Properties of the Product Topology on the Universal Type Space

Daisuke Oyama

Topics in Economic Theory

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Papers

- Lipman, B.L. (2003). "Finite Order Implications of Common Priors," Econometrica 71, 1255-1267.
- Weinstein, J. and M. Yildiz (2007). "A Structure Theorem for Rationalizability with Application to Robust Predictions of Refinements," Econometrica 75, 365-400.

Type Spaces

- Fix the set of states Θ (finite)
- Type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$:
  - T_i: set of i's types (countable)
  - $\pi_i \colon T_i \to \Delta(T_{-i} \times \Theta)$: i's belief
- Universal type space $(T^*, f)_{i=1}^I$, $T^* \subset \prod_{k=0}^{\infty} \Delta(X^k)$

  Endowed with the product topology:
  $\delta^n = (\delta^{n,k})_{k=0}^{\infty} \to \delta = (\delta^k)_{k=0}^{\infty}$ iff $\delta^{n,k} \to \delta^k$ for all k (weakly)
- Each $t_i \in T_i$ is embedded into T^* by:
  - $\hat{\pi}_i^1(t_i)(\theta) = \sum_{t_{-i} \in T_{-i}} \pi_i(t_i)(t_{-i}, \theta)$
  - $\hat{\pi}_i^k(t_i)((\delta_{-i}^{\ell})_{\ell=1}^{k-1}, \theta) = \sum_{t_{-i} \colon \hat{\pi}_{-i}^{\ell}(t_{-i}) = \delta_{-i}^{\ell},\, \ell = 1, \ldots, k-1} \pi_i(t_i)(t_{-i}, \theta)$
  - $\hat{\pi}_i^*(t_i) = (\hat{\pi}_i^k(t_i))_{k=1}^{\infty} \in T^*$

  Identify T_i with $\hat{\pi}_i^*(T_i) \subset T^*$

- $(T_i)_{i=1}^I$, $T_i \subset T^*$, is a belief-closed subspace if $f(t_i)(T_{-i} \times \Theta) = 1$ for all i and all $t_i \in T_i$.

  It is finite if each T_i is finite.

- $t_i \in T^*$ is a finite type if $t_i \in T_i$ for some finite belief-closed subspace $(T_i)_{i=1}^I$.

- $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$ has common support if $\pi_i(t_i)(t_{-i}, \theta) > 0 \iff \pi_j(t_j)(t_{-j}, \theta) > 0$ for all i, j.

- $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$ admits a common prior if there exists $\mu \in \Delta(T \times \Theta)$ such that $\mu(t_i) = \sum_{t_{-i},\theta} \mu((t_i, t_{-i}), \theta) > 0$ for all t_i and

$$\pi_i(t_i)(t_{-i}, \theta) = \frac{\mu((t_i, t_{-i}), \theta)}{\mu(t_i)}$$

  for all t_i, t_{-i}, and θ.

- $t_i \in T^*$ is a weakly consistent (common prior type) if it is from some type space that has common support (admits a common prior).

Denseness of Common Prior Types (Lipman)

- T_{f}: set of finite types
- $T_{\mathrm{f,wc}}$: set of finite and weakly consistent types
- $T_{\mathrm{f,cp}}$: set of finite and common prior types ($\subset T_{\mathrm{f,wc}}$)

Proposition 1

1. *T_{f} is dense in T^*. (Mertens and Zamir)*
2. *$T_{\mathrm{f,wc}}$ is dense in T_{f}.*
3. *$T_{\mathrm{f,cp}}$ is dense in $T_{\mathrm{f,wc}}$. (Lipman)*

Example

- $\Theta = \{\theta^1, \theta^2\}$
- $T_1 = \{t_1\}$, $T_2 = \{t_2\}$
- $\pi_1(t_1)(t_2, \cdot) = (2/3, 1/3)$, $\pi_2(t_2)(t_1, \cdot) = (1/3, 2/3)$
- Lipman's result:

  For each N, there exist a finite common prior type space (T_i', π_i') and $t_i' \in T_i'$ such that $\hat{\pi}_i^k(t_i') = \hat{\pi}_i^k(t_i)$ for all $k \leq N$.

- $N = 2$

  $T_1' = \{1, 2, 3\}$, $T_2' = \{1, 2, 3\}$

  Common prior:

  $\theta = \theta^1$

	1	2	3
1	$\frac{1}{8}$	$\frac{1}{8}$	0
2	0	0	$\frac{2}{8}$
3	0	0	0

  $\theta = \theta^2$

	1	2	3
1	$\frac{1}{8}$	0	0
2	$\frac{1}{8}$	0	0
3	0	$\frac{2}{8}$	0

  (Easier to see with a partition model)

- $\hat{\pi}_1^k(t_1' = 1) = \hat{\pi}_1^k(t_1)$ for all $k \leq 2$

Email Game

- The product topology does not care about the tail of a hierarchy of beliefs.
- It matters for strategic behavior.
- In the Email Game example: for all N,
  - $\hat{\pi}_1^k(t_1 = N) = t_1^{\theta^1,k}$ for all $k \leq N$,
  - $R_1(\hat{\pi}_1^*(t_1 = N)) = \{B\} \neq R_1(t_1^{\theta^1}) = \{A, B\}$.

Generic Uniqueness of Rationalizable Actions (Weinstein and Yildiz)

- A_i: finite set of actions for i
- $g_i \colon A \times \Theta \to \mathbb{R}$: payoff function for i
- $R_i^{\mathcal{T}}(t_i)$: ICR
- Richness Assumption:

  For each i and a_i, there exists $\theta^{a_i} \in \Theta$ such that $g_i(a_i, a_{-i}, \theta^{a_i}) > g_i(a_i', a_{-i}, \theta^{a_i})$ for all $a_i' \neq a_i$ and all a_{-i}.

Proposition 2

Under the Richness Assumption, for any $t \in \prod_{i=1}^{I} T^*$ *and any* $a \in R(t)$*, there exists a sequence of types* t^n *such that*

- $t^n \to t$ *and*
- $R(t^n) = \{a\}$.

Moreover, such types can be taken as common prior types. (Lipman)

Email Game

- $\Theta = \{\theta^1, \theta^A, \theta^B\}$
- θ^1:

	A_2	B_2
A_1	$4, 4$	$0, 3$
B_1	$3, 0$	$2, 2$

  θ^A: A is strictly dominant; θ^B: B is strictly dominant

- t^{θ^1}: common knowledge type of θ^1

  $R_i(t_i^{\theta^1}) = \{A_i, B_i\}$

- "Standard Email Game prior" P^ε:

  $t^n \to t^{\theta^1}$, $R_i(t_i^n) = \{B_i\}$

  $(P^\varepsilon(\theta^1) = 1 - \varepsilon)$

- For A_i:
  - $P'(\theta^A, t_1 = 0, t_2 = 0) = \frac{1+\varepsilon}{2}$
  - $P'(\theta^1, t_1 = 1, t_2 = 0) = \frac{1+\varepsilon}{2}\frac{1-\varepsilon}{2}$
  - $P'(\theta^1, t_1 = 1, t_2 = 1) = \frac{1+\varepsilon}{2}\left(\frac{1-\varepsilon}{2}\right)^2$
  - $P'(\theta^1, t_1 = 2, t_2 = 1) = \frac{1+\varepsilon}{2}\left(\frac{1-\varepsilon}{2}\right)^3$
  - $\cdots$

$\left(P'(\theta^1) = \frac{1-\varepsilon}{2}\right)$

- Alternatively,
  - $P''(\theta^A, t_1 = 0, t_2 = 0) = \varepsilon$
  - $P''(\theta^1, t_1 = 1, t_2 = 0) = \varepsilon \frac{1-\varepsilon}{2}$
  - $P''(\theta^1, t_1 = 1, t_2 = 1) = \varepsilon \left(\frac{1-\varepsilon}{2}\right)^2$
  - $P''(\theta^1, t_1 = 2, t_2 = 1) = \varepsilon \left(\frac{1-\varepsilon}{2}\right)^3$
  - $\dots$
  - $P''(\theta^1, t_1 = \infty, t_2 = \infty) = 1 - \frac{2}{1+\varepsilon}\varepsilon$

  $P''(\theta^1) = 1 - \varepsilon$,
  but the dominance-solvability on the whole subspace is lost.