From Imitation Games to Kakutani

by McLennan and Tourky

Daisuke Oyama

Advanced Economic Theory

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Reference

- McLennan, A. and R. Tourky (2006). "From Imitation Games to Kakutani."

Imitation Games

- A two-player normal form game (A, B), where A and B are $m \times n$ matrices, is an *imitation game* if $m = n$ and $B = I$, where I is the $m \times m$ identity matrix.
- Let $\mathcal{I} = \{1, \ldots, m\}$, and $\Delta^m = \{x \in \mathbb{R}^m_+ \mid x_1 + \cdots + x_m = 1\}$.
- For any $\iota \in \Delta^m$, $\arg\max_{i \in \mathcal{I}} (I\iota)_i \subset \operatorname{supp}(\iota)$.
- For $\rho \in \Delta^m$, let

$$\rho^\circ = \{i \in \mathcal{I} \mid \rho_i = 0\}, \quad \overline{\rho} = \arg\max_{i \in \mathcal{I}} (A\rho)_i.$$

- A is in *general position* if $|\rho^\circ| + |\overline{\rho}| \le m$ for all $\rho \in \Delta^m$.
- A is in general position if and only if (A, I) is nondegenerate.

I-Equilibria

- $\rho \in \Delta^m$ is an I-*equilibrium* of an imitation game (A, I) if $\rho^\circ \cup \overline{\rho} = \mathcal{I}$, or equivalently $\operatorname{supp}(\rho) \subset \overline{\rho}$.
- $\rho \in \Delta^m$ is an I-equilibrium of (A, I) if and only if there exists $\iota \in \Delta^m$ such that (ι, ρ) is a Nash equilibrium of (A, I).
- If (ι, ρ) is a Nash equilibrium of (A, I), then

$$\operatorname{supp}(\rho) \subset \arg\max_{i \in \mathcal{I}} (I\iota)_i \subset \operatorname{supp}(\iota) \subset \arg\max_{i \in \mathcal{I}} (A\rho)_i.$$

- If ρ is an I-equilibrium of (A, I), then let ι be defined by $\iota_i = \dfrac{1}{|\operatorname{supp}(\rho)|}$ for $i \in \operatorname{supp}(\rho)$.

  Then we have $\operatorname{supp}(\rho) \subset \arg\max_{i \in \mathcal{I}} (A\rho)_i$ and

$$\operatorname{supp}(\rho) = \operatorname{supp}(\iota) = \arg\max_{i \in \mathcal{I}} (I\iota)_i.$$

Existence of I-Equilibrium

- The imitation game (A, I) has an I-equilibrium.

  Proof: By Lemke-Howson.

- (We cannot use Kakutani here, as our goal is to prove Kakutani's fixed point theorem.)

Correspondences and Imitation Games

- Let $C \neq \emptyset$ be a closed convex subset of an inner product space.
- Let $F\colon C \to C$ be a nonempty-valued correspondence.
- Fix any $x_1 \in C$.
- Define sequences $x_1, x_2, \ldots \in C$ and $y_1, y_2, \ldots \in C$ as follows:

  Given $x_1, \ldots, x_m$ and $y_1, \ldots, y_{m-1}$:
  - Choose any $y_m \in F(x_m)$.
  - Define an imitation game (A, I) with m actions by
    $$a_{ij} = -\|x_i - y_j\|^2.$$
  - Let $\rho^m \in \Delta^m$ be an I-equilibrium of (A, I).
  - Let
    $$x_{m+1} = \sum_{j=1}^{m} \rho_j^m y_j.$$

Proposition 1

For any m,

$$\operatorname{supp}(\rho^m) \subset \underset{i=1,\ldots,m}{\arg\min} \|x_i - x_{m+1}\|.$$

- Define $G_F \colon C \to C$ by

$$G_F(x) = \bigcap_{\delta>0} \overline{\operatorname{co} B_\delta(F(x))},$$

where

$$B_\delta(F(x)) = \bigcup_{\|x'-x\|<\delta} F(x').$$

Proposition 2

An accumulation point x^* of $\{x_m\}$ is a fixed point of G_F.

Proof

- Let x^* be an accumulation point of $\{x_m\}$.
- Fix any $\delta > 0$ and any $\varepsilon \in (0, \delta/3)$.
- Let m_0 be such that $x_{m_0} \in B_\varepsilon(x^*)$, and take any $m_1 > m_0$ such that $x_{m_1+1} \in B_\varepsilon(x^*)$.
- We have $\|x_{m_0} - x_{m_1+1}\| < (2/3)\delta$.
- Then for all $j \in M = \arg\min_{j=1,\ldots,m_1} \|x_j - x_{m_1+1}\|$, we have $\|x_j - x_{m_1+1}\| < (2/3)\delta$ and hence $\|x_j - x^*\| < \delta$.
- Therefore, $y_j \in B_\delta(F(x^*))$ for all $j \in M$.

- By Proposition 1, $x_{m_1+1} = \sum_{j \in M} \rho_j y_j$, where ρ_j is an I-equilibrium of the imitation game.
- This means that $x_{m_1+1} \in \operatorname{co} B_\delta(F(x^*))$.
- Since $\varepsilon > 0$ has been taken arbitrarily, it follows that $x^* \in \overline{\operatorname{co} B_\delta(F(x^*))}$.
- Since $\delta > 0$ has been taken arbitrarily, it follows that $x^* \in \bigcap_{\delta > 0} \overline{\operatorname{co} B_\delta(F(x^*))}$.

Upper Semi-Continuity

- A correspondence $F\colon C \to C$ is *upper semi-continuous* if for any $x \in C$, for any open neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(U) \subset V$.
- (McLennan-Tourky require $F(x)$ in addition to be closed.)

Kakutani's Fixed Point Theorem

Theorem 1

Suppose that $C \neq \emptyset$ is a compact convex subset of an inner product space, and that $F\colon C \to C$ is a nonempty-, convex-, and compact-valued upper semi-continuous correspondence.

Then F has a fixed point.

Proof

- $\{x^m\}$ has an accumulation point x^* since C is (sequentially) compact.
- x^* is a fixed point of G_F by Proposition 2.
- Show that $G_F = F$.
  - For any $\delta > 0$, there exists $\delta' > 0$ such that $F(B_{\delta'}(x)) \subset B_\delta(F(x))$ by the upper semi-continuity of F.
  - $\operatorname{co} F(B_{\delta'}(x)) = F(B_{\delta'}(x))$ by the convexity of F.
  - $\bigcap_{\delta>0} \overline{B_\delta(F(x))} = F(x)$ by the closedness of F.