Bayes Correlated Equilibrium

Daisuke Oyama

Game Theory I

October 9, 2023

Papers

- Bergemann, D. and S. Morris (2016). "Bayes Correlated Equilibrium and the Comparison of Information Structures in Games," Theoretical Economics 11, 487-522.
- Bergemann, D. and S. Morris (2019). "Information Design: A Unified Perspective," Journal of Economic Literature 57, 44-95.

(We often deviate from these papers in notation.)

Base Game

We fix the base game:

- $I = \{1, \dots, |I|\}$: Set of players
- A_i: Finite set of actions for i $\qquad (A = A_1 \times \cdots \times A_{|I|})$
- Θ: Finite set of states
- $\mu \in \Delta(\Theta)$: Probability distribution over Θ

  Assume full support: $\mu(\theta) > 0$ for all $\theta \in \Theta$
- $u_i \colon A \times \Theta \to \mathbb{R}$: i's payoff function

We identify the base game with $(u_i)_{i \in I}$.

Bayes Correlated Equilibrium (without Prior Information)

- $\nu \in \Delta(A \times \Theta)$: Outcome

Definition 1

$\nu \in \Delta(A \times \Theta)$ is a *Bayes correlated equilibrium* of $(u_i)_{i \in I}$ if it satisfies

1. *consistency*: $\sum_a \nu(a, \theta) = \mu(\theta)$ for all $\theta \in \Theta$; and
2. *obedience*: for all $i \in I$,

$$\sum_{a_{-i}, \theta} \nu((a_i, a_{-i}), \theta) u_i((a_i, a_{-i}), \theta) \geq \sum_{a_{-i}, \theta} \nu((a_i, a_{-i}), \theta) u_i((a_i', a_{-i}), \theta)$$

for all $a_i, a_i' \in A_i$.

Information Structures

- T_i: Set of types of player i (finite or countably infinite)

  $(T = T_1 \times \cdots \times T_{|I|})$
- $\pi \in \Delta(T \times \Theta)$: Common prior, consistent with μ:

$$\sum_t \pi(t, \theta) = \mu(\theta)$$

  for all $\theta \in \Theta$.
- Together with the base game, an information structure $\mathcal{T} = ((T_i)_{i \in I}, \pi)$ defines an incomplete information game.

- A strategy profile $\sigma = (\sigma_i)_{i \in I}$ is a Bayes Nash equilibrium of $\mathcal{T}$ if for all $i \in I$ and all $t_i \in T_i$,

$$\begin{aligned}
&\sigma_i(t_i)(a_i) > 0 \\
&\Rightarrow \sum_{t_{-i},\theta} \pi(t_{-i},\theta|t_i) \sum_{a_{-i}} \left(\prod_{j\neq i} \sigma_j(t_j)(a_j)\right) u_i((a_i,a_{-i}),\theta) \\
&\qquad \geq \sum_{t_{-i},\theta} \pi(t_{-i},\theta|t_i) \sum_{a_{-i}} \left(\prod_{j\neq i} \sigma_j(t_j)(a_j)\right) u_i((a_i',a_{-i}),\theta)
\end{aligned}$$

  for all $a_i, a_i' \in A_i$.
  - $\pi(t_i) = \sum_{t_{-i},\theta} \pi((t_i,t_{-i}),\theta)$
  - $\pi(t_{-i},\theta|t_i) = \frac{\pi((t_i,t_{-i}),\theta)}{\pi(t_i)}$

Partial Implementation

- An information structure $\mathcal{T}$ and a strategy profile σ *induce* an outcome $\nu \in \Delta(A \times \Theta)$ if

$$\nu(a, \theta) = \sum_t \pi(t, \theta) \prod_i \sigma_i(t_i)(a_i)$$

for each $a \in A$ and $\theta \in \Theta$.

Definition 2

$\nu \in \Delta(A \times \Theta)$ is *partially implementable* if there exist an information structure $\mathcal{T}$ and a Bayes Nash equilibrium σ of $\mathcal{T}$ that induce ν.

Partial Implementation

Proposition 1

$\nu \in \Delta(A \times \Theta)$ *is partially implementable if and only if it is a Bayes correlated equilibrium.*

Proof

"Only if" part

- Suppose that an information structure $\mathcal{T} = ((T_i)_{i \in I}, \pi)$ and a Bayes Nash equilibrium σ of $\mathcal{T}$ that induce ν.
- First, ν satisfies consistency: for all $\theta \in \Theta$,

$$\sum_a \nu(a,\theta) = \sum_a \sum_t \pi(t,\theta) \prod_i \sigma_i(t_i)(a_i) = \sum_t \pi(t,\theta) = \mu(\theta).$$

- Second, fix $i \in I$ and $a_i, a_i' \in A$.
- By optimality, for all $t_i \in T_i$, if $\sigma_i(t_i)(a_i) > 0$, then

$$\sum_{t_{-i},\theta} \pi(t_{-i},\theta|t_i) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j)\right) u_i((a_i, a_{-i}),\theta)$$
$$\geq \sum_{t_{-i},\theta} \pi(t_{-i},\theta|t_i) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j)\right) u_i((a_i', a_{-i}),\theta).$$

- Multiply both sides by $\pi(t_i)\sigma_i(t_i)(a_i)$ and sum them over t_i: we have

$$\sum_{a_{-i},\theta}\sum_{t}\pi(t,\theta)\left(\prod_j \sigma_j(t_j)(a_j)\right)u_i((a_i,a_{-i}),\theta)$$
$$\geq \sum_{a_{-i},\theta}\sum_{t}\pi(t,\theta)\left(\prod_j \sigma_j(t_j)(a_j)\right)u_i((a_i',a_{-i}),\theta),$$

i.e.,

$$\sum_{a_{-i},\theta}\nu(a,\theta)u_i((a_i,a_{-i}),\theta) \geq \sum_{a_{-i},\theta}\nu(a,\theta)u_i((a_i',a_{-i}),\theta).$$

This means that ν is a Bayes correlated equilibrium.

“If” part

- Suppose that ν is a Bayes correlated equilibrium.
- Let $\mathcal{T} = ((T_i)_{i \in I}, \pi)$ be the “direct mechanism”:
  - $T_i = \{a_i \in A_i \mid \sum_{a_{-i},\theta} \nu((a_i, a_{-i}), \theta) > 0\}$ for each $i \in I$, and
  - $\pi = \nu$.
- Define σ by $\sigma_i(t_i)(a_i) = 1$ if $t_i = a_i$, and $\sigma_i(t_i)(a_i) = 0$ otherwise.
- Clearly, σ induces ν:

  For all a, θ,

$$\sum_t \pi(t, \theta) \prod_j \sigma_j(t_j)(a_j) = \pi(a, \theta) = \nu(a, \theta).$$

- It remains to show that σ is a Bayes Nash equilibrium of $\mathcal{T}$.

- For any i, $t_i = a_i$, and a_i',
  the interim payoff (multiplied by $\pi(t_i)$) is

$$\sum_{t_{-i},\theta} \pi((t_i, t_{-i}), \theta) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j) \right) u_i((a_i', a_{-i}), \theta)$$
$$= \sum_{a_{-i},\theta} \pi((t_i, a_{-i}), \theta) u_i((a_i', a_{-i}), \theta)$$
$$= \sum_{a_{-i},\theta} \nu((a_i, a_{-i}), \theta) u_i((a_i', a_{-i}), \theta).$$

- For all a_i, a_i',

  since ν is a Bayes correlated equilibrium and hence
  $\sum_{a_{-i},\theta} \nu((a_i, a_{-i}), \theta) u_i((a_i, a_{-i}), \theta) \geq$
  $\sum_{a_{-i},\theta} \nu((a_i, a_{-i}), \theta) u_i((a_i', a_{-i}), \theta)$,

  if $\sigma_i(t_i)(a_i) > 0$ and hence $t_i = a_i$, then
  $\sum_{t_{-i},\theta} \pi((t_i, t_{-i}), \theta) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j) \right) u_i((a_i, a_{-i}), \theta) \geq$
  $\sum_{t_{-i},\theta} \pi((t_i, t_{-i}), \theta) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j) \right) u_i((a_i', a_{-i}), \theta)$.

Best-Case Optimal Information Design

- An information designer chooses an information structure $\mathcal{T}$ to maximize her objective $V: A \times \Theta \to \mathbb{R}$, where she assumes that the best equilibrium (for the designer) is played in $\mathcal{T}$:

$$\sup_{\mathcal{T}} \max_{\sigma \in E(\mathcal{T})} \sum_{t,\theta} \pi(t,\theta) \sum_{a} \left(\prod_i \sigma_i(t_i)(a_i)\right) V(a,\theta),$$

  where $E(\mathcal{T})$ is the set of Bayes Nash equilibria of $\mathcal{T}$.

- By Proposition 1, this is equivalent to

$$\max_{\nu \in BCE} \sum_{a,\theta} \nu(a,\theta) V(a,\theta),$$

  where BCE is the set of Bayes correlated equilibria.

  $\cdots$ Finite linear program

- Linear program:

$$\max_{\nu} \sum_{a,\theta} V(a,\theta)\nu(a,\theta)$$

subject to

$$\sum_{a_{-i},\theta} (u_i((a_i', a_{-i}), \theta) - u_i((a_i, a_{-i}), \theta))\nu((a_i, a_{-i}), \theta) \leq 0 \qquad (i, a_i, a_i')$$

$$\sum_{a} \nu(a,\theta) = \mu(\theta), \qquad (\theta)$$

$$\nu(a,\theta) \geq 0. \qquad (a,\theta)$$

Investment Example: Single Player

- $I = \{1\}$
- $A_1 = \{NI, I\}$
- $\Theta = \{B, G\}$
- $\mu(B) = \mu(G) = \frac{1}{2}$
- Payoffs:

	B	G
NI	0	0
I	-1	x

$0 < x < 1$

- Designer's objective: maximize the probability of I:

$$V(a_1, \theta) = \begin{cases} 0 & \text{if } a_1 = NI, \\ 1 & \text{if } a_1 = I. \end{cases}$$

- Zero information:
  - NI: 0
  - I: $\frac{1}{2} \times (-1) + \frac{1}{2} \times x < 0$

  $\Rightarrow$ Play NI
- Complete information:
  - At $\theta = B$, play NI
  - At $\theta = G$, play I

  $\Rightarrow$ Prob$(I) = \frac{1}{2}$

BCE

u_1	B	G
NI	0	0
I	-1	x

ν	B	G
NI	$\frac{1}{2}(1-p_B)$	$\frac{1}{2}(1-p_G)$
I	$\frac{1}{2}p_B$	$\frac{1}{2}p_G$

- Obedience for I:

$$\frac{1}{2}p_B \times (-1) + \frac{1}{2}p_G \times x \geq 0$$

- Obedience for NI:

$$0 \geq \frac{1}{2}(1-p_B) \times (-1) + \frac{1}{2}(1-p_G) \times x$$

- Since $x < 1$, the obedience condition for NI is not binding.

Optimal BCE

u_1	B	G
NI	0	0
I	-1	x

ν^*	B	G
NI	$\frac{1}{2}(1-x)$	0
I	$\frac{1}{2}x$	$\frac{1}{2}$

- Prob$(I) = \frac{1}{2}(1+x)$

Investment Example: Two Players

- $I = \{1, 2\}$
- $A_1 = A_2 = \{NI, I\}$
- $\Theta = \{B, G\}$, $\mu(B) = \mu(G) = \frac{1}{2}$
- Payoffs:

B	NI	I
NI	0	0
I	-1	$-1 + \varepsilon$

G	NI	I
NI	0	0
I	x	$x + \varepsilon$

$0 < x < 1$, $-x < \varepsilon < \frac{1}{2}(1 - x)$

- Designer's objective: maximize the number of players who invest.

- Zero information: (NI, NI)
- Complete information:
  - At $\theta = B$, (NI, NI)
  - At $\theta = G$, (I, I)

Optimal BCE

By symmetry, we focus on symmetric outcomes:

B	NI	I
NI	p_{0B}	p_{1B}
I	p_{1B}	p_{2B}

G	NI	I
NI	p_{0G}	p_{1G}
I	p_{1G}	p_{2G}

- Obedience for I:

$$p_{1B} \times (-1) + p_{2B} \times (-1 + \varepsilon) + p_{1G} \times x + p_{2G} \times (x + \varepsilon) \geq 0$$

- Obedience for NI:

$$0 \geq p_{0B} \times (-1) + p_{1B} \times (-1 + \varepsilon) + p_{0G} \times x + p_{1G} \times (x + \varepsilon)$$

- Consistency:

$$p_{0B} + 2p_{1B} + p_{2B} = \frac{1}{2}$$
$$p_{0G} + 2p_{1G} + p_{2G} = \frac{1}{2}$$

- Under consistency, obedience for I implies obedience for NI:

$$\begin{aligned}
&-\left[p_{0B} \times (-1) + p_{1B} \times (-1+\varepsilon) + p_{0G} \times x + p_{1G} \times (x+\varepsilon)\right] \\
&-\left[p_{1B} \times (-1) + p_{2B} \times (-1+\varepsilon) + p_{1G} \times x + p_{2G} \times (x+\varepsilon)\right] \\
&= \frac{1}{2}(1-x) - \varepsilon(p_{1B} + p_{2B} + p_{1G} + p_{2G}) \\
&\geq \frac{1}{2}(1-x) - \varepsilon > 0
\end{aligned}$$

Linear Program

- Maximize

$$2p_{1B} + 2p_{2B} + 2p_{1G} + 2p_{2G}$$

- subject to

$$p_{1B} + (1-\varepsilon)p_{2B} - xp_{1G} - (x+\varepsilon)p_{2G} \leq 0 \qquad (I)$$

$$p_{0B} + 2p_{1B} + p_{2B} = \frac{1}{2} \qquad (B)$$

$$p_{0G} + 2p_{1G} + p_{2G} = \frac{1}{2} \qquad (G)$$

$$p_{0B}, p_{1B}, p_{2B}, p_{0G}, p_{1G}, p_{2G} \geq 0$$

One may alternatively solve the dual problem:

- Minimize

$$\frac{1}{2}\lambda_B + \frac{1}{2}\lambda_G$$

- subject to

$$\lambda_B \geq 0 \qquad (p_{0B})$$
$$\lambda_I + 2\lambda_B \geq 2 \qquad (p_{1B})$$
$$(1-\varepsilon)\lambda_I + \lambda_B \geq 2 \qquad (p_{2B})$$
$$\lambda_G \geq 0 \qquad (p_{0G})$$
$$-x\lambda_I + 2\lambda_G \geq 2 \qquad (p_{1G})$$
$$-(x+\varepsilon)\lambda_I + \lambda_G \geq 2 \qquad (p_{2G})$$
$$\lambda_I \geq 0$$

Case of Strategic Complementarity: $\varepsilon > 0$

- Let $p_{0G} = p_{1G} = 0$ and $p_{2G} = \frac{1}{2}$.
- Then let $p_{1B} = 0$, and solve $(1-\varepsilon)p_{2B} - (x+\varepsilon)\frac{1}{2} = 0$ (I).

B	NI	I
NI	$\frac{1-x-2\varepsilon}{2(1-\varepsilon)}$	0
I	0	$\frac{x+\varepsilon}{2(1-\varepsilon)}$

G	NI	I
NI	0	0
I	0	$\frac{1}{2}$

- $V^* = 1 + \frac{x+\varepsilon}{1-\varepsilon}$ $(> 1 + x)$

Case of Strategic Substitutability: $\varepsilon < 0$

Case (i) $\varepsilon \leq \frac{1}{2} - x$ (i.e., $|\varepsilon| \geq x - \frac{1}{2}$)

B	NI	I
NI	$\frac{1-2x-2\varepsilon}{2}$	$\frac{x+\varepsilon}{2}$
I	$\frac{x+\varepsilon}{2}$	0

G	NI	I
NI	0	0
I	0	$\frac{1}{2}$

- $V^* = 1 + x + \varepsilon \; (< 1 + x)$

Case (ii) $\varepsilon \geq \frac{1}{2} - x$ (i.e., $|\varepsilon| \leq x - \frac{1}{2}$)

B	NI	I
NI	0	$\frac{1-x-2\varepsilon}{2(1-2\varepsilon)}$
I	$\frac{1-x-2\varepsilon}{2(1-2\varepsilon)}$	$\frac{-1+2x+2\varepsilon}{2(1-2\varepsilon)}$

G	NI	I
NI	0	0
I	0	$\frac{1}{2}$

- $V^* = 1 + \frac{x}{1-2\varepsilon}$ $(< 1 + x)$