Equilibrium Computation for Two-Player Games in Strategic Form II

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Advanced Economic Theory

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Reference

- von Stengel, B. (2007). "Equilibrium Computation for Two-Player Games in Strategic and Extensive Form," Chapter 3, Algorithmic Game Theory.

Polyhedra/Polytopes

- An *affine combination* of $z_1, \ldots, z_k \in \mathbb{R}^n$ is $\sum_{i=1}^k \lambda_i z_i$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\sum_{i=1}^k \lambda_i = 1$.
- It is called a *convex combination* if $\lambda_i \geq 0$ for all i.
- A subset of $\mathbb{R}^n$ is *convex* if it is closed under convex combinations.
- $z_1, \ldots, z_k$ are *affinely independent* if none of these points is an affine combination of the others, or equivalently, $z_1 - z_k, \ldots, z_{k-1} - z_k$ are linearly independent.
- A convex set has dimension d if has $d + 1$, but no more, affinely independent points.

Polyhedra/Polytopes

- A *polyhedron* P in $\mathbb{R}^d$ is a set $\{z \in \mathbb{R}^d \mid Cz \leq q\}$ for some matrix C and vector q.

  It is called *full-dimensional* if it has dimension d.

- A polyhedron is called a *polytope* if it is bounded.

- A *face* of a polyhedron P is a set $\{z \in P \mid c'z = q_0\}$ for some $c \in \mathbb{R}^d$ and $q_0 \in \mathbb{R}$ such that the inequality $c'z \leq q_0$ holds for all $z \in P$.

- Any nonempty face F of P is written as $\{z \in P \mid c_i z = q_i,\ i \in I\}$ for some rows $\{c_i\}_{i \in I}$ of C.

  $c_i z \leq q_i$ are called *binding inequalities*.

- A *vertex* of P is the unique element of a 0-dimensional face of P.

- An *edge* of P is a 1-dimensional face of P.

Polyhedra/Polytopes

- A *facet* of a d-dimensional polyhedron is a face of dimension $d - 1$.
- A d-dimensional polyhedron P is called *simple* if no point belongs to more than d facets of P.

Best Response Polyhedra/Polytopes

Let a bimatrix game (A, B) is given.

- Best response polyhedron:

$$\overline{P} = \{(x, v) \in \mathbb{R}^M \times \mathbb{R} \mid x \geq \mathbf{0},\ B'x \leq v\mathbf{1},\ \mathbf{1}'x = 1\},$$
$$\overline{Q} = \{(y, u) \in \mathbb{R}^N \times \mathbb{R} \mid Ay \leq u\mathbf{1},\ y \geq \mathbf{0},\ \mathbf{1}'y = 1\}.$$

- Assume, without loss of generality, that A and B' are nonnegative and have no zero column.

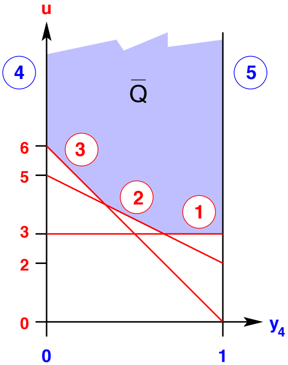
- Best response polytope:

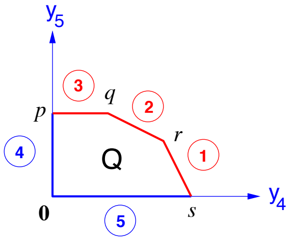
$$P = \{x \in \mathbb{R}^M \mid x \geq \mathbf{0},\ B'x \leq \mathbf{1}\},$$
$$Q = \{y \in \mathbb{R}^N \mid Ay \leq \mathbf{1},\ y \geq \mathbf{0}\}.$$

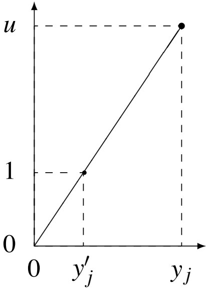
Example

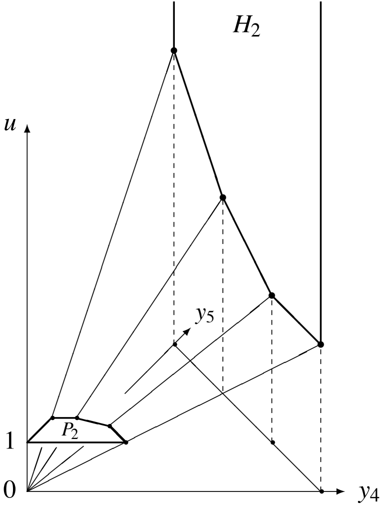
$M = \{1, 2, 3\}$, $N = \{4, 5\}$:

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{bmatrix}.$$









(From von Stengel 2002)

Nash Equilibrium

- For $x \in \Delta^M$ and $y \in \Delta^N$, write

$$\bar{x} = \arg\max_{j \in N} (B'x)_j, \qquad x^\circ = \{i \in M \mid x_i = 0\},$$
$$\bar{y} = \arg\max_{i \in M} (Ay)_i, \qquad y^\circ = \{j \in N \mid y_j = 0\}.$$

3. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if

$$\mathrm{supp}(x) \subset \bar{y}, \quad \mathrm{supp}(y) \subset \bar{x}.$$

3′. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if

$$\bar{y} \cup x^\circ = M, \quad \bar{x} \cup y^\circ = N,$$

or equivalently,

$$(\bar{x} \cup x^\circ) \cup (\bar{y} \cup y^\circ) = M \cup N.$$

Labels

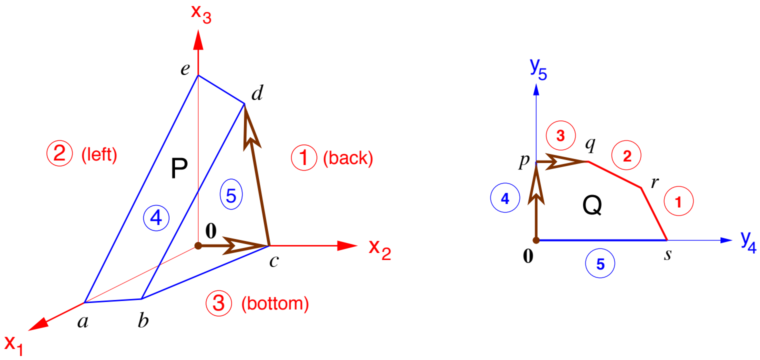
- $(x, v) \in \overline{P}$ has *label* $k \in M \cup N$ if
  - for $k = j \in N$, $(B'x)_j = v$, so that $j \in \bar{x}$, or
  - for $k = i \in M$, $x_i = 0$, so that $i \in x^\circ$.
- $(y, u) \in \overline{Q}$ has *label* $k \in M \cup N$ if
  - for $k = i \in M$, $(Ay)_i = u$, so that $i \in \bar{y}$, or
  - for $k = j \in N$, $y_j = 0$, so that $j \in y^\circ$.
- $((x, v), (y, u)) \in \overline{P} \times \overline{Q}$ is *completely labeled* if every $k \in M \cup N$ appears as a label of either (x, v) or (y, u).

3″. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if $((x, v), (y, u))$ with $u = \max_i (Ay)_i$ and $v = \max_j (B'x)_j$ is completely labeled.

Example

$M = \{1, 2, 3\}$, $N = \{4, 5\}$:

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{bmatrix}.$$



Nondegeneracy

- Recall:

  A two-player game is *nondegenerate* if for any $x \in \Delta^M$ and any $y \in \Delta^N$,

  $$|\bar{x}| \le |\mathrm{supp}(x)|, \quad |\bar{y}| \le |\mathrm{supp}(y)|,$$

  or equivalently,

  $$|x^\circ| + |\bar{x}| \le m, \quad |y^\circ| + |\bar{y}| \le n,$$

  i.e., every $x \in P$ $(y \in Q)$ has no more than m (n) labels.

- If the game is nondegenerate, then in P (Q), only vertices can have m (n) labels.

  $\because$ If a non-vertex point had m labels, it would belong to a face of dimension 1 or larger, and a vertex of it would have additional labels.

Vertex Enumeration

- Input: Nondegenerate bimatrix game
- Output: All Nash equilibria of the game
- Method:

  For each vertex x of $P \setminus \{\mathbf{0}\}$ and each vertex y of $Q \setminus \{\mathbf{0}\}$, check that (x, y) is completely labeled.

- An algorithm for vertex enumeration:

  "lexicographic reverse search"

- *lrs* and its Julia wrapper LRSLib.jl

- "*lrsNash*" (Avis et al. 2010)

  - Enumerate only vertices x of $P \setminus \{0\}$ (assuming $|M| \leq |N|$).
  - For each vertex x of $P \setminus \{0\}$, find the facet given by the missing labels L of x.
  - By nondegeneracy $|L| = n$, and that facet either is empty or consists of a single vertex y.
  - In the latter case, (x, y) is a Nash equilibrium.

If $m = n$, the maximum number of vertices of P is approximately $(27/4)^{n/2} \approx 2.6^n$.