Review on Common Beliefs

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Topics in Economic Theory

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Papers

- Monderer, D. and D. Samet (1989). "Approximating Common Knowledge with Common Beliefs," Games and Economic Behavior 1, 170-190.
- Kajii, A. and S. Morris (1997a). "The Robustness of Equilibria to Incomplete Information," Econometrica 65, 1283-1309.
- Kajii, A. and S. Morris (1997b). "Refinements and Higher Order Beliefs: A Unified Survey."
- Oyama, D. and S. Takahashi (2015). "Generalized Belief Operator and the Impact of Small Probability Events on Higher Order Beliefs."

Type Spaces

- Type space $(T_i, \pi_i)_{i \in I}$:
  - T_i: set of i's types (countable)
  - $\pi_i \colon T_i \to \Delta(T_{-i})$: i's belief
- $T = \prod_{i \in I} T_i$, $T_{-i} = \prod_{j \neq i} T_j$
- If there is a common prior $P \in \Delta(T)$ with $P(t_i) = P(\{t_i\} \times T_{-i}) > 0$ for all i and t_i,

$$\pi_i(t_i)(E_{-i}) = \frac{P(\{t_i\} \times E_{-i})}{P(t_i)}$$

  for $E_{-i} \subset T_{-i}$.
- $\mathcal{T}_i = 2^{T_i}$, $\mathcal{T} = \prod_{i \in I} \mathcal{T}_i$,
  with a generic element $\mathbf{E} = (E_i)_{i \in I} \in \mathcal{T}$.

p-Belief Operator

- $B_i^p \colon \mathcal{T} \to \mathcal{T}_i$:

$$B_i^p(\mathbf{E}) = \{t_i \in T_i \mid t_i \in E_i \text{ and } \pi_i(t_i)(E_{-i}) \geq p\},$$

where $E_{-i} = \prod_{j \neq i} E_j$.

Proposition 1

1. $B_i^p(\mathbf{E}) \subset E_i$.
2. *If* $\mathbf{E} \subset \mathbf{F}$, *then* $B_i^p(\mathbf{E}) \subset B_i^p(\mathbf{F})$.
3. *If* $\mathbf{E}^0 \supset \mathbf{E}^1 \supset \cdots$, *then* $B_i^p(\bigcap_{k=0}^{\infty} \mathbf{E}^k) = \bigcap_{k=0}^{\infty} B_i^p(\mathbf{E}^k)$.

(3. If $E_{-i}^0 \supset E_{-i}^1 \supset \cdots$, then $\pi_i(t_i)(\bigcap_{k=0}^{\infty} E_{-i}^k) = \lim_{k \to \infty} \pi_i(t_i)(E_{-i}^k)$.)

Common $\mathbf{p}$-Belief (Iteration)

- For $\mathbf{p} \in [0,1]^I$,

$$B_*^{\mathbf{p}}(\mathbf{E}) = (B_i^{p_i}(\mathbf{E}))_{i \in I},$$
$$C^{\mathbf{p}}(\mathbf{E}) = \bigcap_{k=1}^{\infty} (B_*^{\mathbf{p}})^k(\mathbf{E}).$$

Definition 1

$\mathbf{E} \in \mathcal{T}$ is *common* $\mathbf{p}$*-belief at* $t \in T$ if $t_i \in C_i^{\mathbf{p}}(\mathbf{E})$ for all $i \in I$.

Common $\mathbf{p}$-Belief (Fixed Point)

Definition 2

$\mathbf{F} \in \mathcal{T}$ is $\mathbf{p}$-*evident* if

$$F_i \subset B_i^{\mathbf{p}}(\mathbf{F}) \text{ for all } i \in I.$$

(Equivalent to the condition with "$F_i = B_i^{\mathbf{p}}(\mathbf{F})$".)

Definition 3

$\mathbf{E} \in \mathcal{T}$ is *common* $\mathbf{p}$-*belief at* $t \in T$ if there exists a $\mathbf{p}$-evident event profile $\mathbf{F}$ such that

$$t_i \in F_i \subset B_i^{\mathbf{p}}(\mathbf{E}) \text{ for all } i \in I.$$

(Equivalent to the condition with "$t_i \in F_i \subset E_i$".)

Equivalence

Proposition 2

$C^{\mathbf{p}}(\mathbf{E})$ *is* $\mathbf{p}$*-evident, i.e.,* $C_i^{\mathbf{p}}(\mathbf{E}) \subset B_i^{\mathbf{p}}(C^{\mathbf{p}}(\mathbf{E}))$ *for all* $i \in I$*.*

Proof.

$C^{\mathbf{p}}(\mathbf{E}) = \bigcap_{k=1}^{\infty} B_*^{\mathbf{p}}((B_*^{\mathbf{p}})^{k-1}(\mathbf{E})) = B_*^{\mathbf{p}}(\bigcap_{k=1}^{\infty}(B_*^{\mathbf{p}})^{k-1}(\mathbf{E}))$. □

Proposition 3

$C^{\mathbf{p}}(\mathbf{E})$ *is the largest* $\mathbf{p}$*-evident event profile in* $\mathbf{E}$*, i.e.,*
if $\mathbf{F} \subset \mathbf{E}$ *and* $\mathbf{F} \subset B_*^{\mathbf{p}}(\mathbf{F})$*, then* $\mathbf{F} \subset C^{\mathbf{p}}(\mathbf{E})$*.*

Proof.

First, $\mathbf{F} \subset B_*^{\mathbf{p}}(\mathbf{F}) \subset B_*^{\mathbf{p}}(\mathbf{E})$.

Suppose $\mathbf{F} \subset (B_*^{\mathbf{p}})^k(\mathbf{E})$. Then
$\mathbf{F} \subset B_*^{\mathbf{p}}(\mathbf{F}) \subset B_*^{\mathbf{p}}((B_*^{\mathbf{p}})^k(\mathbf{E})) = (B_*^{\mathbf{p}})^{k+1}(\mathbf{E})$. □

Equivalence

Proposition 4

The two definitions are equivalent, i.e.,

$$t_i \in C_i^{\mathbf{p}}(\mathbf{E}) \text{ for all } i \in I$$
$$\iff \exists\, \mathbf{F} : \mathbf{p}\text{-evident s.t. } t_i \in F_i \subset B_i^{\mathbf{p}}(\mathbf{E}) \text{ for all } i \in I.$$

Proof.

- "$\Rightarrow$":

  $C^{\mathbf{p}}(\mathbf{E})$ is $\mathbf{p}$-evident by Proposition 2, and $C^{\mathbf{p}}(\mathbf{E}) \subset B_*^{\mathbf{p}}(\mathbf{E})$.

- "$\Leftarrow$":

  $\mathbf{F} \subset C^{\mathbf{p}}(\mathbf{E})$ by Proposition 3.

Example: Email Game

- $T_1 = T_2 = \{0, 1, 2, \ldots\}$
- $\pi_1 \colon T_1 \to \Delta(T_2)$:

$$\pi_1(t_2|t_1) = \begin{cases} 1 & \text{if } t_1 = 0,\ t_2 = 0 \\ \frac{1}{2-\varepsilon} & \text{if } t_1 \geq 1,\ t_2 = t_1 - 1 \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_1 \geq 1,\ t_2 = t_1 \\ 0 & \text{otherwise} \end{cases}$$

  $\pi_2 \colon T_2 \to \Delta(T_1)$:

$$\pi_2(t_1|t_2) = \begin{cases} \frac{1}{2-\varepsilon} & \text{if } t_2 = 0,\ t_1 = 0 \\ \frac{1}{2-\varepsilon} & \text{if } t_2 \geq 1,\ t_1 = t_2 \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } t_2 \geq 0,\ t_1 = t_2 + 1 \\ 0 & \text{otherwise} \end{cases}$$

- Let $E_1 = T_1 \setminus \{0\}$ and $E_2 = T_2$, and $p_i \geq \frac{1}{2}$.

Connection to Games 1

- Type space $(T_i, \pi_i)_{i \in I}$
- Players $I = 1, \ldots, |I|$
- Binary actions $A_i = \{0, 1\}$
- $\mathbf{F} = (F_i)_{i \in i} \in \mathcal{T}$ is identified with the (pure) strategy profile σ such that $\sigma_i(t_i) = 1$ if and only if $t_i \in F_i$.
- Fix $\mathbf{E} \in \mathcal{T}$.
- Incomplete information game $\mathbf{u^p}$:

  If $t_i \in E_i$: for all t_{-i} with $\pi_i(t_i)(t_{-i}) > 0$,

$$u_i^{p_i}(1, a_{-i}, t_i, t_{-i}) = \begin{cases} 1 - p_i & \text{if } a_{-i} = \mathbf{1}_{-i}, \\ -p_i & \text{otherwise}, \end{cases}$$

$$u_i^{p_i}(0, a_{-i}, t_i, t_{-i}) = 0.$$

  If $t_i \notin E_i$: 0 is a dominant action.

- $B_i^{p_i}(E_i, \mathbf{F}_{-i})$ is the (largest) best response to $\mathbf{F}_{-i}$ (play 1 if indifferent).
- $C_i^{\mathbf{p}}(\mathbf{E})$ is the largest strategy that survives the iterated elimination of strictly dominated strategies.
- $\mathbf{F}$ is an equilibrium if and only if $\mathbf{F} \subset \mathbf{E}$ and $\mathbf{F}$ is $\mathbf{p}$-evident.
- $C^{\mathbf{p}}(\mathbf{E})$ is the largest equilibrium.

Connection to Games 2

- Players $I = 1, \ldots, |I|$
- Actions A_i (finite)
- Complete information game $\mathbf{g}$, $g_i \colon A \to \mathbb{R}$
- $a^* \in A$ is a $\mathbf{p}$-*dominant equilibrium* of $\mathbf{g}$ if

  $$a_i^* \in br_i(\lambda_i)$$

  for any $\lambda_i \in \Delta(A_{-i})$ such that $\lambda_i(a_{-i}^*) \geq p_i$.
- Incomplete information game $\mathbf{u}$, $u_i \colon A \times T \to \mathbb{R}$
- Let

  $$T_i^{g_i} = \{t_i \in T_i \mid u_i(a, t_i, t_{-i}) = g_i(a) \text{ for all } a \in A \text{ and } \text{for all } t_{-i} \in T_{-i} \text{ with } \pi_i(t_i)(t_{-i}) > 0\},$$

  and $\mathbf{T}^{\mathbf{g}} = (T_i^{g_i})_{i \in I} \in \mathcal{T}$.

Lemma 5

Suppose that a^* is a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$. Then $\mathbf{u}$ has an equilibrium σ such that $\sigma(a^*|t) = 1$ for all $t \in C^{\mathbf{p}}(T^{\mathbf{g}})$.

Proof

- Σ_i^*: set of all strategies σ_i such that $\sigma_i(a_i^*|t_i) = 1$ for all $t_i \in C_i^{\mathbf{p}}(T^{\mathbf{g}})$

  $\Sigma^* = \prod_{i\in I} \Sigma_i^*$, $\Sigma_{-i}^* \prod_{j\in I} \Sigma_j^*$
- Σ^* is nonempty, convex, and compact (in appropriate topology).
- Define the correspondence $\beta_i^* \colon \Sigma_{-i}^* \to \Sigma_i^*$ by

  $$\beta_i^*(\sigma_{-i}) = \{\sigma_i \in \Sigma_i^* \mid \sigma_i(a_i|t_i) > 0 \Rightarrow a_i \in BR_i(\sigma_{-i})(t_i)\},$$

  and $\beta^* \colon \Sigma^* \to \Sigma^*$ by $\beta^*(\sigma) = \prod_{i\in I} \beta_i^*(\sigma_{-i})$.
- β^* is convex- and compact-valued and upper semi-continuous.

Proof

- It remains to show that $\beta_i^*(\sigma_{-i}) \neq \emptyset$ for all $i \in I$ and all $\sigma_{-i} \in \Sigma_{-i}^*$.
- Let $t_i \in C_i^{\mathbf{p}}(T^{\mathbf{g}})$ ($\subset T_i^{g_i}$) and $\sigma_{-i} \in \Sigma_{-i}^*$.

  We want to show that $a_i^* \in BR_i(\sigma_{-i})(t_i)$.
- $C^{\mathbf{p}}(T^{\mathbf{g}})$ is $\mathbf{p}$-evident by Proposition 3, so that $C_i^{\mathbf{p}}(T^{\mathbf{g}}) \subset B_i^{\mathbf{p}}(C^{\mathbf{p}}(T^{\mathbf{g}}))$.
- Hence,

$$\pi_i(t_i)(\{t_{-i} \mid \sigma_{-i}(a_{-i}^*|t_{-i}) = 1\}) \geq \pi_i(t_i)(C_{-i}^{\mathbf{p}}(T^{\mathbf{g}})) \geq p_i,$$

  where the last inequality follows from $t_i \in B_i^{\mathbf{p}}(C^{\mathbf{p}}(T^{\mathbf{g}}))$.
- Since a^* is $\mathbf{p}$-dominant, this implies that $a_i^* \in BR_i(\sigma_{-i})(t_i)$.

Proof

- Therefore, by Kakutani's Fixed Point Theorem, β^* has a fixed point in Σ^*, which is an equilibrium of $\mathbf{u}$.

Proposition 6

Suppose that a^* *is a strict equilibrium of* $\mathbf{g}$*.*

For any $\delta > 0$*, there exists* $\varepsilon > 0$ *such that*
for any $P \in \Delta(T)$ *such that* $P(C^{\mathbf{p}}(T^{\mathbf{g}})) \geq 1 - \varepsilon$ *for any* $\mathbf{p} \ll \mathbf{1}$*,*
there exists an equilibrium σ *of* $(T, P, \mathbf{u})$ *such that*
$P(\{t \in T \mid \sigma(a^*|t) = 1\}) \geq 1 - \delta$*.*

- A strict equilibrium is $\mathbf{p}$-dominant for some $\mathbf{p} \ll \mathbf{1}$.
- The proposition holds even with non common priors P_i.

Critical Path Theorem (Kajii and Morris 1997a)

- $P \in \Delta(T)$: common prior

Theorem 1

For $\mathbf{p} \in [0,1]^I$*, suppose that* $\sum_{i \in I} p_i < 1$*, and let* $\kappa(\mathbf{p}) = (1 - \min_{i \in I} p_i)/(1 - \sum_{i \in I} p_i)$*.*

Then for any type space $((T_i)_{i \in I}, P)$ *and any* $\mathbf{E} \in \mathcal{T}$*,*

$$P\left(\prod_{i \in I} C_i^{\mathbf{p}}(\mathbf{E})\right) \geq 1 - \kappa(\mathbf{p})\left(1 - P\left(\prod_{i \in I} E_i\right)\right).$$

- If $\sum_{i \in I} p_i < 1$, $P\left(\prod_{i \in I} C_i^{\mathbf{p}}(\mathbf{E})\right) \to 1$ as $P\left(\prod_{i \in I} E_i\right) \to 1$.
- In the Email game example where $p_1, p_2 \geq 1/2$, we have $C_i^{\mathbf{p}}(\mathbf{E}) = \emptyset$ while $P\left(\prod_{i \in I} E_i\right) = 1 - \varepsilon$.