Equilibrium Computation for Two-Player Games in Strategic Form III

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Advanced Economic Theory

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Reference

- von Stengel, B. (2007). "Equilibrium Computation for Two-Player Games in Strategic and Extensive Form," Chapter 3, Algorithmic Game Theory.

Best Response Polyhedra/Polytopes

Let a bimatrix game (A, B) is given.

- Best response polyhedra:

$$\overline{P} = \{(x, v) \in \mathbb{R}^M \times \mathbb{R} \mid x \geq \mathbf{0},\ B'x \leq v\mathbf{1},\ \mathbf{1}'x = 1\},$$
$$\overline{Q} = \{(y, u) \in \mathbb{R}^N \times \mathbb{R} \mid Ay \leq u\mathbf{1},\ y \geq \mathbf{0},\ \mathbf{1}'y = 1\}.$$

- Assume, without loss of generality, that A and B' are nonnegative and have no zero column.

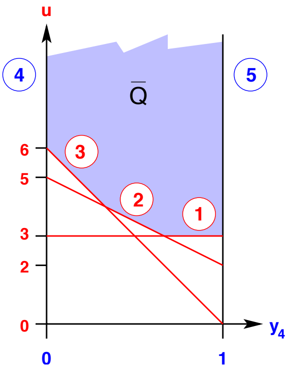
- Best response polytopes:

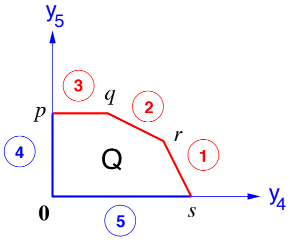
$$P = \{x \in \mathbb{R}^M \mid x \geq \mathbf{0},\ B'x \leq \mathbf{1}\},$$
$$Q = \{y \in \mathbb{R}^N \mid Ay \leq \mathbf{1},\ y \geq \mathbf{0}\}.$$

Example

$M = \{1, 2, 3\}$, $N = \{4, 5\}$:

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{bmatrix}.$$





Labels

- $x \in P$ has *label* $k \in M \cup N$ if
  - for $k = j \in N$, $(B'x)_j = 1$, or
  - for $k = i \in M$, $x_i = 0$.
- $y \in Q$ has *label* $k \in M \cup N$ if
  - for $k = i \in M$, $(Ay)_i = 1$, or
  - for $k = j \in N$, $y_j = 0$.
- $(x, y) \in P \times Q$ is *completely labeled* if every $k \in M \cup N$ appears as a label of either x or y.
- $(\mathbf{0}, \mathbf{0}) \in P \times Q$ is completely labeled.

3″. $(x, y) \in P \times Q$, $(x, y) \neq (\mathbf{0}, \mathbf{0})$, is an ("un-normalized") Nash equilibrium if and only if (x, y) is completely labeled.

- $(\mathbf{0}, \mathbf{0}) \in P \times Q$: "artificial equilibrium"

Example

$M = \{1, 2, 3\}$, $N = \{4, 5\}$:

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B' = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}.$$

- P:

$$3x_1 + 2x_2 + 3x_3 \leq 1 \tag{4}$$
$$2x_1 + 6x_2 + 1x_3 \leq 1 \tag{5}$$

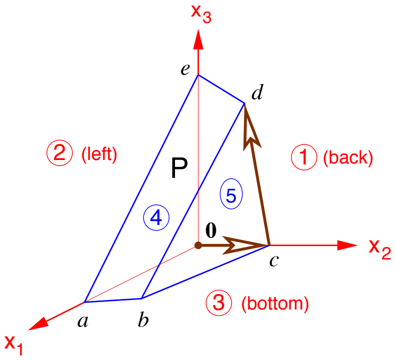
- Q:

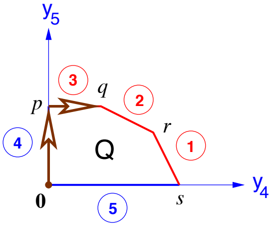
$$3y_4 + 3y_5 \leq 1 \tag{1}$$
$$2y_4 + 5y_5 \leq 1 \tag{2}$$
$$6y_5 \leq 1 \tag{3}$$

Example

$M = \{1, 2, 3\}$, $N = \{4, 5\}$:

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B' = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}.$$





Lemke-Howson

Start with the artificial equilibrium $(\mathbf{0}, \mathbf{0})$ ①②③ ④⑤

1. $(\mathbf{0}, \mathbf{0}) \to (c, \mathbf{0})$ ①③⓹ ④⑤

   Drop label 2 in P for example. Label 5 is picked up.

2. $(c, \mathbf{0}) \to (c, p)$ ①③⑤ ⓷④

   Drop label 5 in Q. Label 3 is picked up.

3. $(c, p) \to (d, p)$ ①⓸⑤ ③④

   Drop label 3 in P. Label 4 is picked up.

4. $(d, p) \to (d, q)$ ①④⑤ ⓶③

   Drop label 4 in Q. Label 2 is picked up.

5. (d, q) is completely labeled, and so it is a Nash equilibrium (after normalization).

Existence, Finiteness, Oddness in Nondegenerate Games

- For $k \in M \cup N$,
  $(x, y) \in P \times Q$ is k-*almost completely labeled*
  if every label in $M \cup N \setminus \{k\}$ appears a label of x or y.
- Suppose that the game is nondegenerate.
- Fix any $k \in M \cup N$.

  Let V_k be the set of k-almost completely labeled vertex pairs.
  - $V_k \neq \emptyset$ as $(\mathbf{0}, \mathbf{0}) \in V_k$.
  - Nash equilibria are all contained in V_k.

Existence, Finiteness, Oddness in Nondegenerate Games

- In V_k, define an edge (x, y)-(x', y') if
  - $|L(x) \cap L(x')| = m - 1$ and $y = y'$, or
  - $x = x'$ and $|L(y) \cap L(y')| = n - 1$.
- If $(x, y) \in V_k$ is completely labeled, then its degree is one.
- If $(x, y) \in V_k$ is such that $L(x) \cap L(y) = \{k\}$, then its degree is two.
- Therefore, the graph only has paths and cycles, and there are an even number of endpoints.
- With $(\mathbf{0}, \mathbf{0})$ excluded, there are an odd number of Nash equilibria.

  In particular, there is at least one Nash equilibrium.

Complementary Pivoting

- Introduce *slack variables*:

$$
\begin{array}{llllll}
3x_1 & +2x_2 & +3x_3 & +s_4 & & =1 \\
2x_1 & +6x_2 & +x_3 & & +s_5 & =1 \\
 & & & & & \\
r_1 & & & +3y_4 & +3y_5 & =1 \\
 & +r_2 & & +2y_4 & +5y_5 & =1 \\
 & & +r_3 & & +6y_5 & =1
\end{array}
$$

  $x \geq \mathbf{0}$, $s \geq \mathbf{0}$, $r \geq \mathbf{0}$, $y \geq \mathbf{0}$.

- A solution (x, s, r, y) is completely labeled if and only if

$$x'r = 0, \quad y's = 0.$$

Complementary Pivoting

- For example:
  $(x, s) = (0, 0, 0, 1, 1)$, $(r, y) = (1, 1, 1, 0, 0)$ is a solution, where $s_4, s_5 > 0$ and $r_1, r_2, r_3 > 0$ are called *basic variables*, while $x_1, x_2, x_3 = 0$ and $y_4, y_5 = 0$ *nonbasic variables*.
- *Pivoting* is a change of the basis where a nonbasic variable *enters* and a basic variable *leaves* the set of basic variables while satisfying the nonnegativity constraint.

Complementary Pivoting

- $\mathbf{0}$:

$$\begin{array}{llllllll} 4 & 3x_1 & +2x_2 & +3x_3 & +s_4 & & =1 \\ 5 & 2x_1 & +6x_2 & +x_3 & & +s_5 & =1 \end{array}$$

- c:

$$\begin{array}{llllllll} 4 & \frac{7}{3}x_1 & & +\frac{8}{3}x_3 & +s_4 & -\frac{1}{3}s_5 & =\frac{2}{3} \\ 2 & \frac{1}{3}x_1 & +x_2 & +\frac{1}{6}x_3 & & +\frac{1}{6}s_5 & =\frac{1}{6} \end{array}$$

- d:

$$\begin{array}{llllllll} 3 & \frac{7}{8}x_1 & & +x_3 & +\frac{3}{8}s_4 & -\frac{1}{8}s_5 & =\frac{1}{4} \\ 2 & \frac{3}{16}x_1 & +x_2 & & -\frac{1}{16}s_4 & +\frac{3}{16}s_5 & =\frac{1}{8} \end{array}$$

Complementary Pivoting

- $\mathbf{0}$:

$$\begin{array}{lllllll}
1 & r_1 & & & +3y_4 & +3y_5 & =1 \\
2 & & +r_2 & & +2y_4 & +5y_5 & =1 \\
3 & & & +r_3 & & +6y_5 & =1
\end{array}$$

- p:

$$\begin{array}{lllllll}
1 & r_1 & & -\frac{1}{2}r_3 & +3y_4 & & =\frac{1}{2} \\
2 & & +r_2 & -\frac{5}{6}r_3 & +2y_4 & & =\frac{1}{6} \\
5 & & & +\frac{1}{6}r_3 & & +y_5 & =\frac{1}{6}
\end{array}$$

- q:

$$\begin{array}{lllllll}
1 & r_1 & -\frac{3}{2}r_2 & +\frac{3}{4}r_3 & & & =\frac{1}{4} \\
4 & & +\frac{1}{2}r_2 & -\frac{5}{12}r_3 & +y_4 & & =\frac{1}{12} \\
5 & & & +\frac{1}{6}r_3 & & +y_5 & =\frac{1}{6}
\end{array}$$

Integer Pivoting

- Computation using floating-point numbers yields rounding errors, and with a large number of pivoting steps they may accumulate to be a large amount.
- If the input data consist of integers, exact computation is possible by *integer pivoting*.

- pivot element = 6:

	x_1	x_2	x_3	s_4	s_5	
4	3	2	3	1	0	1
5	2	$\underline{6}$	1	0	1	1

- Multiply all rows except the pivot row by the pivot element:

	x_1	x_2	x_3	s_4	s_5	
4	18	12	18	6	0	6
5	2	6	1	0	1	1

- Subtract suitable multiples of the pivot row from the other rows to obtain zero entries in the pivot column:

	x_1	x_2	x_3	s_4	s_5	
4	14	0	16	6	-2	4
5	2	6	1	0	1	1

- pivot element $= 6$:

	r_1	r_2	r_3	y_4	y_5	
1	1	0	0	3	3	1
2	0	1	0	2	5	1
3	0	0	1	0	$\underline{6}$	1

- Multiply all rows except the pivot row by the pivot element:

	r_1	r_2	r_3	y_4	y_5	
1	6	0	0	18	18	6
2	0	6	0	12	30	6
3	0	0	1	0	6	1

- Subtract suitable multiples of the pivot row from the other rows to obtain zero entries in the pivot column:

	r_1	r_2	r_3	y_4	y_5	
1	6	0	-3	18	0	3
2	0	6	-5	12	0	1
3	0	0	1	0	6	1

- pivot element = 16:

	x_1	x_2	x_3	s_4	s_5	
4	14	0	$\underline{16}$	6	−2	4
2	2	6	1	0	1	1

- Multiply all rows except the pivot row by the pivot element:

	x_1	x_2	x_3	s_4	s_5	
4	14	0	$\underline{16}$	6	−2	4
2	32	96	16	0	16	16

- Subtract suitable multiples of the pivot row from the other rows to obtain zero entries in the pivot column:

	x_1	x_2	x_3	s_4	s_5	
4	14	0	$\underline{16}$	6	−2	4
2	18	96	0	−6	18	12

- All rows except the pivot row can be divided by the previous pivot element 6:

	x_1	x_2	x_3	s_4	s_5	
4	14	0	$\underline{16}$	6	-2	4
2	3	16	0	-1	3	2

- pivot element $= 12$:

	r_1	r_2	r_3	y_4	y_5	
1	6	0	-3	18	0	3
2	0	6	-5	$\underline{12}$	0	1
5	0	0	1	0	6	1

- Multiply all rows except the pivot row by the pivot element:

	r_1	r_2	r_3	y_4	y_5	
1	72	0	-36	216	0	36
2	0	6	-5	12	0	1
5	0	0	12	0	72	12

- Subtract suitable multiples of the pivot row from the other rows to obtain zero entries in the pivot column:

	r_1	r_2	r_3	y_4	y_5	
1	72	-108	54	0	0	18
2	0	6	-5	12	0	1
5	0	0	12	0	72	12

- All rows except the pivot row can be divided by the previous pivot element 6:

	r_1	r_2	r_3	y_4	y_5	
1	12	-18	9	0	0	3
2	0	6	-5	12	0	1
5	0	0	2	0	12	2

Degenerate Games

- The game is nondegenerate if and only if in any basic feasible solution, the basic variables have positive values.

  Therefore, for nondegenerate games, the minimum ratio test has a unique minimizer.
- If the game is degenerate, the minimum ratio test may have more than one minimizers.
- In this case, arbitrary tie breaking may lead to cycling, so that the algorithm falls into an infinite loop.
- The "lexico-minimum ratio test" is to avoid cycling.
- Suppose that the game is degenerate.

  Instead of considering the original system $B'x + s = \mathbf{1}$, consider

  $$B'x + s = \mathbf{1} + (\varepsilon^1, \ldots, \varepsilon^n)'$$

  with sufficiently small $\varepsilon > 0$.

- After any number of pivoting steps, the system can be written as

$$CB'x + Cs = C\mathbf{1} + C(\varepsilon^1, \ldots, \varepsilon^n)',$$

  where C is the inverse of a basic matrix.

- Write $c_{i0} + c_{i1}\varepsilon^1 + \cdots + c_{in}\varepsilon^n$ for the ith entry of the vector in the right hand, and let d_i be the ith row of the pivoting column.

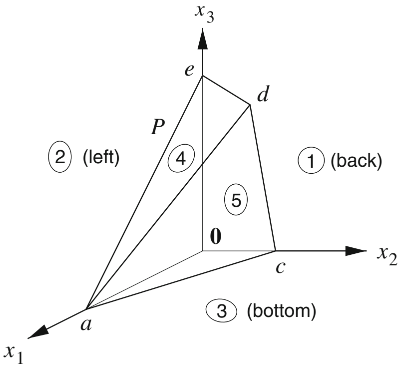
- Lexico-minimum ratio test:
  - Choose the minimizers of c_{i0}/d_i.
  - If more than one, among them choose the minimizers of c_{i1}/d_i.
  - $\ldots$

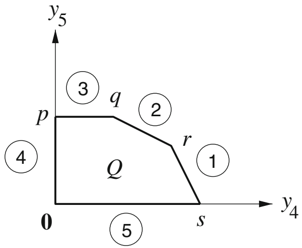
- c_{ik} appears as the coefficient of the slack variable s_k.

Example

$M = \{1, 2, 3\}$, $N = \{4, 5\}$:

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}, \quad B' = \begin{bmatrix} 3 & 2 & 3 \\ \underline{3} & 6 & 1 \end{bmatrix}.$$





(From Avis et al. 2010)

Complementary Pivoting with Lexico-Minimum Test

Initial pivot = 1

- $\mathbf{0}$:

$$
\begin{array}{llllllll}
4 & 3x_1 & +2x_2 & +3x_3 & +s_4 & & & = 1+\varepsilon \\
5 & 3x_1 & +6x_2 & +x_3 & & +s_5 & & = 1+\varepsilon^2
\end{array}
$$

- a:

$$
\begin{array}{lllllll}
4 & & -4x_2 & +2x_3 & +s_4 & -s_5 & = 0+\varepsilon-\varepsilon^2 \\
1 & x_1 & +2x_2 & +\frac{1}{3}x_3 & & +\frac{1}{3}s_5 & = \frac{1}{3}+\frac{1}{3}\varepsilon^2
\end{array}
$$

- a:

$$
\begin{array}{lllllll}
3 & & -2x_2 & +x_3 & +\frac{1}{2}s_4 & -\frac{1}{2}s_5 & = 0+\frac{1}{2}\varepsilon-\frac{1}{2}\varepsilon^2 \\
1 & x_1 & +\frac{8}{3}x_2 & & -\frac{1}{6}s_4 & +\frac{1}{2}s_5 & = \frac{1}{3}-\frac{1}{6}\varepsilon+\frac{1}{2}\varepsilon^2
\end{array}
$$

Complementary Pivoting with Lexico-Minimum Test

- $\mathbf{0}$:

1	r_1			$+3y_4$	$+3y_5$	$=1$
2		$+r_2$		$+2y_4$	$+5y_5$	$=1$
3			$+r_3$		$+6y_5$	$=1$

- p:

1	r_1		$-\frac{1}{2}r_3$	$+3y_4$		$=\frac{1}{2}$
2		$+r_2$	$-\frac{5}{6}r_3$	$+2y_4$		$=\frac{1}{6}$
5			$+\frac{1}{6}r_3$		$+y_5$	$=\frac{1}{6}$

- q:

1	r_1	$-\frac{3}{2}r_2$	$+\frac{3}{4}r_3$			$=\frac{1}{4}$
4		$+\frac{1}{2}r_2$	$-\frac{5}{12}r_3$	$+y_4$		$=\frac{1}{12}$
5			$+\frac{1}{6}r_3$		$+y_5$	$=\frac{1}{6}$

Complementary Pivoting with Lexico-Minimum Test

- d:

$$\begin{array}{llllllll} 3 & \frac{3}{4}x_1 & & +x_3 & +\frac{3}{8}s_4 & -\frac{1}{8}s_5 & = \frac{1}{4}+\frac{3}{8}\varepsilon-\frac{1}{8}\varepsilon^2 \\ 2 & \frac{3}{8}x_1 & +x_2 & & -\frac{1}{16}s_4 & +\frac{3}{16}s_5 & = \frac{1}{8}-\frac{1}{16}\varepsilon+\frac{3}{16}\varepsilon^2 \end{array}$$

Examples of Cycles by "Ad Hoc" Tie Breaking Rules

$M = \{1, 2, 3\}$, $N = \{4, 5, 6\}$

- Tie breaking that picks the variable with the smallest row index in the tableau leads to cycling:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Tie breaking that picks the variable with the smallest variable index leads to cycling:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B' = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$