Robust Equilibria under Non-Common Priors

Daisuke Oyama

Topics in Economic Theory

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Paper

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Robustness of Equilibria under Non-Common Priors

- An analyst analyzes some strategic situation with a complete information game $\mathbf{g}$ and a Nash equilibrium a^* thereof.
- He knows that it is a good approximation, but he also thinks that there may be "small" payoff uncertainty among players in the real world and does not know about the uncertainty structure;

  in particular, he has no reason to assume that the players share a common prior in the real incomplete information game.
- Is the Nash equilibrium a^* "close" to some Bayesian Nash equilibrium of any incomplete information game "close" to $\mathbf{g}$ where players have possibly different priors?
- What kind of equilibrium will be robust under non-common priors?

Results

- Point-valued concept:

  In generic games, a NE is robust under non-common priors $\iff$ it is a unique rationalizable action profile (unique action profile that survives iterated elimination of dominated actions).

- Set-valued concept:

  In generic games, a smallest robust set exists and coincides with the set of a posteriori equilibria.

Complete Information Games

- Set of players $I = \{1, \dots, |I|\}$
- Action set A_i (finite)
- Payoff function $g_i \colon A \to \mathbb{R}$

  Fix players and actions, and identify the complete information game with $\mathbf{g} = (g_i)_{i \in I}$.
- g_i is extended to $\Delta(A_{-i})$ by

$$g_i(a_i, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i, a_{-i}) \qquad (\pi_i \in \Delta(A_{-i})).$$

- The set of i's best responses to $\pi_i \in \Delta(A_{-i})$:

$$br_i(\pi_i) = \{a_i \in A_i \mid g_i(a_i, \pi_i) \geq g_i(a_i', \pi_i) \ \forall\, a_i' \in A_i\}.$$

(Correlated) Rationalizability

- $S_i^0 = A_i$
- $S_i^k = \{a_i \in A_i \mid a_i \in br_i(\pi_i) \; \exists \, \pi_i \in \Delta(S_{-i}^{k-1})\}$
  where $S_{-i}^{k-1} = \prod_{j \neq i} S_j^{k-1}$
- $S_i^\infty = \bigcap_{k=0}^\infty S_i^k$

Never Strict Best Response

- $W_i^0 = A_i$
- $W_i^k = \{a_i \in A_i \mid \{a_i\} = br_i(\pi_i) \; \exists \, \pi_i \in \Delta(W_{-i}^{k-1})\}$

  where $W_{-i}^{k-1} = \prod_{j \neq i} W_j^{k-1}$
- $W_i^\infty = \bigcap_{k=0}^{\infty} W_i^k$

Subjective Correlated Equilibrium/A Posteriori Equilibrium

Definition 1

A profile of action distributions $(\mu_i)_{i=1}^I \in (\Delta(A))^I$ is a *subjective correlated equilibrium* of $\mathbf{g}$ if for all i and all a_i,

$$\mu_i(a_i) > 0 \Rightarrow a_i \in br_i(\mu_i(\cdot|a_i)).$$

Definition 2

- $(\mu_i)_{i=1}^I \in (\Delta(A))^I$ is an N*-subjective correlated equilibrium* of $\mathbf{g}$ if it is a subjective correlated equilibrium of $\mathbf{g}$ and $\mu_i(S^N) = 1$ for all i.
- $(\mu_i)_{i=1}^I \in (\Delta(A))^I$ is an *a posteriori equilibrium* of $\mathbf{g}$ if it is a subjective correlated equilibrium of $\mathbf{g}$ and $\mu_i(S^\infty) = 1$ for all i.

Undominated/Strict A Posteriori Equilibrium

Definition 3

- $(\mu_i)_{i=1}^I \in (\Delta(A))^I$ is an *undominated N-subjective correlated equilibrium* of $\mathbf{g}$ if it is an N-subjective correlated equilibrium of $\mathbf{g}$ such that $\mu_i(W^N) = 1$ for all i.
- $(\mu_i)_{i=1}^I \in (\Delta(A))^I$ is an *undominated a posteriori equilibrium* of $\mathbf{g}$ if it is an a posteriori equilibrium of $\mathbf{g}$ such that $\mu_i(W^\infty) = 1$ for all i.
- $(\mu_i)_{i=1}^I \in (\Delta(A))^I$ is a *strict a posteriori equilibrium* of $\mathbf{g}$ if it is an a posteriori equilibrium of $\mathbf{g}$ such that for all i and all a_i,

$$\mu_i(a_i) > 0 \Rightarrow \{a_i\} = br_i(\mu_i(\cdot|a_i)).$$

Type Spaces

- Type space $\mathcal{T} = ((T_i, P_i)_{i \in I})$:
  - $T_i = \{0, 1, 2, \ldots\}$: set of i's types
  - $P_i \in \Delta(T)$: i's prior

    Assume $P_i(t_i) = P_i(\{t_i\} \times T_{-i}) > 0$ for all i and t_i.
- Let

$$P_i(E_{-i}|t_i) = \frac{P_i(\{t_i\} \times E_{-i})}{P_i(t_i)}$$

  for $t_i \in T_i$ and $E_{-i} \subset T_{-i}$.
- For $E \in \mathcal{S}$,

$$K_i(E) = \{t_i \in T_i \mid t_i \in E_i \text{ and } P_i(E_{-i}|t_i) = 1\}.$$

- $P_0 \in \Delta(T)$: the analyst's prior

Incomplete Information Games

- Fix I, $(A_i)_{i \in I}$, and $(T_i)_{i \in I}$.
- Incomplete information game $(\mathbf{u}, (P_i)_{i \in I})$: $u_i \colon A \times T \to \mathbb{R}$
- i's strategy: $\sigma_i \colon T_i \to \Delta(A_i)$; set of all strategies Σ_i
- $U_i(a_i, \sigma_{-i}|t_i) = \sum_{t_{-i} \in T_{-i}} P_i(t_{-i}|t_i) u_i((a_i, \sigma_{-i}(t_{-i})), (t_i, t_{-i}))$
- For a strategy profile σ and $i \in \{0\} \cup I$,
  write $\sigma_{P_i} \in \Delta(A)$ for the induced action distribution:
  $\sigma_{P_i}(a) = \sum_{t \in T} P_i(t) \sigma(a|t)$.
- Given $\mathbf{g}$ and $(\mathbf{u}, (P_i)_{i \in I})$, let

$$T_i^{g_i} = \{t_i \in T_i \mid u_i(a, t_i, t_{-i}) = g_i(a) \text{ for all } a \in A \text{ and for all } t_{-i} \in T_{-i} \text{ with } P_i(t_{-i}|t_i) > 0\},$$

and $T^{\mathbf{g}} = \prod_{i=1}^{I} T_i^{g_i}$.

Robust Equilibria I

- $(\mathbf{u}, (P_i)_{i \in I})$ is an (ε, N)-*perturbation* of $\mathbf{g}$ if $P_i(\bigcap_{n=1}^{N}(K_*)^n(T^{\mathbf{g}})) \geq 1 - \varepsilon$ for all $i \in I$.

Definition 4

- $\mu = (\mu_i)_{i \in I} \in (\Delta(A))^I$ is N-*robust* in $\mathbf{g}$ if for any $\delta > 0$, there exists $\varepsilon > 0$ such that any (ε, N)-perturbation of $\mathbf{g}$ has a BNE σ such that $\|\mu_i - \sigma_{P_i}\| \leq \delta$ for all $i \in I$.
- $\mu = (\mu_i)_{i \in I} \in (\Delta(A))^I$ is *robust* in $\mathbf{g}$ if there exists $N \geq 0$ such that μ is N-robust.

Robust Equilibria II

- $((\mathbf{u}, (P_i)_{i \in I}), P_0)$ is an (ε, N)-*elaboration* of $\mathbf{g}$ if $P_0(\bigcap_{n=1}^{N}(K_*)^n(T^{\mathbf{g}})) \geq 1 - \varepsilon$ for all $i \in I$.

Definition 5

- $\xi \in \Delta(A)$ is N-*robust* in $\mathbf{g}$ if for any $\delta > 0$, there exists $\varepsilon > 0$ such that any (ε, N)-elaboration of $\mathbf{g}$, $((\mathbf{u}, (P_i)_{i \in I}), P_0)$, $(\mathbf{u}, (P_i)_{i \in I})$ has a BNE σ such that $\|\xi - \sigma_{P_0}\| \leq \delta$.
- $\xi \in \Delta(A)$ is *robust* in $\mathbf{g}$ if there exists $N \geq 0$ such that ξ is N-robust.

Point-Valued Robustness

Theorem 1

Suppose that $S^\infty = W^\infty$ *in* $\mathbf{g}$.

$\mathbf{g}$ *has a robust equilibrium if and only if* $\mathbf{g}$ *is dominance solvable.*

Iterative Dominance Purification of A Posteriori Equilibrium

Lemma 1

Let $(\mu_i)_{i \in I}$ *be a strict a posteriori equilibrium of* $\mathbf{g}$ *with common support.*

Then for any $\varepsilon > 0$ *and* $N \geq 0$*, there exists an* (ε, N)*-perturbation of* $\mathbf{g}$ *such that there is a unique rationalizable strategy profile* σ *and it satisfies* $\sigma_{P_i} = \mu_i$ *for all* $i \in I$*.*

Corollary 2

Let $(\mu_i)_{i \in I}$ *be an undominated a posteriori equilibrium of* $\mathbf{g}$*.*

Then for any $\delta > 0$*,* $\varepsilon > 0$*, and* $N \geq 0$*, there exists an* (ε, N)*-perturbation of* $\mathbf{g}$ *such that there is a unique rationalizable strategy profile* σ *and it satisfies* $\|\sigma_{P_i} - \mu_i\| \leq \delta$ *for all* $i \in I$*.*

Example: Matching Pennies

- Let $(\mu_1, \mu_2) \in \Delta(A) \times \Delta(A)$ be any strict a posteriori equilibrium with full support, where $A = \{H_1, T_1\} \times \{H_2, T_2\}$.
- Fix any $\varepsilon > 0$ and $N \geq 0$, and
  construct a dominance solvable (ε, N)-perturbation such that the unique rationalizable strategy σ satisfies $\sigma_{P_i} = \mu_i$ for all $i \in I$.
- $T_1 = \{(0, H_1), (0, T_1), (1, H_1), (1, T_1), (2, H_1), (2, T_1), \ldots\}$
  $T_2 = \{(0, H_2), (0, T_2), (1, H_2), (1, T_2), (2, H_2), (2, T_2), \ldots\}$
- $P_1((k, a_1), (k-1, a_2)) = \varepsilon_k \mu_1(a_1, a_2)$
  $P_1((k, a_1), (k+1, a_2)) \approx 0$
  ($\varepsilon_k \approx \tilde{\varepsilon}(1-\tilde{\varepsilon})^k$, $\tilde{\varepsilon} = 1 - (1-\varepsilon)^{1/(N+1)}$)

Proof of the Necessity Part

- Suppose that $W^\infty \neq \emptyset$.
- If $|W^\infty| > 1$, then we can take distinct a posteriori equilibria (μ_i) and (μ_i') whose support is W^∞.
- Applying Lemma 1 to each of (μ_i) and (μ_i') shows that there is no robust equilibrium.

Discussion

- Kajii and Morris (1997):

  A unique correlated equilibrium is robust under common prior.
  $\because \bigcap_{\varepsilon>0} \overline{\{\text{equilibrium action distributions of } \varepsilon\text{-perturbations}\}}$
  $= \{\text{correlated equilibria}\}$

- Under non-common priors:

  $\bigcap_{\varepsilon>0} \overline{\{\text{rationalizable action distributions of } \varepsilon\text{-perturbations}\}}$
  $= \{\text{a posteriori equilibria}\}$

  $+$

  Each a posteriori equilibrium can be "contagious" in some ε-perturbation with non-common priors.

  ("Iterative dominance purification of a posteriori equilibrium")

- Weinstein and Yildiz (2007): "Interim approach"

  For any type t and any rationalizable action a^* of t, there exist a dominance solvable incomplete information game and a sequence of types from this game such that

  1. this sequence converges to t in product topology, and
  2. each type of this sequence plays a^*.

  Moreover, by Lipman (2003, 2010), such an incomplete information game can be one with a common prior.

- This paper: "Ex ante approach"

  The above set of properties is incompatible with the requirement that the ex ante probability that the payoffs are close to those of t must be small.

- It is impossible in general to have an (ε, N)-perturbation such that
  1. it has a common prior,
  2. it is dominance solvable, and
  3. the unique rationalizable strategy induces an action distribution that is close to the given a posteriori equilibrium.

Set-Valued Robustness I

Definition 6

- A product set of action distribution profiles $M = \prod_{i \in I} M_i \subset (\Delta(A))^I$ is N-*robust* in $\mathbf{g}$ if it is closed, and for all $\delta > 0$, there exists $\varepsilon > 0$ such that any (ε, N)-perturbation of $\mathbf{g}$ has a BNE σ such that for all $i \in I$, there exists $\mu_i \in M_i$ such that $\|\mu_i - \sigma_{P_i}\| \leq \delta$.
- M is *robust* in $\mathbf{g}$ if there exists $N \geq 0$ such that M is N-robust.

Theorem 2

Suppose that $S^\infty = W^\infty$ in $\mathbf{g}$.

The set of a posteriori equilibria of $\mathbf{g}$ is the smallest robust set of $\mathbf{g}$.

Set-Valued Robustness II

Definition 7

- A set of action distributions $\Xi \subset \Delta(A)$ is N-*robust* in $\mathbf{g}$ if it is closed, and for all $\delta > 0$, there exists $\varepsilon > 0$ such that any (ε, N)-elaboration of $\mathbf{g}$, $((\mathbf{u}, (P_i)_{i\in I}), P_0)$, $(\mathbf{u}, (P_i)_{i\in I})$ has a BNE σ such that there exists $\xi \in \Xi$ such that$\|\xi - \sigma_{P_0}\| \leq \delta$.
- Ξ is *robust* in $\mathbf{g}$ if there exists $N \geq 0$ such that Ξ is N-robust.

Theorem 3

Suppose that $S^\infty = W^\infty$ *in* $\mathbf{g}$.

$\Delta(S^\infty)$ *is the smallest robust set of* $\mathbf{g}$.

Uniform Bound on the Heterogeneity in Priors

- A measure of heterogeneity in priors:

$$\rho((P_i)_{i \in I}) = \max_{i \neq j} \sup_{t \in T} \frac{P_i(t)}{P_j(t)},$$

where $q/0 = \infty$ for $q > 0$ and $0/0 = 1$.

- $\mu = (\mu_i)_{i \in I} \in (\Delta(A))^I$ is r-*robust* in $\mathbf{g}$ if for any $\delta > 0$, there exists $\varepsilon > 0$ such that any ε-perturbation of $\mathbf{g}$ with $\rho((P_i)_{i \in I}) \leq r$ has a BNE σ such that $\|\mu_i - \sigma_{P_i}\| \leq \delta$ for all $i \in I$.

Critical Path Theorem

Proposition 3

For any $r > 1$*, if* $p < 1/\{1 + r(|I| - 1)\}$*, then in any type space* $(T_i, P_i)_{i \in I}$ *with* $\rho((P_i)_{i \in I}) \leq r$*, any simple event* E *satisfies*

$$P_j(CB^p(E)) \geq 1 - \frac{1 - p}{1 - \{1 + r(|I| - 1)\}p} \max_{i \in I}(1 - P_i(E))$$

for all $j \in I$*.*

Robustness with Uniform Bound

Proposition 4

Suppose that a^* *is a* p*-dominant equilibrium of* $\mathbf{g}$ *with* $p < 1/\{1 + r(|I| - 1)\}$.

Then $[a^*]^I$ *is* r*-robust.*