Equilibrium Computation for Two-Player Games in Strategic Form II

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Advanced Economic Theory

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Polyhedra/Polytopes

- An affine combination of $z_1, \ldots, z_k \in \mathbb{R}^n$ is $\sum_{i=1}^{k} \lambda_i z_i$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\sum_{i=1}^{k} \lambda_i = 1$.

- It is called a convex combination if $\lambda_i \geq 0$ for all $i$.

- A subset of $\mathbb{R}^n$ is convex if it is closed under convex combinations.

- $z_1, \ldots, z_k$ are affinely independent if none of these points is an affine combination of the others, or equivalently, $z_1 - z_k, \ldots, z_{k-1} - z_k$ are linearly independent.

- A convex set has dimension $d$ if has $d + 1$, but no more, affinely independent points.
Polyhedra/Polytopes

- A polyhedron $P$ in $\mathbb{R}^d$ is a set $\{z \in \mathbb{R}^d \mid Cz \leq q\}$ for some matrix $C$ and vector $q$.

  It is called full-dimensional if it has dimension $d$.

- A polyhedron is called a polytope if it is bounded.

- A face of a polyhedron $P$ is a set $\{z \in P \mid c'z = q_0\}$ for some $c \in \mathbb{R}^d$ and $q_0 \in \mathbb{R}$ such that the inequality $c'z \leq q_0$ holds for all $z \in P$.

- Any nonempty face $F$ of $P$ is written as $\{z \in P \mid c_i z = q_i, \ i \in I\}$ for some rows $\{c_i\}_{i \in I}$ of $C$.

  $c_i z \leq q_i$ are called binding inequalities.

- A vertex of $P$ is the unique element of a 0-dimensional face of $P$.

- An edge of $P$ is a 1-dimensional face of $P$. 
A facet of a $d$-dimensional polyhedron is a face of dimension $d - 1$.

A $d$-dimensional polyhedron $P$ is called simple if no point belongs to more than $d$ facets of $P$. 
Best Response Polyhedra/Polytopes

Let a bimatrix game \((A, B)\) is given.

- **Best response polyhedron:**

  \[
  \overline{P} = \{(x, v) \in \mathbb{R}^M \times \mathbb{R} \mid x \geq 0, B'x \leq v1, 1'x = 1\}, \\
  \overline{Q} = \{(y, u) \in \mathbb{R}^N \times \mathbb{R} \mid Ay \leq u1, y \geq 0, 1'y = 1\}.
  \]

- Assume, without loss of generality, that \(A\) and \(B'\) are nonnegative and have no zero column.

- **Best response polytope:**

  \[
  P = \{x \in \mathbb{R}^M \mid x \geq 0, B'x \leq 1\}, \\
  Q = \{y \in \mathbb{R}^N \mid Ay \leq 1, y \geq 0\}.
  \]
Example

\[ M = \{1, 2, 3\}, \quad N = \{4, 5\} : \]

\[
A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{bmatrix}.
\]

The left picture in Fig. 3.1 shows \( \bar{Q} \) for our example, for \( 0 \leq y^4 \leq 1 \) which uniquely determines \( y^5 \) as \( 1 - y^4 \). The circled numbers indicate the facets of \( \bar{Q} \), which are either the strategies \( i \in M \) of the other player 1 or the own strategies \( j \in N \). Facets 1, 2, 3 of player 1 indicate his best responses together with his expected payoff \( u \). For example, 1 is a best response when \( y^4 \geq 2/3 \). Facets 4 and 5 of player 2 tell when the respective own strategy has probability zero, namely \( y^4 = 0 \) or \( y^5 = 0 \).

We say a point \((y, u)\) of \( \bar{Q} \) has label \( k \in M \cup N \) if the \( k \)th inequality in the definition of \( \bar{Q} \) is binding, which for \( k = i \in M \) is the \( i \)th binding inequality \( \sum_{j \in N} a_{ij} y_j = u \) (meaning \( i \) is a best response to \( y \)), or for \( k = j \in N \) the binding inequality \( y_j = 0 \). In the example, \((y^4, y^5, u) = (2/3, 1/3, 3)\) has labels 1 and 2, so rows 1 and 2 are best responses to \( y^4 \) with expected payoff 3 to player 1. The labels of a point \((x, v)\) of \( P \) are defined correspondingly: It has label \( i \in M \) if \( x^i = 0 \), and label \( j \in N \) if \( \sum_{i \in M} b_{ij} x^i = v \). With these labels, an equilibrium is a pair \((x, y)\) of mixed strategies so that with the corresponding expected payoffs \( v \) and \( u \), the pair \((x, v) , (y, u)\) in \( P \times Q \) is completely labeled, which means that every label \( k \in M \cup N \) appears either as a label of \((x, v)\) or of \((y, u)\). This is equivalent to the best response condition (3.2): A missing label would mean a pure strategy of a player, for example \( i \) of player 1, that does not have probability zero, so \( x^i > 0 \), and is also not a best response, since \( \sum_{j \in N} a_{ij} y_j < u \), because the respective inequality \( i \) is not binding in \( P \) or \( Q \). But this is exactly when the best response condition is violated. Conversely, if every label appears in \( P \) or \( Q \), then each pure
These bijections are not linear. However, they preserve the face incidences since a binding inequality in $H_1$ corresponds to a binding inequality in $P_1$ and vice versa. In particular, vertices have the same labels defined by the binding inequalities, which are some of the $m+n$ inequalities defining $P_1$ and $P_2$ in (2.18).

Figure 2.5. The map $H_2 \rightarrow P_2$, $\left( y_j, u \right) \mapsto y_j \cdot \left( \frac{1}{u} \right)$ as a projective transformation with projection point $(0,0)$. The left-hand side shows this for a single component $y_j$ of $y$, the right-hand side shows how $P_2$ arises in this way from $H_2$ in the example (2.15).

Figure 2.5 shows a geometric interpretation of the bijection $\left( y, u \right) \mapsto y \cdot \left( \frac{1}{u} \right)$ as a projective transformation (see Ziegler, 1995, Sect. 2.6). On the left-hand side, the pair $\left( y_j, u \right)$ is shown as part of $\left( y, u \right)$ in $H_2$ for any component $y_j$ of $y$. The line connecting this pair to $(0,0)$ contains the point $y_j' = y_j / u$. Thus, $P_2 \times \{1\}$ is the intersection of the lines connecting any $\left( y, u \right)$ in $H_2$ with $(0,0)$ in $\mathbb{R}^{m+n} \times \{1\}$.

The vertices $0$ of $P_1$ and $P_2$ do not arise as such projections, but correspond to $H_1$ and $H_2$ “at infinity”.

2.5. Complementary pivoting

Traversing a polyhedron along its edges has a simple algebraic implementation known as pivoting. The constraints defining the polyhedron are thereby represented as linear equations with nonnegative variables. For $P_1 \times P_2$, these have the form

$$Ay + r = 1$$

$$Bx + s = 1$$

(2.20)

(From von Stengel 2002)
Nash Equilibrium

- For $x \in \Delta^M$ and $y \in \Delta^N$, write

$$\bar{x} = \arg\max_{j \in N} (B'x)_j, \quad x^o = \{i \in M \mid x_i = 0\},$$

$$\bar{y} = \arg\max_{i \in M} (Ay)_i, \quad y^o = \{j \in N \mid y_j = 0\}.$$

3. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if

$$\text{supp}(x) \subset \bar{y}, \quad \text{supp}(y) \subset \bar{x}.$$

3'. $(x, y) \in \Delta^M \times \Delta^N$ is a Nash equilibrium if and only if

$$\bar{y} \cup x^o = M, \quad \bar{x} \cup y^o = N,$$

or equivalently,

$$(\bar{x} \cup x^o) \cup (\bar{y} \cup y^o) = M \cup N.$$
Labels

- \((x, v) \in \overline{P}\) has label \(k \in M \cup N\) if
  - for \(k = j \in N\), \((B'x)_j = v\), so that \(j \in \bar{x}\), or
  - for \(k = i \in M\), \(x_i = 0\), so that \(i \in x^\circ\).

- \((y, u) \in \overline{Q}\) has label \(k \in M \cup N\) if
  - for \(k = i \in M\), \((Ay)_i = u\), so that \(i \in \bar{y}\), or
  - for \(k = j \in N\), \(y_j = 0\), so that \(j \in y^\circ\).

- \(((x, v), (y, u)) \in \overline{P} \times \overline{Q}\) is completely labeled if every \(k \in M \cup N\) appears as a label of either \((x, v)\) or \((y, u)\).

3′′. \((x, y) \in \Delta^M \times \Delta^N\) is a Nash equilibrium if and only if \(((x, v), (y, u))\) with \(u = \max_i (Ay)_i\) and \(v = \max_j (B'x)_j\) is completely labeled.
Example

\[ M = \{1, 2, 3\}, \quad N = \{4, 5\}: \]

\[
A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{bmatrix}.
\]

Fig. 3.2. The best response polytopes \( P \) (with vertices \( a, b, c, d, e \)) and \( Q \) for the game in (3.3). The arrows describe the Lemke–Howson algorithm, see Section 3.4.
Nondegeneracy

- Recall:

  A two-player game is nondegenerate if for any \( x \in \Delta^M \) and any \( y \in \Delta^N \),

  \[
  |\bar{x}| \leq |\operatorname{supp}(x)|, \quad |\bar{y}| \leq |\operatorname{supp}(y)|,
  \]

  or equivalently,

  \[
  |x^\circ| + |\bar{x}| \leq m, \quad |y^\circ| + |\bar{y}| \leq n,
  \]

  i.e., every \( x \in P \) \( (y \in Q) \) has no more than \( m \) \( (n) \) labels.

- If the game is nondegenerate, then in \( P \) \( (Q) \), only vertices can have \( m \) \( (n) \) labels.

  \[\because\] If a non-vertex point had \( m \) labels, it would belong to a face of dimension 1 or larger, and a vertex of it would have additional labels.
Vertex Enumeration

- Input: Nondegenerate bimatrix game
- Output: All Nash equilibria of the game
- Method:
  For each vertex \(x\) of \(P \setminus \{0\}\) and each vertex \(y\) of \(Q \setminus \{0\}\), check that \((x, y)\) is completely labeled.
An algorithm for vertex enumeration:

“lexicographic reverse search”

- *lrs* and its Julia wrapper *LRSLib.jl*

- “*lrsNash*” (Avis et al. 2010)
  - Enumerate only vertices $x$ of $P \setminus \{0\}$ (assuming $|M| \leq |N|$).
  - For each vertex $x$ of $P \setminus \{0\}$, find the facet given by the missing labels $L$ of $x$.
  - By nondegeneracy $|L| = n$, and that facet either is empty or consists of a single vertex $y$.
  - In the latter case, $(x, y)$ is a Nash equilibrium.

If $m = n$, the maximum number of vertices of $P$ is approximately $(27/4)^{n/2} \approx 2.6^n$. 