Approximating Common Knowledge with Common Beliefs

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Papers


Type Spaces

- Type space \( \mathcal{T} = (T_i, \pi_i)_{i=1}^I \):
  - \( T_i \): set of \( i \)'s types (countable)
  - \( \pi_i : T_i \to \Delta(T_{-i}) \): \( i \)'s belief

- \( T = \prod_{i=1}^I T_i \), \( T_{-i} = \prod_{j \neq i} T_j \)

- If there is a common prior \( P \in \Delta(T) \) with \( P(t_i) = P(\{t_i\} \times T_{-i}) > 0 \) for all \( i \) and \( t_i \),

\[
\pi_i(t_i)(E_{-i}) = \frac{P(\{t_i\} \times E_{-i})}{P(t_i)}
\]

for \( E_{-i} \subset T_{-i} \).

- An event \( E \subset T \) is simple if \( E = \prod_{i=1}^I E_i \) for some \( E_i \subset T_i \), \( i = 1, \ldots, I \).

Let \( S \subset 2^T \) denote the set of simple events.
\( p \)-Belief Operator

\( B^p_i : S \rightarrow 2^{T_i} : \)

\[
B^p_i (E) = \{ t_i \in T_i \mid t_i \in E_i \text{ and } \pi_i(t_i)(E_{-i}) \geq p \}.
\]

Proposition 1

1. \( B^p_i (E) \subset E_i \).

2. If \( E \subset F \), then \( B^p_i (E) \subset B^p_i (F) \).

3. If \( E^0 \supset E^1 \supset \cdots \), then \( B^p_i (\bigcap_{k=0}^{\infty} E^k) = \bigcap_{k=0}^{\infty} B^p_i (E^k) \).

(3. If \( E^0 \supset E^1 \supset \cdots \), then \( \pi_i(t_i)\left(\bigcap_{k=0}^{\infty} E^k_{-i}\right) = \lim_{k \to \infty} \pi_i(t_i)(E^k_{-i}) \).)
Common \( p \)-Belief (Iteration)

- For \( p \in [0, 1]^I \),

\[
B^p_\ast(E) = \prod_{i=1}^{I} B^p_{i} (E),
\]

\[
C^p(E) = \bigcap_{k=1}^{\infty} (B^p_\ast)^k (E).
\]

**Definition 1**

\( E \in S \) is common \( p \)-belief at \( t \in T \) if \( t \in C^p(E) \).
Common $p$-Belief (Fixed Point)

Definition 2

$E \in S$ is $p$-evident if

$$E \subset B^p_*(E).$$

(Equivalent to the condition with “$E = B^p_*(E)$”.)

Definition 3

$E \in S$ is common $p$-belief at $t \in T$ if there exists a $p$-evident event $F$ such that

$$t \in F \subset B^p_*(E).$$

(Equivalent to the condition with “$t \in F \subset E$”.)
Equivalence

Proposition 2

$C_P(E)$ is $p$-evident, i.e., $C_P(E) \subset B^*_P(C_P(E))$.

Proof.

$C_P(E) = \bigcap_{k=1}^{\infty} B^*_P((B^*_P)^{k-1}(E)) = B^*_P(\bigcap_{k=1}^{\infty} (B^*_P)^{k-1}(E))$. \hfill \Box$

Proposition 3

$C_P(E)$ is the largest $p$-evident event in $E$, i.e., if $F \subset E$ and $F \subset B^*_P(F)$, then $F \subset C_P(E)$.

Proof.

First, $F \subset B^*_P(F) \subset B^*_P(E)$.

Suppose $F \subset (B^*_P)^n(E)$. Then $F \subset B^*_P(F) \subset B^*_P((B^*_P)^n(E)) = (B^*_P)^{n+1}(E)$. \hfill \Box
Equivalence

Proposition 4

The two definitions are equivalent, i.e.,

\[ t \in C^p(E) \iff \exists F : F \subset B^p_*(F) \text{ and } t \in F \subset B^p_*(E). \]

Proof.

- **“Only if”:**

  \( C^p(E) \) is \( p \)-evident by Proposition 2, and
  \( C^p(E) \subset B^p_*(C^p(E)) \).

- **“If”:**

  \( F \subset C^p(E) \) by Proposition 3.
Example: Email Game

- $T_1 = T_2 = \{0, 1, 2, \ldots\}$
- $\pi_1 : T_1 \to \Delta(T_2)$:
  \[
  \pi_1(t_2 | t_1) = \begin{cases} 
  1 & \text{if } t_1 = 0, t_2 = 0 \\
  \frac{1}{2-\varepsilon} & \text{if } t_1 \geq 1, t_2 = t_1 - 1 \\
  \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_1 \geq 1, t_2 = t_1 \\
  0 & \text{otherwise}
  \end{cases}
  \]

- $\pi_2 : T_2 \to \Delta(T_1)$:
  \[
  \pi_2(t_1 | t_2) = \begin{cases} 
  \frac{1}{2-\varepsilon} & \text{if } t_2 = 0, t_1 = 0 \\
  \frac{1}{2-\varepsilon} & \text{if } t_2 \geq 1, t_1 = t_2 \\
  \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_2 \geq 0, t_1 = t_2 + 1 \\
  0 & \text{otherwise}
  \end{cases}
  \]

- Let $E_1 = T_1 \setminus \{0\}$ and $E_2 = T_2$, and $p_i \geq \frac{1}{2}$. 
Connection to Games 1

- Type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$
- Players $1, \ldots, I$
- Binary actions $A_i = \{0, 1\}$
- $F \in S$ is identified with the (pure) strategy profile $\sigma$ such that $\sigma_i(t_i) = 1$ if and only if $t_i \in F_i$.
- Fix $E \in S$.
- Incomplete information game $u^p$:
  
  If $t_i \in E_i$: for all $t_{-i}$ with $\pi_i(t_i)(t_{-i}) > 0$,
  
  $$u^p_i(1, a_{-i}, t_i, t_{-i}) = \begin{cases} 1 - p_i & \text{if } a_{-i} = 1_{-i}, \\ -p_i & \text{otherwise}, \end{cases}$$
  $$u^p_i(0, a_{-i}, t_i, t_{-i}) = 0.$$

  If $t_i \notin E_i$: $0$ is a dominant action.
- $B^p_i(E_i \times F_{-i})$ is the (largest) best response to $F_{-i}$ (play 1 if indifferent).
- $1 \in R_i(t_i)$ if and only if $t_i \in C^p_i(E)$.
- $F$ is an equilibrium if and only if $F \subset E$ and $F$ is $p$-evident.
- $C^p(E)$ is the largest equilibrium.
Connection to Games 2

- Players 1, \ldots, I
- Actions $A_i$ (finite)
- Complete information game $g$, $g_i : A \rightarrow \mathbb{R}$
- $a^* \in A$ is a $p$-dominant equilibrium of $g$ if
  $$a^*_i \in br_i(\lambda_i)$$
  for any $\lambda_i \in \Delta(A_{-i})$ such that $\lambda_i(a^*_i) \geq p_i$.
- Incomplete information game $u$, $u_i : A \times T \rightarrow \mathbb{R}$
- Let
  $$T^{g_i}_i = \{t_i \in T_i \mid u_i(a, t_i, t_{-i}) = g_i(a) \text{ for all } a \in A \text{ and for all } t_{-i} \in T_{-i} \text{ with } \pi_i(t_i)(t_{-i}) > 0\},$$
  and $T^g = \prod_{i=1}^I T^{g_i}_i$. 

Lemma 5

Suppose that $a^*$ is a $p$-dominant equilibrium of $g$. Then $u$ has an equilibrium $\sigma$ such that $\sigma(t)(a^*) = 1$ for all $t \in C^p(T^g)$. 
Proof

- For each $i$, let $F_i = B_i^{p_i}(C^p(T^g)) \subset T_i^{g_i}$.
  Then $C^p(T^g) \subset F$ (in fact $C^p(T^g) = F$).

- Consider the modified game $u'$ where each player $i$ must play $a_i^*$ if $t_i \in F_i$.
  Let $\sigma^*$ be any equilibrium of $u'$.
  We want to show that $\sigma^*$ is also an equilibrium of $u$.

- For $t_i \in T_i \setminus F_i$, $\sigma_i^*(t_i)$ is a best response to $\sigma_{-i}^*$ by construction.

- Suppose $t_i \in F_i$.
  Then by definition, $\pi_i(t_i)(C^p(T^g)) \geq p_i$, and hence $i$ assigns probability at least $p_i$ to the others playing $a_{-i}^*$.
  Therefore, $\sigma_i^*(t_i) = a_i^*$ is a best response to $\sigma_{-i}^*$. 
Proposition 6

Suppose that $a^*$ is a strict equilibrium of $g$.

For any $\delta > 0$, there exists $\varepsilon > 0$ such that for any $P \in \Delta(T)$ such that $P(C^p(Tg)) \geq 1 - \varepsilon$ for all $p \ll 1$, there exists an equilibrium $\sigma$ of $(T, P, u)$ such that $P(\{t \in T \mid \sigma(t)(a^*) = 1\}) \geq 1 - \delta$.

- A strict equilibrium is $p$-dominant for some $p \ll 1$.
- The proposition holds even with non common priors $P_i$. 