Nonnegative Matrices II

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Topics in Economic Theory

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References

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Hawkins-Simon Condition I

Proposition 1

For an $n \times n$ matrix B such that $b_{ij} \leq 0$ for all $i \neq j$, the following conditions are equivalent:

1. *For any $c \geq 0$, there exists $x \geq 0$ such that $Bx = c$.*
2. *There exist $c \gg 0$ and $x \geq 0$ such that $Bx = c$.*
3. *There exist $c \gg 0$ and $x \gg 0$ such that $Bx = c$.*
4. *$|B_k| > 0$ for all $k = 1, \ldots, n$. ("Hawkins-Simon Condition")*
5. *There exist lower and upper triangular matrices L and U with positive diagonals and nonpositive off-diagonals such that $B = LU$.*
6. *B is nonsingular, and $B^{-1} \geq O$.*

Proof

$1 \Rightarrow 2$

- Obvious.

$2 \Rightarrow 3$

- Suppose that $By = d$ for some $y \geq 0$ and $d \gg 0$.
- Let $\alpha > 0$ small enough that $d + \alpha(B\mathbf{1}) \gg 0$, and let $c = d + \alpha(B\mathbf{1}) \gg 0$ and $x = y + \alpha\mathbf{1} \gg 0$.
- Then we have $Bx = By + \alpha(B\mathbf{1}) = c$.

$3 \Rightarrow 4$

- "3 ⇒ 4" holds for $n = 1$.
- Assume that "3 ⇒ 4" holds for $n - 1$.
- We have $b_{11} = \left(c_1 - \sum_{j \neq 1} b_{1j} x_j\right) / x_1 > 0$ since $b_{1j} \leq 0$ for all $j \neq 1$.
- Let

$$D = \begin{pmatrix} 1 & & & & 0 \\ -\frac{b_{21}}{b_{11}} & 1 & & & \\ -\frac{b_{31}}{b_{11}} & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ -\frac{b_{n1}}{b_{11}} & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

- Then

$$DB = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ 0 & & \\ \vdots & B' & \\ 0 & & \end{pmatrix},$$

  where $b'_{ij} = b_{ij} - \frac{b_{i1}}{b_{11}} b_{1j} \leq 0$ for all $i, j \geq 2$, $i \neq j$.

- Also $(Dc)_i = c_i - \frac{b_{i1}}{b_{11}} c_1 > 0$ for all $i \geq 2$.
- Letting $y = (x_2, \ldots, x_n)' \gg 0$ and $d = ((Dc)_2, \ldots, (Dc)_n)' \gg 0$, we have $B'y = d$.
- Therefore, by the induction hypothesis, $|B'_\ell| > 0$ for all $\ell = 1, \ldots, n-1$.
- Hence, for all $k = 1, \ldots, n$, we have $|B_k| = b_{11} |B'_{k-1}| > 0$.

$4 \Rightarrow 5$

- "$4 \Rightarrow 5$" holds for $n = 1$.
- Assume that "$4 \Rightarrow 5$" holds for $n - 1$.
- Suppose that B satisfies 4.
- Let D and B' be as in the previous proof.
- Since $|B'_\ell| = \frac{1}{b_{11}}|B_{\ell+1}| > 0$ for all $\ell = 1, \dots, n-1$, B' is written as $B' = L'U'$ by the induction hypothesis.

- Let

$$L = D^{-1}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & L' & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & L' & \\ * & & & \end{pmatrix},$$

$$U = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ 0 & & \\ \vdots & U' & \\ 0 & & \end{pmatrix},$$

where $\ell_{i1} = \frac{b_{i1}}{b_{11}} \leq 0$ for $i = 2, \ldots, n$.

- Then $B = LU$.

5 ⇒ 6

- L^{-1} and U^{-1} exist and are nonnegative.
- Therefore, $B^{-1} = U^{-1}L^{-1} \geq O$.

6 ⇒ 1

- For any $c \geq 0$, $B^{-1}c \geq 0$. Let $x = B^{-1}c$.

Frobenius Roots

- Let A be an $n \times n$ nonnegative matrix.
- Let

$$M = \{\rho > 0 \mid \rho I - A \text{ satisfies HS}\}.$$

Lemma 1

$M = (m, \infty)$ *for some* m.

- $M \neq \emptyset$.

  Fix any $x \gg 0$. Then for a large $\rho > 0$, we have $\rho x - Ax \gg 0$. For any such ρ, we have $\rho \in M$.

- If $\rho \in M$ and $\rho' > \rho$, then $\rho' \in M$.

  By $\rho \in M$, $(\rho I - A)x \gg 0$ for some $x \geq 0$. Then $(\rho' I - A)x = (\rho I - A)x + (\rho' - \rho)x \gg 0$. Hence, $\rho' \in M$.

- M is open. Because $M = \bigcup_{x \gg 0} \{\rho > 0 \mid (\rho I - A)x \gg 0\}$.

Proposition 2

Let A be an $n \times n$ nonnegative matrix, and M be as defined above. Then there exists $x \geq 0$, $x \neq 0$, such that $Ax = (\inf M)x$.

Proof

- Fix any $c \gg 0$.
- For $\rho \in M$, define $x(\rho) = (\rho I - A)^{-1} c \geq 0$.
- We claim that if $\rho < \rho'$, then $x(\rho) \geq x(\rho')$.

  Indeed, since $(\rho I - A)(x(\rho) - x(\rho')) = (\rho' - \rho)x(\rho') \geq 0$, we have $x(\rho) - x(\rho') = (\rho' - \rho)(\rho I - A)^{-1} x(\rho') \geq 0$.

- Now, let $\{\rho_k\} \subset M$ be such that $\rho_k \searrow \inf M$.

  Then $x(\rho_k) \leq x(\rho_{k+1})$.

- Let $\eta_k = \|x(\rho_k)\|$, where $\eta_k \leq \eta_{k+1}$.

- If $\{\eta_k\}$ is bounded above, then $\lim_{k\to\infty} x(\rho_k) = x \geq 0$ exists.

  Then by letting $k \to \infty$ in $(\rho_k I - A)x(\rho_k) = c$, we have $((\inf M)I - A)x = c$, which implies that $\inf M \in M$, contradicting the openness of M.

- Thus, $\eta_k \to \infty$.

- Let $y_k = x(\rho_k)/\eta_k$.

  Since $\{y_k\}$ is contained in a compact set, it has a convergent subsequence, again denoted by $\{y_k\}$, with a limit $y \geq 0, \neq 0$.

- By construction, $(\rho_k I - A)y_k = c/\eta_k$.

  Let $k \to \infty$. Then we have $((\inf M)I - A)y = 0$, or $Ay = (\inf M)y$.

Perron-Frobenius Theorem I

Proposition 3

Let A be an $n \times n$ nonnegative matrix.

1. $\lambda(A)$ *is an eigenvalue of* A*, and there exists a nonnegative eigenvector that belongs to* $\lambda(A)$*.*
2. *If* $A \geq B \geq O$*, then* $\lambda(A) \geq \lambda(B)$*.*

Proof

1.

- Let $\omega \in \mathbb{C}$ be any eigenvalue of A, and $z \in \mathbb{C}^n$, $z \neq 0$, be its eigenvector.

  It suffices to show that $|\omega| \leq \inf M$.

- For all $i = 1, \ldots, n$, $|\omega||z_i| = |\omega z_i| = \left|\sum_j a_{ij} z_j\right| \leq \sum_j a_{ij}|z_j|$, we have $A\hat{z} \geq |\omega|\hat{z}$, or $(|\omega|I - A)\hat{z} \leq 0$, where $\hat{z} = (|z_1|, \ldots, |z_n|)'$.

- If $|\omega| \in M$, then $(|\omega|I - A)^{-1}$ would exist and be nonnegative, so that $\hat{z} \leq 0$, contradicting $\hat{z} \geq 0$, $\hat{z} \neq 0$.

- Therefore, $|\omega| \notin M$, and hence $|\omega| \leq \inf M$.

2.

- Let $A \geq B \geq O$.

  It suffices to show that $M(A) \subset M(B)$.

- Let $\rho \in M(A)$. Then $(\rho I - A)x \gg 0$ for some $x \geq 0$.

- For any such $x \geq 0$, $\rho x \gg Ax \geq Bx$, or $(\rho I - B)x \gg 0$, which implies that $\rho \in M(B)$.

Hawkins-Simon Condition II

Proposition 4

For an $n \times n$ nonnegative matrix A and a positive real number ρ, the following conditions are equivalent:

1. $\rho I - A$ *satisfies the Hawkins-Simon condition.*
2. $\lambda(A) < \rho$.

Hawkins-Simon Condition III

Proposition 5

For an $n \times n$ nonnegative matrix A and a positive real number ρ, the following conditions are equivalent:

1. $\rho I - A$ *satisfies the Hawkins-Simon condition.*
2. $\lim_{k\to\infty} \frac{1}{\rho} \sum_{\ell=0}^{k} \frac{1}{\rho^\ell} A^\ell$ *exists (which is equal to* $(\rho I - A)^{-1}$*).*

Proof

$1 \Rightarrow 2$

- By 1, $(\rho I - A)^{-1}$ exists and is nonnegative.
- Let $T_k = \frac{1}{\rho} \sum_{\ell=0}^{k} \frac{1}{\rho^\ell} A^\ell$.
  Then $T_k \leq T_{k+1}$.
- Since $(\rho I - A)T_k = T_k(\rho I - A) = I - \frac{1}{\rho^{k+1}} A^{k+1} \leq I$, $T_k \leq (\rho I - A)^{-1}$.
- Therefore, $\lim_{k\to\infty} T_k = T$ exists.
- Then $\lim_{k\to\infty} \frac{1}{\rho^k} A^k = \lim_{k\to\infty} \rho(T_k - T_{k-1}) = O$.
- Therefore, we have $(\rho I - A)T = T(\rho I - A) = I$, and hence $T = (\rho I - A)^{-1}$.

$2 \Rightarrow 1$

- By 2, $T = \lim_{k\to\infty} T_k$ exists, which is nonnegative.
- As previously, $\lim_{k\to\infty} \frac{1}{\rho^k} A^k = \lim_{k\to\infty} \rho(T_k - T_{k-1}) = O$.
- Then letting $k \to \infty$ in $(\rho I - A)T_k = T_k(\rho I - A) = I - \frac{1}{\rho^{k+1}} A^{k+1}$, we have $(\rho I - A)T = T(\rho I - A) = I$, and hence $(\rho I - A)^{-1} = T \geq 0$.

Perron-Frobenius Theorem II

Proposition 6

Let A be an $n \times n$ irreducible nonnegative matrix.

1. $\lambda(A) > 0$, $\lambda(A)$ *is an eigenvalue of* A, *and there exists a positive eigenvector that belongs to* $\lambda(A)$.
2. *An eigenvector that belongs to* $\lambda(A)$ *is unique (up to multiplication).*
3. *If* $Ay = \mu y$, $\mu \geq 0$, *for some* $y \geq 0$, $y \neq 0$, *then* $\mu = \lambda(A)$.
4. *If* $A \geq B \geq O$ *and* $A \neq B$, *then* $\lambda(A) > \lambda(B)$.

Proof

1.

- Since A is irreducible, $A\mathbf{1} \gg 0$, or $\min_i(A\mathbf{1})_i > 0$.

  Since $(\min_i(A\mathbf{1})_i)I\mathbf{1} \leq A\mathbf{1}$, $\min_i(A\mathbf{1})_i \notin M$.
  ($\because$ If $\min_i(A\mathbf{1})_i \in M$, then $(\min_i(A\mathbf{1})_i)I - A)^{-1} \geq O$, and we would have $\mathbf{1} \leq 0$.)

  Therefore, $\lambda(A) = \inf M \geq \min_i(A\mathbf{1})_i > 0$.

- Let $x \geq 0$, $x \neq 0$, be such that $Ax = \lambda(A)x$.

  Then $(I + A)x = (1 + \lambda(A))x$.

  Since $I + A$ is irreducible and aperiodic, $(I + A)^m \gg O$ for some m.

  Since $(1 + \lambda(A))^m x = (I + A)^m x \gg 0$, where $1 + \lambda > 0$, we have $x \gg 0$.

2.

- Let $Ax = \lambda(A)x$, $x \gg 0$, and $Ay = \lambda(A)y$, $y \neq 0$.
- Let $\theta = \min_i \frac{y_i}{x_i}$, and let $z = y - \theta x \geq 0$, where $z \not\gg 0$.
- Then $Az = \lambda(A)z$.
- As we have shown, if $z \neq 0$, we would have $z \gg 0$.
- Therefore, $z = 0$, so that $y = \theta x$.

3.

- Since A' is also irreducible, we can take an $x \gg 0$ such that $A'x = \lambda(A')x$, where $\lambda(A') = \lambda(A)$.
- Let $Ay = \mu y$, $y \geq 0$, $y \neq 0$.
- Then

$$\mu(x{\cdot}y) = x{\cdot}\mu y = x{\cdot}Ay = (A'x){\cdot}y = (\lambda(A)x){\cdot}y = \lambda(A)(x{\cdot}y),$$

  and hence $(\mu - \lambda(A))(x \cdot y) = 0$.
- Since $x \gg 0$, $y \geq 0$, and $y \neq 0$, $x \cdot y \neq 0$.

  Hence $\mu = \lambda(A)$.

Periodicity and Eigenvalues of a Nonnegative Matrix

Proposition 7

Let A be an $n \times n$ irreducible nonnegative matrix with period d. Then A has d distinct eigenvalues ρ such that $|\rho| = \lambda(A)$.