Generalized Belief Operator and the Impact of Small Probability Events on Higher Order Beliefs

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Paper

- Oyama, D. and S. Takahashi (2015). "Generalized Belief Operator and the Impact of Small Probability Events on Higher Order Beliefs."

Sufficient Conditions for Robustness

- Higher order beliefs approach

  Kajii and Morris (1997)
  A $\mathbf{p}$-dominant equilibrium with $\sum_i p_i < 1$ is robust.

- Potential approach

  Ui (2002)
  A potential maximizer is robust (to canonical elaborations).

- Potential $+$ monotonicity/supermodularity

  Morris and Ui (2005)
  A monotone potential maximizer is robust if $\mathbf{g}$ or v is supermodular.

- Potential $+$ monotonicity/supermodularity $+$ iteration

  Oyama and Tercieux (2009)
  An iterated monotone potential maximizer is robust if $\mathbf{g}$ or v^k is supermodular.

This Page

- Show that in generic binary supermodular games,
  a^* is a robust equilibrium if and only if it is an MP-maximizer.

- Higher order beliefs approach

  Introduce a generalized belief operator: game $\mathbf{f} \leftrightarrow$ "$\mathbf{f}$-belief"

  (Kajii and Morris: $\mathbf{p}$-dominant equilibrium $\leftrightarrow$ $\mathbf{p}$-belief)

- If $\mathbf{f}$ admits a monotone potential,
  then a version of the Critical Path Theorem holds.

- If $\mathbf{f}$ does not admit a monotone potential,
  then a contagion result holds.

Binary Supermodular Games

- Set of players $I = \{1, \ldots, |I|\}$
  - $\mathcal{I} = 2^I$
  - $\mathcal{I}_{-i} = 2^{I \setminus \{i\}}$
- Action set $A_i = \{0, 1\}$
- Identify $S \in \mathcal{I}_{-i}$ with i's opponents' action profile such that j plays action 1 if and only if $j \in S$.
- *Payoff function* $f_i \colon \mathcal{I}_{-i} \to \mathbb{R}$:
  - if i plays 1 and the opponents play $S \in \mathcal{I}_{-i}$, i's payoff is $f_i(S)$;
  - if i plays 0, i's payoff is 0 (regardless of the opponents' play).
- Assumption: f_i is weakly increasing, i.e.,
  $S \subset S' \Rightarrow f_i(S) \leq f_i(S')$.

Type Spaces

- Type space $\mathcal{T} = ((T_i)_{i \in I}, P)$:
  - $T_i = \{0, 1, 2, \ldots\}$: set of i's types
  - $P \in \Delta(T)$: common prior

    Assume $P(t_i) = P(\{t_i\} \times T_{-i}) > 0$ for all i and t_i.
- Let

$$P(E_{-i}|t_i) = \frac{P(\{t_i\} \times E_{-i})}{P(t_i)}$$

for $t_i \in T_i$ and $E_{-i} \subset T_{-i}$.

- For $i \in I$, $t_i \in T_i$, and $\mathbf{E}_{-i} = (E_j)_{j \neq i}$ with $E_j \subseteq T_j$, define $P^{\mathbf{E}_{-i}}(\cdot|t_i) = (P^{\mathbf{E}_{-i}}(S|t_i))_{S \in \mathcal{I}_{-i}} \in \Delta(\mathcal{I}_{-i})$ by

$$
\begin{aligned}
P^{\mathbf{E}_{-i}}(S|t_i) &= P(\{t_{-i} \in T_{-i} \mid \{j \neq i \mid t_j \in E_j\} = S\}|t_i) \\
&= P\left(\prod_{j \in S} E_j \times \prod_{j \notin S \cup \{i\}} (T_j \setminus E_j) \,\middle|\, t_i\right)
\end{aligned}
$$

for $S \in \mathcal{I}_{-i}$.

- For $q_i \in \Delta(\mathcal{I}_{-i})$, define

$$
\langle q_i, f_i \rangle = \sum_{S \in \mathcal{I}_{-i}} q_i(S) f_i(S).
$$

f_i-Belief Operator

- i's f_i-*belief operator* $B_i^{f_i} \colon \prod_{j \in I} 2^{T_j} \to 2^{T_i}$ is defined by

$$B_i^{f_i}(\mathbf{E}) = \{t_i \in T_i \mid t_i \in E_i \text{ and } \langle P^{\mathbf{E}_{-i}}(\cdot|t_i), f_i \rangle \geq 0\}$$

for $\mathbf{E} = (E_j)_{j \in I}$ with $E_j \subseteq T_j$.

- Note:

$$\langle P^{\mathbf{E}_{-i}}(\cdot|t_i), f_i \rangle = \sum_{S \in \mathcal{I}_{-i}} P^{\mathbf{E}_{-i}}(S|t_i) f_i(S)$$

$$= \sum_{S \in \mathcal{I}_{-i}} P(\{t_{-i} \in T_{-i} \mid \{j \neq i \mid t_j \in E_j\} = S\}|t_i) f_i(S)$$

$$= \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) f_i(\{j \neq i \mid t_j \in E_j\})$$

Example 1 (p-Belief)

$$B_i^{f_i}(\mathbf{E}) = \{t_i \in T_i \mid t_i \in E_i \text{ and } \langle P^{\mathbf{E}_{-i}}(\cdot|t_i), f_i\rangle \geq 0\}$$

- Let $f_i^{p_i}(S) = \begin{cases} 1 - p_i & \text{if } S = I \setminus \{i\}, \\ -p_i & \text{otherwise.} \end{cases}$
- Then,

$$\sum_{S \in \mathcal{I}_{-i}} P^{\mathbf{E}_{-i}}(S|t_i) f_i^{p_i}(S) = P^{\mathbf{E}_{-i}}(I \setminus \{i\}|t_i) - p_i$$

$$= P(E_{-i}|t_i) - p_i,$$

so $t_i \in B_i^{f_i^{p_i}}(\mathbf{E}) \iff t_i \in B_i^{p_i}(E)$.

- Thus, p-belief only captures the case where all agents belong to E_j or not.

Proposition 1

1. $B_i^{f_i}(\mathbf{E}) \subset E_i$.
2. *If* $E_j \subset F_j$ *for all* j, *then* $B_i^{f_i}(\mathbf{E}) \subset B_i^{f_i}(\mathbf{F})$.
3. *If* $E_j^0 \supset E_j^1 \supset \cdots$ *for all* j, *then* $B_i^{f_i}((\bigcap_{k=0}^{\infty} E_j^k)_{j \in I}) = \bigcap_{k=0}^{\infty} B_i^{f_i}((E_j^k)_{j \in I})$.

- For each $S \in \mathcal{I}_{-i}$,

$$P(\{t_{-i} \in T_{-i} \mid \{j \neq i \mid t_j \in \bigcap_{k=0}^{\infty} E_j^k\} \subset S\} | t_i) = \lim_{k \to \infty} P(\{t_{-i} \in T_{-i} \mid \{j \neq i \mid t_j \in E_j^k\} \subset S\} | t_i)$$

Common $\mathbf{f}$-Belief

- Define

$$B_i^{\mathbf{f},0}(\mathbf{E}) = E_i,$$
$$B_i^{\mathbf{f},k+1}(\mathbf{E}) = B_i^{\mathbf{f}}((B_j^{\mathbf{f},k}(\mathbf{E}))_{j\in I}),$$

  and

$$CB_i^{\mathbf{f}}(\mathbf{E}) = \bigcap_{k=1}^{\infty} B_i^{\mathbf{f},k}(\mathbf{E}).$$

- $\mathbf{E}$ is *common* $\mathbf{f}$*-belief* at $t \in T$ if $t_i \in CB_i^{\mathbf{f}}(\mathbf{E})$ for each $i \in I$.

$\mathbf{f}$-Evidence

- $\mathbf{F}$ is $\mathbf{f}$-*evident* if $F_i \subset B_i^{\mathbf{f}}(\mathbf{F})$ for each $i \in I$.

Proposition 2

$(CB_i^{\mathbf{f}}(\mathbf{E}))_{i \in I}$ *is the largest* $\mathbf{f}$*-evident event in* $\mathbf{E}$*.*

Connection to Games 1

- Type space $\mathcal{T} = ((T_i)_{i=1}^{I}, P)$
- Set of players $I = \{1, \ldots, |I|\}$
- Action set $A_i = \{0, 1\}$
- $\mathbf{F}$ is identified with the (pure) strategy profile σ such that $\sigma_i(t_i) = 1$ if and only if $t_i \in F_i$.
- Fix $\mathbf{E}$.
- Incomplete information game $\mathbf{u}^{\mathbf{f}}$:

  If $t_i \in E_i$: for all t_{-i} with $P(t_{-i}|t_i) > 0$,

$$u_i^{f_i}(1, a_{-i}, t_i, t_{-i}) = f_i(\{j \neq i \mid a_j = 1\}),$$
$$u_i^{f_i}(0, a_{-i}, t_i, t_{-i}) = 0.$$

  If $t_i \notin E_i$: 0 is a dominant action.

- $B_i^{f_i}((E_i, \mathbf{F}_{-i}))$ is the (largest) best response to $\mathbf{F}_{-i}$ (play 1 if indifferent).
- $1 \in R_i(t_i)$ if and only if $t_i \in CB_i^{\mathbf{f}}(\mathbf{E})$.
- $\mathbf{F}$ is an equilibrium if and only if $F_i \subset E_i$ and $\mathbf{F}$ is $\mathbf{f}$-evident.
- $(CB_i^{\mathbf{f}}(\mathbf{E}))_{i \in I}$ is the largest equilibrium.

Connection to Games 2

- Type space $\mathcal{T} = ((T_i)_{i=1}^I, P)$
- Set of players $I = \{1, \ldots, |I|\}$
- Action set $A_i = \{0, 1\}$
- Complete information game $\mathbf{f}$ (supermodular)
- Incomplete information game $\mathbf{u}$:
- Define $T_i^{f_i} \subset T_i$ by the following:

  $t_i \in T_i^{f_i}$ if and only if for all t_{-i} with $P(t_{-i}|t_i) > 0$ and all $a_{-i} \in A_{-i}$,

$$u_i^{f_i}(1, a_{-i}, t_i, t_{-i}) = f_i(\{j \neq i \mid a_j = 1\}),$$
$$u_i^{f_i}(0, a_{-i}, t_i, t_{-i}) = 0.$$

  Write $\mathbf{T}^{\mathbf{f}} = (T_i^{f_i})_{i \in I}$.

Lemma 3

$\mathbf{u}$ *has an equilibrium* σ *such that for all* $i \in I$, $\sigma_i(t_i)(1) = 1$ *for all* $t_i \in CB_i^{\mathbf{f}}(\mathbf{T}^{\mathbf{f}})$.

Proof

- For each i, let $F_i = B_i^{f_i}((CB_j^{\mathbf{f}}(\mathbf{T}^{\mathbf{f}}))_{j \in I})$ ($\subset T_i^{f_i}$).

  Then $CB_i^{\mathbf{f}}(\mathbf{T}^{\mathbf{f}}) \subset F_i$ (in fact $CB_i^{\mathbf{f}}(\mathbf{T}^{\mathbf{f}}) = F_i$).
- Consider the modified game $\mathbf{u}'$ where each player i must play 1 if $t_i \in F_i$.

  Let σ^* be any equilibrium of $\mathbf{u}'$.

  We want to show that σ^* is also an equilibrium of $\mathbf{u}$.
- For $t_i \in T_i \setminus F_i$,
  1 is a best response to σ^*_{-i} by construction.

- Suppose $t_i \in F_i$.

  Then by definition, $\langle P^{(CB_j^{\mathbf{f}}(\mathbf{T^f}))_{j\neq i}}(\cdot|t_i), f_i\rangle \geq 0$.

- The expected payoff from playing action 1 is:

$$
\begin{aligned}
&U_i(1, \sigma^*_{-i}|t_i) \\
&\quad = \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i) \sum_{a_{-i}\in A_{-i}} u_i(1, a_{-i}, t_i, t_{-i}) \prod_{j\neq i} \sigma_j^*(a_j|t_j) \\
&\quad = \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i) f_i(\{j \neq i \mid \sigma_j^*(1|t_j) = 1\}) \\
&\quad \leq \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i) f_i(\{j \neq i \mid t_j \in CB_j^{\mathbf{f}}(\mathbf{T^f})\}) \\
&\quad = \langle P^{(CB_j^{\mathbf{f}}(\mathbf{T^f}))_{j\neq i}}(\cdot|t_i), f_i\rangle \geq 0.
\end{aligned}
$$

  Thus playing 1 is a best response to σ^*_{-i} for $t_i \in F_i$.

Potential

Definition 1

$\mathbf{f} = (f_i)_{i \in I}$ admits a potential $v \colon \mathcal{I} \to \mathbb{R}$ if

$$f_i(S) = v(S \cup \{i\}) - v(S)$$

for any $i \in I$ and $S \in \mathcal{I}_{-i}$.

Characterizations

- $\mathbf{f} = (f_i)_{i \in I}$ admits a potential if and only if

  $$(*) \quad f_i(S) + f_j(S \cup \{i\}) = f_j(S) + f_i(S \cup \{j\})$$

  for any $i \neq j$ and $S \subset I \setminus \{i, j\}$.

- If $(*)$ holds, then the potential is determined uniquely up to constants:

  $$v(S) = v(\emptyset) + \sum_{\ell=1}^{k} f_{i_\ell}(\{i_1, \ldots, i_{\ell-1}\})$$

  for $S = \{i_1, \ldots, i_k\} \in \mathcal{I}$,
  where the summation is independent of the order of players in S.

Example 1 (p-Belief)

- Let $f_i^{p_i}(S) = \begin{cases} 1 - p_i & \text{if } S = I \setminus \{i\}, \\ -p_i & \text{otherwise.} \end{cases}$
- For $S \subset I \setminus \{i, j\}$, we have

$$f_i(S) + f_j(S \cup \{i\}) = f_j(S) + f_i(S \cup \{j\})$$
$$= \begin{cases} 1 - p_i - p_j & \text{if } S = I \setminus \{i, j\}, \\ -p_i - p_j & \text{if } S \subsetneq I \setminus \{i, j\}. \end{cases}$$

- A potential is given by

$$v(S) = \begin{cases} 1 - \sum_{i \in I} p_i & \text{if } S = I, \\ -\sum_{i \in S} p_i & \text{otherwise.} \end{cases}$$

$\Rightarrow f_i(S) = v(S \cup \{i\}) - v(S) \; \forall\, i \in I, \forall\, S \in \mathcal{I}_{-i}$.

- Note that v is uniquely maximized at I iff $\sum_{i \in I} p_i < 1$.

  $\cdots$ the condition for the Critical Path Result.
- We will argue that this is not a coincidence.

Example 2 (Anonymity and Symmetry)

- Let $f_i^{m_i}(S) = g(|S|) - m_i$,

  where $g\colon \{0, 1, 2, \ldots, |I| - 1\} \to \mathbb{R}$ and $m_i \in \mathbb{R}$.

- $\mathbf{f}$ has a potential:

$$v(S) = \sum_{n=0}^{|S|-1} g(n) - \sum_{i \in S} m_i.$$

$\cdots$ "integral" of g $(+$ constants).

Monotone Potential

Definition 2

$\mathbf{f} = (f_i)_{i \in I}$ admits a monotone potential $v\colon \mathcal{I} \to \mathbb{R}$ if there exists $\boldsymbol{\lambda} = (\lambda_i)_{i \in I}$ with $\lambda_i > 0$ such that

$$f_i(S) \geq \lambda_i \big(v(S \cup \{i\}) - v(S)\big)$$

for any $i \in I$ and $S \in \mathcal{I}_{-i}$.

- If $\langle q_i, f_i \rangle < 0$, then $\langle q_i, v(\cdot \cup \{i\}) - v(\cdot) \rangle < 0$.
- Theorems will be about implications of existence/non-existence of a monotone potential that is uniquely maximized at I.

Example 3 (Unanimity)

- Let $f_i^{a_i,b_i}(S) = \begin{cases} a_i & \text{if } S = I \setminus \{i\}, \\ -b_i & \text{if } S = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \qquad (a_i, b_i > 0)$
- $\mathbf{f}$ has a potential uniquely maximized at I $\iff a_1 = \cdots = a_{|I|} > b_1 = \cdots = b_{|I|}$.
- $\mathbf{f}$ has a monotone potential uniquely maximized at I $\iff a_i a_j > b_i b_j$ for all $i \neq j$.

Critical Path Theorem for Generalized Beliefs

Theorem 1

If $\mathbf{f} = (f_i)_{i \in I}$ *admits a monotone potential* v *uniquely maximized at* I*, then for any type space* (T, P) *and for any* $\mathbf{E} = (E_i)_{i \in I}$*,* $E_i \subset T_i$*,*

$$P\left(CB^{\mathbf{f}}(\mathbf{E})\right) \geq 1 - \xi(v)(1 - P(E)),$$

where

$$\xi(v) = 1 + \frac{M}{v(I) - v'},$$

$$v' = \max_{S \subsetneq I} v(S),$$

$$M = \max_{S \subseteq S' \subsetneq I} (v(S) - v(S')).$$

Sketch of Proof

The goal is to show that $P(E), P(B^{\mathbf{f}}(\mathbf{E})), P(B^{\mathbf{f}}(\mathbf{B}^{\mathbf{f}}(\mathbf{E}))), \ldots$ do not decrease too fast.

For a technical reason (that will be clear later), we iterate belief operators sequentially:

$$E_i^1 = E_i,$$
$$E_i^{n+1} = \begin{cases} B_i^{f_i}(\mathbf{E}^n) & \text{if } i \equiv n \pmod{|I|}, \\ E_i^n & \text{if } i \not\equiv n \pmod{|I|}. \end{cases}$$

We have

$$CB_i^{\mathbf{f}}(\mathbf{E}) = E_i^1 \cap E_i^{i+1} \cap E_i^{i+|I|+1} \cap E_i^{i+2|I|+1} \cap \cdots$$

We partition T_i into $\{D_i^n\}_{n=0,1,\dots,\infty}$, where

$$D_i^0 = T_i \setminus E_i^1$$
$$D_i^n = E_i^n \setminus E_i^{n+1} = \{t_i \in T_i \mid t_i \text{ is eliminated at step } n\},$$
$$D_i^\infty = CB_i^{\mathbf{f}}(\mathbf{E}).$$

For $\mathbf{n} = (n_i)_{i \in I} \in (\mathbb{N} \cup \{\infty\})^I$, we let $\pi(\mathbf{n}) = P(\prod_i D_i^{n_i})$.

Illustration for $I = \{1, 2\}$

	D_2^0	D_2^2	D_2^4	D_2^6	$\cdots$
D_1^0	$\pi(0,0)$	$\pi(0,2)$	$\pi(0,4)$	$\pi(0,6)$	
D_1^1	$\pi(1,0)$	$\pi(1,2)$	$\pi(1,4)$	$\pi(1,6)$	
D_1^3	$\pi(3,0)$	$\pi(3,2)$	$\pi(3,4)$	$\pi(3,6)$	
D_1^5	$\pi(5,0)$	$\pi(5,2)$	$\pi(5,4)$	$\pi(5,6)$	
$\vdots$					

where

$$E_1 = D_1^1 \cup D_1^3 \cup D_1^5 \cup \cdots ,$$
$$E_2 = D_2^2 \cup D_2^4 \cup D_2^6 \cup \cdots .$$

	D_2^0	D_2^2	D_2^4	D_2^6	$\cdots$
D_1^0	$\pi(0,0)$	$\pi(0,2)$	$\pi(0,4)$	$\pi(0,6)$	
D_1^1	$\pi(1,0)$	$\pi(1,2)$	$\pi(1,4)$	$\pi(1,6)$	
D_1^3	$\pi(3,0)$	$\pi(3,2)$	$\pi(3,4)$	$\pi(3,6)$	
D_1^5	$\pi(5,0)$	$\pi(5,2)$	$\pi(5,4)$	$\pi(5,6)$	
$\vdots$					

We have

$$\pi(1,0)f_1(\emptyset) + (\pi(1,2) + \pi(1,4) + \cdots)f_1(\{2\}) \leq 0$$
$$\Rightarrow \pi(1,0)(v(\{1\}) - v(\emptyset))$$
$$+ (\pi(1,2) + \pi(1,4) + \cdots)(v(\{1,2\}) - v(\{2\})) \leq 0.$$

Similarly,

$$(\pi(0,2) + \pi(1,2))f_2(\emptyset) + (\pi(3,2) + \pi(5,2) + \cdots)f_2(\{1\}) \leq 0$$
$$\Rightarrow (\pi(0,2) + \pi(1,2))(v(\{2\}) - v(\emptyset))$$
$$+ (\pi(3,2) + \pi(5,2) + \cdots)(v(\{1,2\}) - v(\{1\})) \leq 0.$$

Lemma 4

If $F_i \subset T_i \setminus B_i^{f_i}(\mathbf{E})$*, then*
$\sum_{S \in \mathcal{I}_{-i}} P\left(F_i \times \prod_{j \in S} E_j \times \prod_{j \notin S \cup \{i\}} (T_j \setminus E_j)\right) f_i(S) \leq 0.$

Let $S(k, \mathbf{n}) = \{i \in I \mid n_i > k\}$.

Claim 1

For all $i \in I$ and $1 \leq k < \infty$,

$$\sum_{\mathbf{n}: n_i = k} \pi(\mathbf{n})(v(S(n_i, \mathbf{n}) \cup \{i\}) - v(S(n_i, \mathbf{n}))) \leq 0.$$

Proof

If $k \equiv i \pmod{|I|}$, then

$$\sum_{\mathbf{n}:n_i=k} \pi(\mathbf{n}) f_i(S(n_i,\mathbf{n}))$$

$$= \sum_{S\in\mathcal{I}_{-i}} \sum_{\mathbf{n}:n_i=k, S(k,\mathbf{n})=S} P\left(D_i^k \times \prod_{j\neq i} D_j^{n_j}\right) f_i(S)$$

$$= \sum_{S\in\mathcal{I}_{-i}} P\left(D_i^k \times \prod_{j\in S} E_j^k \times \prod_{j\notin S\cup\{i\}} (T_j \setminus E_j^k)\right) f_i(S) \leq 0.$$

The claim then follows from the definition of monotone potential v.

- Let $x_i(0) = 0$ and for $1 \le k < \infty$,

$$\varepsilon = 1 - P(E) = \sum_{\mathbf{n}:\min(\mathbf{n})=0} \pi(\mathbf{n}),$$

$$x_i(k) = \sum_{\mathbf{n}:1\le n_i=\min(\mathbf{n})\le k} \pi(\mathbf{n}),$$

$$x(k) = \sum_{i\in I} x_i(k) = \sum_{\mathbf{n}:1\le\min(\mathbf{n})\le k} \pi(\mathbf{n}).$$

- For $1 \le n < \infty$,

$$1 - P\left(\prod_{i\in I} \tilde{B}_i^{\mathbf{f},n}(\mathbf{E})\right) \le 1 - P\left(\prod_{i\in I} E_i^{n|I|+1}\right)$$
$$= \sum_{\mathbf{n}:\min(\mathbf{n})\le n|I|} \pi(\mathbf{n}) = \varepsilon + x(n|I|).$$

Claim 2

For $1 \leq k < \infty$,

$$x(k) \leq \frac{M}{v(I) - v'}\varepsilon.$$

The key identity: for any $\mathbf{n}$,

$$\sum_{i:\ell \leq n_i \leq k} v(S(n_i, \mathbf{n}) \cup \{i\}) - v(S(n_i, \mathbf{n}))$$
$$= v(S(\ell - 1, \mathbf{n})) - v(S(k, \mathbf{n})).$$

Proof

For $1 \leq k < \infty$, we have

$$
\begin{aligned}
0 &\geq \sum_{i \in I} \sum_{\ell=1}^{k} \sum_{\mathbf{n}: n_i = \ell} \pi(\mathbf{n})(v(S(n_i, \mathbf{n}) \cup \{i\}) - v(S(n_i, \mathbf{n}))) \\
&= \sum_{\mathbf{n}: \min(\mathbf{n}) \leq k} \sum_{i : 1 \leq n_i \leq k} \pi(\mathbf{n})(v(S(n_i, \mathbf{n}) \cup \{i\}) - v(S(n_i, \mathbf{n}))) \\
&= \sum_{\mathbf{n}: \min(\mathbf{n}) \leq k} \pi(\mathbf{n})(v(S(0, \mathbf{n})) - v(S(k, \mathbf{n}))) \\
&= \sum_{\mathbf{n}: \min(\mathbf{n}) \leq k} \pi(\mathbf{n})(v(\underbrace{S(0, \mathbf{n})}_{= I \text{ if } \min(\mathbf{n}) \geq 1}) - v(S(k, \mathbf{n}))) \\
&\geq x(k)(v(I) - v') - M\varepsilon.
\end{aligned}
$$

Implication of Non-Existence of Potential Maximized at I

Recall the definition of monotone potential.

Definition 3 (Coefficients on Left)

$\mathbf{f} = (f_i)_{i \in I}$ admits a monotone potential $v : \mathcal{I} \to \mathbb{R}$ if there exists $\boldsymbol{\lambda}' = (\lambda'_i)_{i \in I}$ with $\lambda'_i > 0$ such that

$$\lambda'_i f_i(S) \geq v(S \cup \{i\}) - v(S)$$

for any $i \in I$ and $S \in \mathcal{I}_{-i}$.

We characterize when $\mathbf{f}$ admits such a monotone potential.

In what follows, we "get rid of" v first, and then $\boldsymbol{\lambda}'$.

If v is uniquely maximized at I, then

$$
\begin{aligned}
&\lambda'_i f_i(I \setminus \{i\}) \geq v(I) - v(I \setminus \{i\}) > 0, \\
&\lambda'_i f_i(I \setminus \{i\}) + \lambda'_j f_j(I \setminus \{i,j\}) \\
&\ \geq (v(I) - v(I \setminus \{i\})) + (v(I \setminus \{i\}) - v(I \setminus \{i,j\})) \\
&\ = v(I) - v(I \setminus \{i,j\}) > 0, \ldots
\end{aligned}
$$

$\gamma = (i_1, \ldots, i_k)$: a finite sequence of distinct players in I.

Γ: the set of all such sequences.

Lemma 5 (Fixed $\boldsymbol{\lambda}'$)

$\mathbf{f} = (f_i)_{i \in I}$ *admits a monotone potential with* $\boldsymbol{\lambda}' = (\lambda'_i)_{i \in I}$ *that is uniquely maximized at* I *if and only if*

$$\sum_{\ell=1}^{k} \lambda'_{i_\ell} f_{i_\ell}(I \setminus \{i_1, \ldots, i_\ell\}) > 0$$

for every $\gamma \in \Gamma$.

For $i \in I$, let $\Gamma_i \subset \Gamma$ be the set of sequences that contain i.

For $\gamma \in \Gamma_i$, let $S(i, \gamma) \in \mathcal{I}_{-i}$ be the set of player i's opponents who are not listed in γ earlier than i.

We can apply a version of Farkas' lemma to "get rid of" $\boldsymbol{\lambda}'$.

Lemma 6

$\mathbf{f} = (f_i)_{i \in I}$ admits a monotone potential that is uniquely maximized at I if and only if there is no $\mu \in \Delta(\Gamma)$ such that

$$\sum_{S \in \mathcal{I}_{-i}} \mu(\{\gamma \in \Gamma_i : S(i, \gamma) = S\}) f_i(S) \leq 0$$

for any $i \in I$.

Contagion Result

Theorem 2

Suppose that $\mathbf{f} = (f_i)_{i \in I}$ *is generic and does not admit a monotone potential that is strictly maximized at* I*. Then for any* $\varepsilon \in (0, 1]$*, there exist a type space* $((T_i)_{i \in I}, P)$ *and a profile* $\mathbf{E} = (E_i)_{i \in I}$ *with* $E = \prod_{i \in I} E_i$ *such that* $P(E) = 1 - \varepsilon$ *and*

$$P\left(CB^{\mathbf{f}}(\mathbf{E})\right) = 0.$$

Proof.

By genericity, for sufficiently small $\varepsilon > 0$, we have

$$\sum_{S \in \mathcal{I}_{-i}} (1-\varepsilon)^{|S|} \mu(\{\gamma \in \Gamma_i : S(i,\gamma) = S\}) f_i(S) < 0$$

for any $i \in I$ such that $\mu(\Gamma_i) > 0$.

Given such small ε, we consider the following type space.

- $T_i = \{1, 2, 3, \ldots\} \cup \{\infty\}$
- $t_i = \theta + \eta_i$, where
  - θ follows a geometric distribution, $\theta = m$ with probability $\varepsilon(1-\varepsilon)^m$ for $m = 0, 1, 2, \ldots$.
  - Independently of θ, $\gamma \in \Gamma$ is drawn according to μ, and $\eta_i = \ell$ if i is listed at the ℓ-th place of γ ($\eta_i = \infty$ if i is not listed in γ).

Let $E_i = \{|I|, |I|+1, \ldots, \infty\}$ for each $i \in I$.

Then each time $B_i^{f_i}$ applies, types $|I|, |I|+1, \ldots$ get eliminated. □

Robustness in Binary Supermodular Games

- The above results imply:
  in generic binary supermodular games
  $\mathbf{1}$ is a robust equilibrium if and only if $\mathbf{1}$ is an MP-maximizer.

- "If" part (MP-max $\Rightarrow$ robust):
  Follows from Critical Path Result.

  $\cdots$ Already known by Ui (2001) and Morris and Ui (2005) via the "potential maximizing strategy" proof.

  We give an alternative, "higher order beliefs" proof.

- "Only if" part (not MP-max $\Rightarrow$ not robust):
  Implication of non-existence of a potential.

  If $\mathbf{1}$ is not an MP-maximizer, then one can construct an ε-elaboration such that $P(\text{type profiles playing } \mathbf{1}) = 0$.

  $\cdots$ New construction in the literature.

Proof: "If" part

- Suppose that the game $\mathbf{f}$ has a monotone potential v maximized at I.
- Fix any $\delta > 0$, let $\varepsilon > 0$ small enough, and consider any ε-elaboration of $\mathbf{f}$.
- Let $T^{\mathbf{f}}$ be the set of type profiles whose payoffs are given by $\mathbf{f}$.

  By definition, $P(T^{\mathbf{f}}) = 1 - \varepsilon$.
- Recall our Critical Path Theorem:

$$P(\underbrace{CB^{\mathbf{f}}(T^{\mathbf{f}})}_{\exists \text{ BNE } \sigma^* \text{ playing } \mathbf{1} \text{ on } CB^{\mathbf{f}}(T^{\mathbf{f}})}) \geq 1 - \xi(v)\big(\underbrace{1 - P(T^{\mathbf{f}})}_{= \varepsilon}\big).$$

- Hence,

$$P(\sigma^* \text{ plays } \mathbf{1}) \geq P(CB^{\mathbf{f}}(T^{\mathbf{f}})) \geq 1 - \xi(v) \times \varepsilon \geq 1 - \delta$$

  if ε is small enough that $\varepsilon \leq \delta/\xi(v)$.

Proof: "Only if" part

- Suppose that the game $\mathbf{f}$ does not have a monotone potential maximized at I.
- For any $\varepsilon > 0$, we can construct an ε-elaboration that has a unique rationalizable strategy σ^*, which satisfies

  $$P(t|\sigma(t) = \mathbf{1}) = 0.$$

  This implies that $\mathbf{1}$ is not robust.

Example: Unanimity Games

- Consider the game $\mathbf{f}$ given by

$$f_i(S) = \begin{cases} a_i & \text{if } S = I, \\ -b_i & \text{if } S = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \qquad (a_i, b_i > 0)$$

- $\mathbf{1}$ is an MP-maximizer
$\iff a_i a_j > b_i b_j$ for all $i \neq j$.

Proposition 7

$\mathbf{1}$ *is robust to incomplete information*
$\iff a_i a_j > b_i b_j$ *for all* $i \neq j$.