Correlated Equilibrium

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Complete Information Games

- $I = \{1, \ldots, |I|\}$: set of players
- A_i: finite set of actions for i
- $g_i \colon A \to \mathbb{R}$: payoff function for i

We identify the game with $\mathbf{g} = (g_i)$.

Nash Equilibrium

- A profile of mixed actions $x = (x_i)_{i \in I} \in \prod_i \Delta(A_i)$ is a Nash equilibrium of $\mathbf{g}$ if for all i,

$$\sum_a \prod_j x_j(a_j) g_i(a) \geq \sum_a x_i'(a_i) \prod_{j \neq i} x_j(a_j) g_i(a)$$

for all $x_i' \in \Delta(A_i)$, or equivalently, if for all i,

$$x_i(a_i) > 0$$
$$\Rightarrow \sum_{a_{-i}} \prod_{j \neq i} x_j(a_j) g_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \prod_{j \neq i} x_j(a_j) g_i(a_i', a_{-i})$$

for all $a_i, a_i' \in A_i$.

Correlated Equilibrium

Definition 1

An action distribution $\xi \in \Delta(A)$ is a correlated equilibrium of $\mathbf{g}$ if for all i,

$$\sum_{a_{-i} \in A_{-i}} \xi(a_i, a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \xi(a_i, a_{-i}) g_i(a_i', a_{-i})$$

for all $a_i, a_i' \in A_i$.

- A profile of mixed actions $x = (x_i)_{i \in I} \in \prod_i \Delta(A_i)$ is a Nash equilibrium of $\mathbf{g}$ if and only if the action distribution $\xi \in \Delta(A)$ defined by $\xi(a) = \prod_i x_i(a_i)$ is a correlated equilibrium of $\mathbf{g}$.

Information Structures

$\mathcal{T} = ((T_i)_{i \in I}, \pi)$

- T_i: Set of types of player i (finite or countably infinite)
- $\pi \in \Delta(T)$: Common prior

Incomplete Information Games

- Payoff functions $\mathbf{g}$ and information structure $\mathcal{T}$ define an incomplete information game (with no payoff uncertainty).
- Player i's strategy: $\sigma_i \colon T_i \to \Delta(A_i)$

  Σ_i: set of i's strategies
- Player i's ex ante payoff for strategy profile $\sigma = (\sigma_j)_{j \in I}$:
  $\sum_t \pi(t) \sum_a \left(\prod_j \sigma_j(t_j)(a_j) \right) g_i(a)$
- Type t_i's interim payoff for a_i and σ_{-i}:
  $\sum_{t_{-i}} \pi(t_{-i}|t_i) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j) \right) g_i(a_i, a_{-i})$
  - $\pi(t_i) = \sum_{t_{-i}} \pi(t_i, t_{-i})$
  - $\pi(t_{-i}|t_i) = \frac{\pi(t_i, t_{-i})}{\pi(t_i)}$

- A strategy profile σ induces an action distribution $\xi \in \Delta(A)$ by

$$\xi(a) = \sum_t \pi(t) \left(\prod_j \sigma_j(t_j)(a_j) \right).$$

Bayes-Nash Equilibrium

- A strategy profile σ is a Bayes-Nash equilibrium of $(\mathbf{g}, \mathcal{T})$ if for all i,

$$\sum_t \pi(t) \sum_a \left(\prod_j \sigma_j(t_j)(a_j)\right) g_i(a)$$
$$\geq \sum_t \pi(t) \sum_a \sigma_i'(t_i)(a_i) \left(\prod_{j\neq i} \sigma_j(t_j)(a_j)\right) g_i(a)$$

for all $\sigma_i' \in \Sigma_i$, or equivalently, if for all i and all $t_i \in T_i$,

$$\sigma_i(t_i)(a_i) > 0$$
$$\Rightarrow \sum_{t_{-i}} \pi(t_{-i}|t_i) \sum_{a_{-i}} \left(\prod_{j\neq i} \sigma_j(t_j)(a_j)\right) g_i(a_i, a_{-i})$$
$$\geq \sum_{t_{-i}} \pi(t_{-i}|t_i) \sum_{a_{-i}} \left(\prod_{j\neq i} \sigma_j(t_j)(a_j)\right) g_i(a_i', a_{-i})$$

for all $a_i, a_i' \in A_i$.

Characterization of Correlated Equilibrium

- Fix $\mathbf{g}$.
- What action distributions are induced by some Bayes-Nash equilibrium of some information structure?
- $=$ Correlated equilibria of $\mathbf{g}$

Characterization of Correlated Equilibrium

Proposition 1

For any information structure $\mathcal{T}$ *and any Bayes-Nash equilibrium* σ *of* $(\mathbf{g}, \mathcal{T})$*, the action distribution* ξ *induced by* $(\mathcal{T}, \sigma)$ *is a correlated equilibrium of* $\mathbf{g}$*.*

Proof

- Take any information structure $\mathcal{T}$ and any Bayes-Nash equilibrium σ of $(\mathbf{g}, \mathcal{T})$.
- Fix $i \in I$ and $a_i, a_i' \in A_i$.
- By optimality, for all $t_i \in T_i$, if $\sigma_i(t_i)(a_i) > 0$, then

$$\sum_{t_{-i}} \pi(t_{-i}|t_i) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j)\right) g_i(a_i, a_{-i})$$
$$\geq \sum_{t_{-i}} \pi(t_{-i}|t_i) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j)\right) g_i(a_i', a_{-i}).$$

- Multiply both sides by $\pi(t_i)\sigma_i(t_i)(a_i)$ and sum them over t_i:

$$\sum_{a_{-i}} \sum_{t} \pi(t) \left(\prod_j \sigma_j(t_j)(a_j)\right) g_i(a_i, a_{-i})$$
$$\geq \sum_{a_{-i}} \sum_{t} \pi(t) \left(\prod_j \sigma_j(t_j)(a_j)\right) g_i(a'_i, a_{-i}).$$

- Let $\xi \in \Delta(A)$ be induced by $(\mathcal{T}, \sigma)$:

$$\xi(a) = \sum_{t} \pi(t) \left(\prod_j \sigma_j(t_j)(a_j)\right).$$

- Thus, we have

$$\sum_{a_{-i}} \xi(a_i, a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \xi(a_i, a_{-i}) g_i(a'_i, a_{-i}).$$

This means that ξ is a correlated equilibrium of $\mathbf{g}$.

Characterization of Correlated Equilibrium

Proposition 2

For any correlated equilibrium $\xi \in \Delta(A)$ *of* $\mathbf{g}$*, there exist an information structure* $\mathcal{T}$ *and a Bayes-Nash equilibrium* σ *of* $(\mathbf{g}, \mathcal{T})$ *that induce* ξ*.*

Proof

- Take any correlated equilibrium $\xi \in \Delta(A)$ of $\mathbf{g}$.
- Define the information structure $\mathcal{T} = ((T_i)_{i \in I}, \pi)$ by
  - $T_i = \{a_i \in A_i \mid \sum_{a_{-i}} \xi(a_i, a_{-i}) > 0\}$ for each $i \in I$, and
  - $\pi = \xi$.
- Define σ by $\sigma_i(t_i)(a_i) = 1$ if $t_i = a_i$, and $\sigma_i(t_i)(a_i) = 0$ otherwise.
- Clearly, σ induces ξ:

  For all a,

  $$\sum_t \pi(t) \prod_j \sigma_j(t_j)(a_j) = \pi(a) = \xi(a).$$

- It remains to show that σ is a Bayes-Nash equilibrium of $(\mathbf{g}, \mathcal{T})$.

- For any i, $t_i = a_i$, and a'_i,
  the interim payoff (multiplied by $\pi(t_i)$) is

$$\sum_{t_{-i}} \pi(t_i, t_{-i}) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j)\right) g_i(a'_i, a_{-i})$$
$$= \sum_{a_{-i}} \pi(t_i, a_{-i}) g_i(a'_i, a_{-i}) = \sum_{a_{-i}} \xi(a_i, a_{-i}) g_i(a'_i, a_{-i}).$$

- For all a_i, a'_i,

  since ξ is a correlated equilibrium and hence
  $\sum_{a_{-i}} \xi(a_i, a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \xi(a_i, a_{-i}) g_i(a'_i, a_{-i})$,

  if $\sigma_i(t_i)(a_i) > 0$ and hence $t_i = a_i$, then
  $\sum_{t_{-i}} \pi(t_i, t_{-i}) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j)\right) g_i(a_i, a_{-i}) \geq$
  $\sum_{t_{-i}} \pi(t_i, t_{-i}) \sum_{a_{-i}} \left(\prod_{j \neq i} \sigma_j(t_j)(a_j)\right) g_i(a'_i, a_{-i})$.

"Revelation Principle"

- $T_i = \{a_i \in A_i \mid \sum_{a_{-i}} \xi(a_i, a_{-i}) > 0\}$ in the information structure constructed in the proof:

  Types, or signals, are interpreted as "action recommendations".

  $\cdots$ "Direct mechanism"

- The condition in the definition of correlated equilibrium: $\sum_{a_{-i}} \xi(a_i, a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \xi(a_i, a_{-i}) g_i(a_i', a_{-i})$

  $\cdots$ Incentive to follow, or obey, the action recommendation

  $\cdots$ "Obedience" condition