$.

For any $\delta > 0$, there exists $\varepsilon > 0$ such that for any $P \in \Delta(T)$ such that $P(C^{\mathbf{p}}(T^{\mathbf{g}})) \geq 1 - \varepsilon$ for all $\mathbf{p} \ll \mathbf{1}$, there exists an equilibrium σ of $(T, P, \mathbf{u})$ such that $P(\{t \in T \mid \sigma(t)(a^*) = 1\}) \geq 1 - \delta$.

- A strict equilibrium is $\mathbf{p}$-dominant for some $\mathbf{p} \ll \mathbf{1}$.
- The proposition holds even with non common priors P_i.