Nonnegative Matrices I

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Topics in Economic Theory

September 26, 2017

References

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Directed Graphs and Matrices

- A *directed graph* $G = (V, E)$ consists of
  - a nonempty finite set V of *vertices* (or *nodes*) and
  - a subset E of $V \times V$, whose elements are called *edges* (or *arcs*).
- An *undirected graph* can be seen as a special case of a directed graph where $(u, v) \in E$ if and only if $(v, u) \in E$.
- The *adjacency matrix* of a directed graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ is the $n \times n$ matrix A such that $a_{ij} = 1$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$ otherwise.
- The directed graph of an $n \times n$ matrix A is the directed graph (V, E) with $V = \{1, \ldots, n\}$ such that $(i, j) \in E$ if and only if $a_{ij} \neq 0$.

Strongly Connected Components of a Directed Graph

- We say that $u \in V$ has access to $v \in V$ and denote $u \to v$ if $u = v$, or there are $v_0, \ldots, v_k \in V$ with $v_0 = u$ and $v_k = v$ such that $(v_i, v_{i+1}) \in E$ for all $i = 0, \ldots, k-1$.
  - The sequence of edges $(v_0, v_1), \ldots, (v_{k-1}, v_k)$ is called a *walk* of length k.
  - By convention, for all $v \in V$ there is a walk of length 0 from v to v (even if $(v, v) \notin E$).
- The binary relation $\to$ on V is a preorder (or quasi-order), i.e., it is reflexive ($u \to u$ for all $u \in V$) and transitive.
- Define the binary relation $\leftrightarrow$ on V by $u \leftrightarrow v$ if $u \to v$ and $v \to u$ (in which case we say that u and v communicate).

  Then $\leftrightarrow$ is an equivalence relation, i.e., it is symmetric ($u \leftrightarrow v$ if and only if $v \leftrightarrow u$), reflexive, and transitive.

- For each v, its *equivalent class* with respect to $\leftrightarrow$ is

$$[v] = \{u \in V \mid u \leftrightarrow v\}.$$

- The *quotient set* of V by $\leftrightarrow$ is

$$V/\!\leftrightarrow\, = \{[v] \mid v \in V\}.$$

- Elements of $V/\!\leftrightarrow$ are called *strongly connected components* (SCCs) of G.
- If G has a unique SCC, then it is called *strongly connected*.

- The *reduced directed graph* (or *condensation directed graph*) of $G = (V, E)$ is the directed graph $R(G) = (V', E')$ given as follows:
  - $V' = V/\leftrightarrow$,
  - $(V_i, V_j) \in E'$ if and only if $V_i \neq V_j$ and $(v_i, v_j) \in E$ for some $v_i \in V_i$ and $v_j \in V_j$.
- For $V_i, V_j \in V'$,
  $V_i \to V_j$ if and only if $v_i \to v_j$ for some (in fact, for all) $v_i \in V_i$ and $v_j \in V_j$.
- The access preorder $\to$ for $R(G)$ is also a partial order, i.e., it is also anti-symmetric (if $U \to V$ and $V \to U$, then $U = V$).
- For $R(G) = (V', E')$, elements of V' can be ordered as $V_1, \ldots, V_k$ so that if $(V_i, V_j) \in E'$, then $i < j$.

Irreducibility of a Matrix

- Let A be an $n \times n$ matrix, and $G(A)$ the directed graph of A.
- For $n \geq 2$,
  we define A to be *irreducible* if $G(A)$ is strongly connected,
  and *reducible* otherwise.
- For $n = 1$,
  we define A to be irreducible if $A \neq O$, and reducible if
  $A = O$.

- Let $n \geq 2$.

  If A is reducible with $k \geq 2$ SCCs, then there exists a permutation matrix P such that

$$PAP' = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ O & A_{22} & \cdots & A_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ O & \cdots & O & A_{kk} \end{pmatrix},$$

  where $A_{11}, \ldots, A_{kk}$ are square irreducible matrices, which are called the irreducible components of A.

  (The form above is called a Frobenius normal form of A.)

Periodicity of a Strongly Connected Directed Graph

(Definitions may differ across different textbooks.)

Let $G = (V, E)$ be a directed graph.

- A *walk* of length k is a sequence of edges $(v_0, v_1), \ldots, (v_{k-1}, v_k)$ of length k.
- A walk is *closed* if $v_k = v_0$.
- A *simple walk* or *path* is a walk such that all vertices, possibly except v_0 and v_k, are distinct.
- A *cycle* is a simple walk such that $v_k = v_0$.

  A cycle of length k is called a k-cycle.

Let $G = (V, E)$ be a *strongly connected* directed graph with $|V| \geq 2$.

- The *period* of a vertex $v \in V$ is the greatest common divisor of the lengths of all closed walks containing v.

  It equals the GCD of the lengths of all cycles containing v.

- If some vertex of G has period d, then all vertices of G (being strongly connected) have period d.

  The period of G is the period of some (in fact, any) vertex of G.

- G is *aperiodic* if its period is 1, and *periodic* otherwise.

- G is *primitive* if there exists a positive integer k such that for any $u, v \in V$, there exists a walk of length k from u to v.

  It is *imprimitive* if it is not primitive.

- The following are equivalent:
  - G is primitive.
  - G is aperiodic.
  - There exists a positive integer k such that for any $t \geq k$ and for any $u, v \in V$, there exists a walk of length t from u to v.

- The period of G is also called the *index of imprimitivity*.

- Suppose that a strongly connected directed graph G has period d.

  Fix any vertex $u_0 \in V$.

  For each $m = 0, \ldots, d-1$, let V_m be the set of vertices v such that there exists a walk of length $kd + m$ from u_0 to v for some k.

  These sets $V_0, \ldots, V_{d-1}$ constitute a partition of V and are called *cyclic components*.

Periodicity of an Irreducible Matrix

Let A be an $n \times n$ *irreducible* matrix, and $G(A)$ the directed graph of A.

- The *period*, or the *index of imprimitivity*, of A is the period of the index of imprimitivity of $G(A)$.
- A is *aperiodic* if its period is 1, or equivalently, if $G(A)$ is aperiodic.
- If A has period d, then there exists a permutation matrix P such that

$$PAP' = \begin{pmatrix} O & A_0 & O & \cdots & O \\ O & O & A_1 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_{d-2} \\ A_{d-1} & O & O & \cdots & O \end{pmatrix}$$

(The form above is called a *cyclic normal form* of A.)

Powers of an Irreducible Nonnegative Matrix

Let A be an $n \times n$ *nonnegative* matrix.

For a nonnegative integer k, write $a_{ij}^{(k)}$ for the (i, j) entry of A^k (where $A^0 = I$).

- $a_{ij}^{(k)} > 0$ if and only if there is a walk of length k from i to j.

Let A be an $n \times n$ *irreducible* nonnegative matrix.

- For each (i, j), there exists $k = 0, \ldots, n - 1$ such that $a_{ij}^{(k)} > 0$.
- A is *primitive* if there exists k such that $A^k \gg O$.
- A is primitive if and only if it is aperiodic.