Equilibrium Computation for Two-Player Games in Strategic Form I

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Advanced Economic Theory

September 27, 2016

Reference

- von Stengel, B. (2007). "Equilibrium Computation for Two-Player Games in Strategic and Extensive Form," Chapter 3, Algorithmic Game Theory.

Bimatrix Games

- Two players: 1 and 2
- 1's action space: $M = \{1, \ldots, m\}$

  2's action space: $N = \{m+1, \ldots, m+n\}$
- 1's payoff matrix: $A \in \mathbb{R}^{M \times N}$

  2's payoff matrix: $B \in \mathbb{R}^{M \times N}$
- For $x \in \Delta^M$ and $y \in \Delta^N$,
  - 1's expected payoff: $x'Ay$
  - 2's expected payoff: $x'By$

  where $\Delta^L = \{x \in \mathbb{R}^L_+ \mid \sum_{\ell \in L} x_\ell = 1\}$, $L = M, N$.

Nash Equilibrium

1. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if

$$x'Ay \geq \tilde{x}'Ay \text{ for all } \tilde{x} \in \Delta^M, \tag{1}$$

$$x'By \geq x'B\tilde{y} \text{ for all } \tilde{y} \in \Delta^N. \tag{2}$$

2. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if

$$(Ay)_i = \max_{i' \in M} (Ay)_{i'} \text{ for all } i \in \operatorname{supp}(x), \tag{3}$$

$$(B'x)_j = \max_{j' \in N} (B'x)_{j'} \text{ for all } j \in \operatorname{supp}(y). \tag{4}$$

Proof

- Fix $y \in \Delta^N$, and let $u = \max_{i \in M} (Ay)_i$. Then
  - $e'_{i^*} Ay = u$ for some $i^* \in M$, and
  - $\tilde{x}' Ay \leq \sum_{i \in M} \tilde{x}_i u = u$ for any $\tilde{x} \in \Delta^M$.
- Consider any $x \in \Delta^M$. We have

$$0 \leq \sum_{i \in M} x_i (u - (Ay)_i) = u - x' Ay.$$

- Therefore,

$$(1) \iff x' Ay \geq u \iff (3).$$

Nash Equilibrium

- For $x \in \Delta^M$ and $y \in \Delta^N$, write

$$\bar{x} = \arg\max_{j \in N} (B'x)_j, \qquad x^\circ = \{i \in M \mid x_i = 0\},$$

$$\bar{y} = \arg\max_{i \in M} (Ay)_i, \qquad y^\circ = \{j \in N \mid y_j = 0\}.$$

3. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if

$$\mathrm{supp}(x) \subset \bar{y}, \quad \mathrm{supp}(y) \subset \bar{x}.$$

3′. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if

$$\bar{y} \cup x^\circ = M, \quad \bar{x} \cup y^\circ = N,$$

or equivalently,

$$(\bar{x} \cup x^\circ) \cup (\bar{y} \cup y^\circ) = M \cup N.$$

Example

$M = \{1, 2, 3\}$, $N = \{4, 5\}$:

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{bmatrix}.$$

Nondegenerate Games

- A two-player game is *nondegenerate* if for any $x \in \Delta^M$ and any $y \in \Delta^N$,

$$|\bar{x}| \leq |\mathrm{supp}(x)|, \quad |\bar{y}| \leq |\mathrm{supp}(y)|,$$

or equivalently,

$$|x^\circ| + |\bar{x}| \leq m, \quad |y^\circ| + |\bar{y}| \leq n.$$

- If (x, y) is a Nash equilibrium of a nondegenerate game, then

$$|\mathrm{supp}(x)| = |\mathrm{supp}(y)|.$$

$\because |\mathrm{supp}(x)| \leq |\bar{y}| \leq |\mathrm{supp}(y)| \leq |\bar{x}| \leq |\mathrm{supp}(x)|.$

Example of a Degenerate Game

$M = \{1, 2, 3\}$, $N = \{4, 5\}$:

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & \underline{3} \\ 2 & 6 \\ 3 & 1 \end{bmatrix}.$$

Recall

- $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if

$$(Ay)_i = \max_{i' \in M} (Ay)_{i'} \text{ for all } i \in \text{supp}(x), \tag{3}$$

$$(B'x)_j = \max_{j' \in N} (B'x)_{j'} \text{ for all } j \in \text{supp}(y). \tag{4}$$

Support Enumeration

- Input: Nondegenerate bimatrix game
- Output: All Nash equilibria of the game
- Method:

  For each $k = 1, \ldots, \min\{m, n\}$ and each pair (I, J), $I \subset M$ and $J \subset N$, such that $|I| = |J| = k$, solve the systems of linear equations

$$\sum_{j \in J} a_{ij} y_j = u \text{ for } i \in I, \quad \sum_{j \in J} y_j = 1,$$
$$\sum_{i \in I} b_{ij} x_i = v \text{ for } j \in J, \quad \sum_{i \in I} x_i = 1.$$

  Check

  - $x_i > 0$ for all $i \in I$ and $y_j > 0$ for all $j \in J$,
  - $u \geq \sum_{j \in J} a_{ij} y_j$ for all $i \notin I$ and $v \geq \sum_{i \in I} b_{ij} x_i$ for all $j \notin J$.

- The systems of equations are written in matrix form as

$$\begin{pmatrix} A_{IJ} & -\mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix} \begin{pmatrix} y_J \\ u \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix},$$
$$\begin{pmatrix} B'_{IJ} & -\mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix} \begin{pmatrix} x_I \\ v \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix},$$

  where

  - $A_{IJ} = (a_{ij})_{i \in I, j \in J}$, $B_{IJ} = (b_{ij})_{i \in I, j \in J}$,
  - $\mathbf{0} = \begin{pmatrix} 0 \cdots 0 \end{pmatrix}' \in \mathbb{R}^k$, $\mathbf{1} = \begin{pmatrix} 1 \cdots 1 \end{pmatrix}' \in \mathbb{R}^k$.

- If $m = n$, the number of equal-sized support pairs is

$$\sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n} - 1 \approx \frac{4^n}{\sqrt{\pi n}}$$

  ("$\approx$" by Stirling's formula $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$).