Generalized Belief Operator and the Impact of Small Probability Events on Higher Order Beliefs

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Topics in Economic Theory

October 2, 2015
Sufficient Conditions for Robustness

- Higher order beliefs approach
  Kajii and Morris (1997)
  A p-dominant equilibrium with $\sum_i p_i < 1$ is robust.

- Potential approach
  Ui (2002)
  A potential maximizer is robust (to canonical elaborations).

- Potential + monotonicity/supermodularity
  Morris and Ui (2005)
  A monotone potential maximizer is robust if $g$ or $v$ is supermodular.

- Potential + monotonicity/supermodularity + iteration
  Oyama and Tercieux (2009)
  An iterated monotone potential maximizer is robust if $g$ or $v^k$ is supermodular.
This Paper

- Show that in generic binary supermodular games, $a^*$ is a robust equilibrium if and only if it is an MP-maximizer.

- Higher order beliefs approach

  Introduce a generalized belief operator: game $f \leftrightarrow \text{"f-belief"}$

  (Kajii and Morris: $p$-dominant equilibrium $\leftrightarrow p$-belief)

- If $f$ admits a monotone potential,
  then a version of the Critical Path Theorem holds.

- If $f$ does not admit a monotone potential,
  then a contagion result holds.
Binary Supermodular Games

- Set of players $I = \{1, \ldots, |I|\}$
  - $\mathcal{I} = 2^I$
  - $\mathcal{I}_{-i} = 2^I \setminus \{i\}$
- Action set $A_i = \{0, 1\}$
- Identify $S \in \mathcal{I}_{-i}$ with $i$’s opponents’ action profile such that $j$ plays action 1 if and only if $j \in S$.
- Payoff function $f_i : \mathcal{I}_{-i} \rightarrow \mathbb{R}$:
  - if $i$ plays 1 and the opponents play $S \in \mathcal{I}_{-i}$, $i$’s payoff is $f_i(S)$;
  - if $i$ plays 0, $i$’s payoff is 0 (regardless of the opponents’ play).
- Assumption: $f_i$ is weakly increasing, i.e.,
  $S \subset S' \Rightarrow f_i(S) \leq f_i(S')$. 
Type Spaces

- Type space $\mathcal{T} = ((T_i)_{i \in I}, P)$:
  - $T_i = \{0, 1, 2, \ldots\}$: set of $i$'s types
  - $P \in \Delta(T)$: common prior
    - Assume $P(t_i) = P(\{t_i\} \times T_{-i}) > 0$ for all $i$ and $t_i$.

- Let
  \[
  P(E_{-i}|t_i) = \frac{P(\{t_i\} \times E_{-i})}{P(t_i)}
  \]
  for $t_i \in T_i$ and $E_{-i} \subset T_{-i}$. 

For $i \in I$, $t_i \in T_i$, and $E_{-i} = (E_j)_{j \neq i}$ with $E_j \subseteq T_j$, define $P^{E_{-i}}(\cdot|t_i) = (P^{E_{-i}}(S|t_i))_{S \in \mathcal{I}_{-i}} \in \Delta(\mathcal{I}_{-i})$ by

$$P^{E_{-i}}(S|t_i) = P(\{t_{-i} \in T_{-i} \mid \{j \neq i \mid t_j \in E_j\} = S\}|t_i)$$

$$= P \left( \prod_{j \in S} E_j \times \prod_{j \notin S \cup \{i\}} (T_j \setminus E_j) \mid t_i \right)$$

for $S \in \mathcal{I}_{-i}$.

For $q_i \in \Delta(\mathcal{I}_{-i})$, define

$$\langle q_i, f_i \rangle = \sum_{S \in \mathcal{I}_{-i}} q_i(S) f_i(S).$$
\( f_i \)-Belief Operator

- \( i \)'s \( f_i \)-belief operator \( B^f_i : \prod_{j \in I} 2^{T_j} \rightarrow 2^{T_i} \) is defined by

\[
B^f_i(E) = \{ t_i \in T_i \mid t_i \in E_i \text{ and } \langle P^{E-i}(\cdot|t_i), f_i \rangle \geq 0 \}
\]

for \( E = (E_j)_{j \in I} \) with \( E_j \subseteq T_j \).

- Note:

\[
\langle P^{E-i}(\cdot|t_i), f_i \rangle = \sum_{S \in \mathcal{I}_{-i}} P^{E-i}(S|t_i)f_i(S)
\]

\[
= \sum_{S \in \mathcal{I}_{-i}} P(\{t_{-i} \in T_{-i} \mid \{j \neq i \mid t_j \in E_j\} = S\}|t_i)f_i(S)
\]

\[
= \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)f_i(\{j \neq i \mid t_j \in E_j\})
\]
Example 1 ($p$-Belief)

$$B_{fi}^p(E) = \{t_i \in T_i \mid t_i \in E_i \text{ and } \langle P_{E-i}(\cdot|t_i), f_i \rangle \geq 0\}$$

- Let $f_{pi}^p(S) = \begin{cases} 1 - p_i & \text{if } S = I \setminus \{i\}, \\ -p_i & \text{otherwise.} \end{cases}$

- Then,

$$\sum_{S \in \mathcal{I}_{-i}} P_{E-i}(S|t_i) f_{pi}^p(S) = P_{E-i}(I \setminus \{i\}|t_i) - p_i$$

$$= P(E_{-i}|t_i) - p_i,$$

so $t_i \in B_{fi}^p(E) \iff t_i \in B_{pi}^p(E)$.

- Thus, $p$-belief only captures the case where all agents belong to $E_j$ or not.
Proposition 1

1. \( B_{f_i}^i(E) \subset E_i. \)

2. If \( E_j \subset F_j \) for all \( j \), then \( B_{f_i}^i(E) \subset B_{f_i}^i(F) \).

3. If \( E_j^0 \supset E_j^1 \supset \cdots \) for all \( j \), then
\[
B_{f_i}^i((\bigcap_{k=0}^{\infty} E_j^k)_{j \in I}) = \bigcap_{k=0}^{\infty} B_{f_i}^i((E_j^k)_{j \in I}).
\]

For each \( S \in \mathcal{I}_{-i} \),
\[
P(\{ t_{-i} \in T_{-i} \mid \{ j \neq i \mid t_j \in \bigcap_{k=0}^{\infty} E_j^k \} \subset S \}| t_i )
\]
\[
= \lim_{k \to \infty} P(\{ t_{-i} \in T_{-i} \mid \{ j \neq i \mid t_j \in E_j^k \} \subset S \}| t_i )
\]
Common $\mathfrak{f}$-Belief

- Define

\[ B^f_i,0(E) = E_i, \]
\[ B^f_i,k+1(E) = B^f_i((B^f_j,k(E))_{j \in I}), \]

and

\[ CB^f_i(E) = \bigcap_{k=1}^{\infty} B^f_i,k(E). \]

- $E$ is common $\mathfrak{f}$-belief at $t \in T$ if $t_i \in CB^f_i(E)$ for each $i \in I$. 

f-Evidence

- **F** is f-evident if $F_i \subset B^f_i(F)$ for each $i \in I$.

**Proposition 2**

$\left(CB^f_i(E)\right)_{i \in I}$ is the largest f-evident event in $E$. 
Connection to Games 1

- Type space $\mathcal{T} = ((T_i)_{i=1}^I, P)$
- Set of players $I = \{1, \ldots, |I|\}$
- Action set $A_i = \{0, 1\}$

$\mathcal{F}$ is identified with the (pure) strategy profile $\sigma$ such that $\sigma_i(t_i) = 1$ if and only if $t_i \in F_i$.

- Fix $E$.

- Incomplete information game $u^f_i$:
  
  If $t_i \in E_i$: for all $t_{-i}$ with $P(t_{-i}|t_i) > 0$,
  
  $u^f_{i}(1, a_{-i}, t_i, t_{-i}) = f_i(\{j \neq i \mid a_j = 1\})$,
  
  $u^f_{i}(0, a_{-i}, t_i, t_{-i}) = 0$.

  If $t_i \notin E_i$: 0 is a dominant action.
$B^f_i(\langle E_i, F_{-i} \rangle)$ is the (largest) best response to $F_{-i}$ (play 1 if indifferent).

1 $\in R_i(t_i)$ if and only if $t_i \in CB^f_i(E)$.

$F$ is an equilibrium if and only if $F_i \subset E_i$ and $F$ is $f$-evident.

$(CB^f_i(E))_{i \in I}$ is the largest equilibrium.
Connection to Games 2

- Type space $\mathcal{T} = ((T_i)_{i=1}^I, P)$
- Set of players $I = \{1, \ldots, |I|\}$
- Action set $A_i = \{0, 1\}$
- Complete information game $f$ (supermodular)
- Incomplete information game $u$:

Define $T_i^{f_i} \subset T_i$ by the following:

\[ t_i \in T_i^{f_i} \text{ if and only if for all } t_{-i} \text{ with } P(t_{-i}|t_i) > 0 \text{ and all } a_{-i} \in A_{-i}, \]

\[ u_i^{f_i}(1, a_{-i}, t_i, t_{-i}) = f_i(\{j \neq i \mid a_j = 1\}), \]

\[ u_i^{f_i}(0, a_{-i}, t_i, t_{-i}) = 0. \]

Write $T^f = (T_i^{f_i})_{i \in I}$. 
Lemma 3

\( \mathbf{u} \) has an equilibrium \( \sigma \) such that for all \( i \in I \), \( \sigma_i(t_i)(1) = 1 \) for all \( t_i \in CB_i^f(T^f) \).
Proof

- For each $i$, let $F_i = B_{i}^{f_i}((C B_{j}^{f}(T^{f}))_{j \in I}) \subseteq T_{i}^{f_i}$.
  Then $C B_{i}^{f}(T^{f}) \subseteq F_i$ (in fact $C B_{i}^{f}(T^{f}) = F_i$).

- Consider the modified game $u'$ where each player $i$ must play 1 if $t_i \in F_i$.
  Let $\sigma^*$ be any equilibrium of $u'$.
  We want to show that $\sigma^*$ is also an equilibrium of $u$.

- For $t_i \in T_i \setminus F_i$, 1 is a best response to $\sigma^*_{-i}$ by construction.
Suppose \( t_i \in F_i \).

Then by definition, \( \langle P^{(CB_j^f(T^f))_{j \neq i}}(\cdot | t_i), f_i \rangle \geq 0 \).

The expected payoff from playing action 1 is:

\[
U_i(1, \sigma_{-i}^* | t_i) = \sum_{t_{-i} \in T_{-i}} P(t_{-i} | t_i) \sum_{a_{-i} \in A_{-i}} u_i(1, a_{-i}, t_i, t_{-i}) \prod_{j \neq i} \sigma_j^*(a_j | t_j)
\]

\[
= \sum_{t_{-i} \in T_{-i}} P(t_{-i} | t_i) \times \langle f_i(\{ j \neq i \mid \sigma_j^*(1 | t_j) = 1 \}) \rangle 
\]

Thus playing 1 is a best response to \( \sigma_{-i}^* \) for \( t_i \in F_i \).
Definition 1

\( f = (f_i)_{i \in I} \) admits a potential \( v: \mathcal{I} \to \mathbb{R} \) if

\[
    f_i(S) = v(S \cup \{i\}) - v(S)
\]

for any \( i \in I \) and \( S \in \mathcal{I}_{-i} \).
Characterizations

\[ f = (f_i)_{i \in I} \text{ admits a potential if and only if} \]

\[ (* ) \quad f_i(S) + f_j(S \cup \{i\}) = f_j(S) + f_i(S \cup \{j\}) \]

for any \( i \neq j \) and \( S \subset I \setminus \{i, j\} \).

\[ \text{If} \ (* ) \ \text{holds, then the potential is determined uniquely up to constants:} \]

\[ v(S) = v(\emptyset) + \sum_{\ell=1}^{k} f_{i_\ell}(\{i_1, \ldots, i_{\ell-1}\}) \]

for \( S = \{i_1, \ldots, i_k\} \in \mathcal{I} \),

where the summation is independent of the order of players in \( S \).
Example 1 ($p$-Belief)

Let \( f_{pi}(S) = \begin{cases} 1 - p_i & \text{if } S = I \setminus \{i\}, \\ -p_i & \text{otherwise.} \end{cases} \)

For \( S \subset I \setminus \{i, j\} \), we have

\[
 f_i(S) + f_j(S \cup \{i\}) = f_j(S) + f_i(S \cup \{j\}) \\
= \begin{cases} 1 - p_i - p_j & \text{if } S = I \setminus \{i, j\}, \\ -p_i - p_j & \text{if } S \not\subset I \setminus \{i, j\}. \end{cases}
\]

A potential is given by

\[
v(S) = \begin{cases} 1 - \sum_{i \in I} p_i & \text{if } S = I, \\ -\sum_{i \in S} p_i & \text{otherwise.} \end{cases}
\]

\[ \Rightarrow f_i(S) = v(S \cup \{i\}) - v(S) \forall i \in I, \forall S \in \mathcal{I}_{-i}. \]

Note that \( v \) is uniquely maximized at \( I \) iff \( \sum_{i \in I} p_i < 1 \). 

\[ \cdots \] the condition for the Critical Path Result.

We will argue that this is not a coincidence.
Example 2 (Anonymity and Symmetry)

Let $f_i^{m_i}(S) = g(|S|) - m_i$,

where $g: \{0, 1, 2, \ldots, |I| - 1\} \rightarrow \mathbb{R}$ and $m_i \in \mathbb{R}$.

$f$ has a potential:

$$v(S) = \sum_{n=0}^{|S| - 1} g(n) - \sum_{i \in S} m_i.$$

\[ \ldots \text{“integral” of } g \ (\text{+ constants}). \]
Monotone Potential

Definition 2

\( f = (f_i)_{i \in I} \) admits a monotone potential \( v : \mathcal{I} \to \mathbb{R} \) if there exists \( \lambda = (\lambda_i)_{i \in I} \) with \( \lambda_i > 0 \) such that

\[
    f_i(S) \geq \lambda_i (v(S \cup \{i\}) - v(S))
\]

for any \( i \in I \) and \( S \in \mathcal{I}_{-i} \).

- If \( \langle q_i, f_i \rangle < 0 \), then \( \langle q_i, v(\cdot \cup \{i\}) - v(\cdot) \rangle < 0 \).

- Theorems will be about implications of existence/non-existence of a monotone potential that is uniquely maximized at \( I \).
Example 3 (Unanimity)

Let \( f^a,b_i(S) = \begin{cases} 
  a_i & \text{if } S = I \setminus \{i\}, \\
  -b_i & \text{if } S = \emptyset, \\
  0 & \text{otherwise.} 
\end{cases} \quad (a_i, b_i > 0) \)

\( f \) has a potential uniquely maximized at \( I \)
\[ \iff \quad a_1 = \cdots = a_{|I|} > b_1 = \cdots = b_{|I|}. \]

\( f \) has a monotone potential uniquely maximized at \( I \)
\[ \iff \quad a_ia_j > b_ib_j \text{ for all } i \neq j. \]
Critical Path Theorem for Generalized Beliefs

Theorem 1

If \( f = (f_i)_{i \in I} \) admits a monotone potential \( v \) uniquely maximized at \( I \), then for any type space \( (T, P) \) and for any \( E = (E_i)_{i \in I} \), \( E_i \subset T_i \),

\[
P \left( CB^f(E) \right) \geq 1 - \xi(v)(1 - P(E)),
\]

where

\[
\xi(v) = 1 + \frac{M}{v(I) - v'},
\]

\[
v' = \max_{S \subset I} v(S),
\]

\[
M = \max_{S \subseteq S' \subsetneq I} (v(S) - v(S')).
\]
Sketch of Proof

The goal is to show that $P(E), P(B^f(E)), P(B^f(B^f(E))), \ldots$ do not decrease too fast.

For a technical reason (that will be clear later), we iterate belief operators sequentially:

$$E_i^1 = E_i,$$
$$E_i^{n+1} = \begin{cases} 
B_i^f(E^n) & \text{if } i \equiv n \pmod{|I|}, \\
E_i^n & \text{if } i \not\equiv n \pmod{|I|}.
\end{cases}$$

We have

$$CB_i^f(E) = E_i^1 \cap E_i^{i+1} \cap E_i^{i+|I|+1} \cap E_i^{i+2|I|+1} \cap \ldots$$
We partition $T_i$ into $\{D^m_i\}_{n=0,1,\ldots,\infty}$, where

\[
D^0_i = T_i \setminus E^1_i \\
D^n_i = E^n_i \setminus E^{n+1}_i = \{t_i \in T_i \mid t_i \text{ is eliminated at step } n\}, \\
D^\infty = CB^f_i(E).
\]

For $\mathbf{n} = (n_i)_{i \in I} \in (\mathbb{N} \cup \{\infty\})^I$, we let $\pi(\mathbf{n}) = P(\prod_i D^{n_i}_i)$. 
Illustration for $I = \{1, 2\}$

<table>
<thead>
<tr>
<th>$D_1^0$</th>
<th>$D_2^0$</th>
<th>$D_2^2$</th>
<th>$D_2^4$</th>
<th>$D_2^6$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(0, 0)$</td>
<td>$\pi(0, 2)$</td>
<td>$\pi(0, 4)$</td>
<td>$\pi(0, 6)$</td>
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<tr>
<td>$\pi(3, 0)$</td>
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<td>$\pi(3, 4)$</td>
<td>$\pi(3, 6)$</td>
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</tr>
<tr>
<td>$\pi(5, 0)$</td>
<td>$\pi(5, 2)$</td>
<td>$\pi(5, 4)$</td>
<td>$\pi(5, 6)$</td>
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</tr>
</tbody>
</table>

where

$E_1 = D_1^1 \cup D_1^3 \cup D_1^5 \cup \ldots$,

$E_2 = D_2^2 \cup D_2^4 \cup D_2^6 \cup \ldots$. 
We have

\[
\pi(1, 0)f_1(\emptyset) + (\pi(1, 2) + \pi(1, 4) + \cdots) f_1(\{2\}) \leq 0
\]

\[
\Rightarrow \pi(1, 0)(v(\{1\}) - v(\emptyset))
\]

\[
+ (\pi(1, 2) + \pi(1, 4) + \cdots)(v(\{1, 2\}) - v(\{2\})) \leq 0.
\]

Similarly,

\[
(\pi(0, 2) + \pi(1, 2)) f_2(\emptyset) + (\pi(3, 2) + \pi(5, 2) + \cdots) f_2(\{1\}) \leq 0
\]

\[
\Rightarrow (\pi(0, 2) + \pi(1, 2))(v(\{2\}) - v(\emptyset))
\]

\[
+ (\pi(3, 2) + \pi(5, 2) + \cdots)(v(\{1, 2\}) - v(\{1\})) \leq 0.
\]
Lemma 4

If $F_i \subset T_i \setminus B_i^f (E)$, then
\[
\sum_{S \in I \setminus i} P \left( F_i \times \prod_{j \in S} E_j \times \prod_{j \notin S \cup \{i\}} (T_j \setminus E_j) \right) f_i(S) \leq 0.
\]

Let $S(k, n) = \{ i \in I \mid n_i > k \}$.

Claim 1

For all $i \in I$ and $1 \leq k < \infty$,
\[
\sum_{n : n_i = k} \pi(n)(v(S(n_i, n) \cup \{i\}) - v(S(n_i, n))) \leq 0.
\]
Proof

If \( k \equiv i \pmod{|I|} \), then

\[
\sum_{n: n_i = k} \pi(n) f_i(S(n_i, n))
\]

\[
= \sum_{S \in \mathcal{I}_{-i}} \sum_{n: n_i = k, S(k, n) = S} P \left( D_i^k \times \prod_{j \neq i} D_j^{n_j} \right) f_i(S)
\]

\[
= \sum_{S \in \mathcal{I}_{-i}} P \left( D_i^k \times \prod_{j \in S} E_j^k \times \prod_{j \notin S \cup \{i\}} (T_j \setminus E_j^k) \right) f_i(S) \leq 0.
\]

The claim then follows from the definition of monotone potential \( \nu \).
Let \( x_i(0) = 0 \) and for \( 1 \leq k < \infty \),

\[
\varepsilon = 1 - P(E) = \sum_{n: \min(n)=0} \pi(n),
\]

\[
x_i(k) = \sum_{n: 1 \leq n_i = \min(n) \leq k} \pi(n),
\]

\[
x(k) = \sum_{i \in I} x_i(k) = \sum_{n: 1 \leq \min(n) \leq k} \pi(n).
\]

For \( 1 \leq n < \infty \),

\[
1 - P \left( \prod_{i \in I} \tilde{B}_{i}^{f,n}(E) \right) \leq 1 - P \left( \prod_{i \in I} E_{i}^{n|I|+1} \right)
\]

\[
= \sum_{n: \min(n) \leq n|I|} \pi(n) = \varepsilon + x(n|I|).
\]
Claim 2

For $1 \leq k < \infty$,

$$x(k) \leq \frac{M}{v(I) - v'} \varepsilon.$$

The key identity: for any $n$,

$$\sum_{i: \ell \leq n_i \leq k} v(S(n_i, n) \cup \{i\}) - v(S(n_i, n)) = v(S(\ell - 1, n)) - v(S(k, n)).$$
For $1 \leq k < \infty$, we have

\[
0 \geq \sum_{i \in I} \sum_{\ell=1}^{k} \sum_{\mathbf{n}: n_i = \ell} \sum_{\mathbf{n}: \min(n) \leq k} \pi(\mathbf{n})(v(S(n_i, \mathbf{n}) \cup \{i\}) - v(S(n_i, \mathbf{n})))
\]

\[
= \sum_{\mathbf{n}: \min(n) \leq k} \sum_{i: 1 \leq n_i \leq k} \pi(\mathbf{n})(v(S(n_i, \mathbf{n}) \cup \{i\}) - v(S(n_i, \mathbf{n})))
\]

\[
= \sum_{\mathbf{n}: \min(n) \leq k} \pi(\mathbf{n})(v(S(0, \mathbf{n})) - v(S(k, \mathbf{n})))
\]

\[
= \sum_{\mathbf{n}: \min(n) \leq k} \pi(\mathbf{n})(v(S(0, \mathbf{n}) \cup \{\} ) - v(S(k, \mathbf{n})))
= I \text{ if } \min(n) \geq 1
\]

\[
\geq x(k)(v(I) - v') - M\varepsilon.
\]
Recall the definition of monotone potential.

**Definition 3 (Coefficients on Left)**

\( f = (f_i)_{i \in I} \) admits a monotone potential \( \nu : \mathcal{I} \to \mathbb{R} \) if there exists \( \lambda' = (\lambda'_i)_{i \in I} \) with \( \lambda'_i > 0 \) such that

\[
\lambda'_i f_i(S) \geq \nu(S \cup \{i\}) - \nu(S)
\]

for any \( i \in I \) and \( S \in \mathcal{I}_{-i} \).

We characterize when \( f \) admits such a monotone potential. In what follows, we “get rid of” \( \nu \) first, and then \( \lambda' \).
If $v$ is uniquely maximized at $I$, then

\[
\lambda'_i f_i (I \setminus \{i\}) \geq v(I) - v(I \setminus \{i\}) > 0,
\]
\[
\lambda'_i f_i (I \setminus \{i\}) + \lambda'_j f_j (I \setminus \{i, j\})
\]
\[
\geq (v(I) - v(I \setminus \{i\})) + (v(I \setminus \{i\}) - v(I \setminus \{i, j\}))
\]
\[
= v(I) - v(I \setminus \{i, j\}) > 0,
\]

\[
\gamma = (i_1, \ldots, i_k): \text{a finite sequence of distinct players in } I.
\]

$\Gamma$: the set of all such sequences.

**Lemma 5 (Fixed $\lambda'$)**

$f = (f_i)_{i \in I}$ admits a monotone potential with $\lambda' = (\lambda'_i)_{i \in I}$ that is uniquely maximized at $I$ if and only if

\[
\sum_{\ell=1}^{k} \lambda'_{i_\ell} f_{i_\ell} (I \setminus \{i_1, \ldots, i_\ell\}) > 0
\]

for every $\gamma \in \Gamma$. 
For $i \in I$, let $\Gamma_i \subset \Gamma$ be the set of sequences that contain $i$. For $\gamma \in \Gamma_i$, let $S(i, \gamma) \in \mathcal{I}_-i$ be the set of player $i$’s opponents who are not listed in $\gamma$ earlier than $i$.

We can apply a version of Farkas’ lemma to “get rid of” $\lambda'$.

**Lemma 6**

$f = (f_i)_{i \in I}$ admits a monotone potential that is uniquely maximized at $I$ if and only if there is no $\mu \in \Delta(\Gamma)$ such that

$$\sum_{S \in \mathcal{I}_-i} \mu(\{\gamma \in \Gamma_i : S(i, \gamma) = S\}) f_i(S) \leq 0$$

for any $i \in I$. 
Contagion Result

Theorem 2

Suppose that \( f = (f_i)_{i \in I} \) is generic and does not admit a monotone potential that is strictly maximized at \( I \). Then for any \( \varepsilon \in (0, 1] \), there exist a type space \( ((T_i)_{i \in I}, P) \) and a profile \( E = (E_i)_{i \in I} \) with \( E = \prod_{i \in I} E_i \) such that \( P(E) = 1 - \varepsilon \) and

\[
P \left( CB^f(E) \right) = 0.
\]
Proof.

By genericity, for sufficiently small $\varepsilon > 0$, we have

$$\sum_{S \in \mathcal{I}_-} (1 - \varepsilon)^{|S|} \mu(\{\gamma \in \Gamma_i : S(i, \gamma) = S\}) f_i(S) < 0$$

for any $i \in I$ such that $\mu(\Gamma_i) > 0$.

Given such small $\varepsilon$, we consider the following type space.

- $T_i = \{1, 2, 3, \ldots\} \cup \{\infty\}$
- $t_i = \theta + \eta_i$, where
  - $\theta$ follows a geometric distribution, $\theta = m$ with probability $\varepsilon(1 - \varepsilon)^m$ for $m = 0, 1, 2, \ldots$.
  - Independently of $\theta$, $\gamma \in \Gamma$ is drawn according to $\mu$, and $\eta_i = \ell$ if $i$ is listed at the $\ell$-th place of $\gamma$ ($\eta_i = \infty$ if $i$ is not listed in $\gamma$).

Let $E_i = \{|I|, |I| + 1, \ldots, \infty\}$ for each $i \in I$.

Then each time $B_i^{f_i}$ applies, types $|I|, |I| + 1, \ldots$ get eliminated.
Robustness in Binary Supermodular Games

The above results imply:
in generic binary supermodular games
1 is a robust equilibrium if and only if 1 is an MP-maximizer.

“If” part (MP-max ⇒ robust):
Follows from Critical Path Result.

Already known by Ui (2001) and Morris and Ui (2005) via the “potential maximizing strategy” proof.

We give an alternative, “higher order beliefs” proof.

“Only if” part (not MP-max ⇒ not robust):
Implication of non-existence of a potential.

If 1 is not an MP-maximizer, then one can construct an ε-elaboration such that \( P(\text{type profiles playing 1}) = 0 \).

New construction in the literature.
Proof: “If” part

- Suppose that the game $f$ has a monotone potential $v$ maximized at $I$.

- Fix any $\delta > 0$, let $\varepsilon > 0$ small enough, and consider any $\varepsilon$-elaboration of $f$.

- Let $T_f$ be the set of type profiles whose payoffs are given by $f$.

By definition, $P(T_f) = 1 - \varepsilon$.

- Recall our Critical Path Theorem:

$$P(\underbrace{CB_f(T_f)}_{\exists \text{BNE } \sigma^* \text{ playing 1 on } CB_f(T_f)}) \geq 1 - \xi(v)(1 - P(T_f)) = \varepsilon$$

- Hence,

$$P(\sigma^* \text{ plays 1}) \geq P(CB_f(T_f)) \geq 1 - \xi(v) \times \varepsilon \geq 1 - \delta$$

if $\varepsilon$ is small enough that $\varepsilon \leq \delta/\xi(v)$. 
Proof: “Only if” part

- Suppose that the game $f$ does not have a monotone potential maximized at $I$.
- For any $\varepsilon > 0$, we can construct an $\varepsilon$-elaboration that has a unique rationalizable strategy $\sigma^*$, which satisfies

$$P(t|\sigma(t) = 1) = 0.$$  

This implies that $1$ is not robust.
Example: Unanimity Games

Consider the game $f$ given by

$$f_i(S) = \begin{cases} 
  a_i & \text{if } S = I, \\
  -b_i & \text{if } S = \emptyset, \\
  0 & \text{otherwise.}
\end{cases} \quad (a_i, b_i > 0)$$

1 is an MP-maximizer

$$\iff a_i a_j > b_i b_j \text{ for all } i \neq j.$$ 

Proposition 7

1 is robust to incomplete information

$$\iff a_i a_j > b_i b_j \text{ for all } i \neq j.$$