Email Game

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Game Theory I

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2×2 Coordination Games of Complete Information

- $I = \{1, 2\}$
- $A_1 = A_2 = \{NI, I\}$
- Payoffs $\mathbf{g} = (g_i)_{i \in I}$:

	NI	I
NI	0	0
I	$\theta^* - 1$	θ^*

$\frac{1}{2} < \theta^* < 1$

- (I, I): Risk-dominant equilibrium

  i.e., I is a strict best response to $\frac{1}{2}[NI] + \frac{1}{2}[I]$.
- (NI, NI): Risk-dominated equilibrium

Email Game Incomplete Information Perturbations

- $T_1 = T_2 = \{1, 2, \dots\}$
- $P \in \Delta(T)$:

$t_1 \backslash t_2$	1	2	3	4	$\cdots$
1	ε				
2	$\varepsilon(1-\varepsilon)$	$\varepsilon(1-\varepsilon)^2$			
3		$\varepsilon(1-\varepsilon)^3$	$\varepsilon(1-\varepsilon)^4$		
4			$\varepsilon(1-\varepsilon)^5$	$\varepsilon(1-\varepsilon)^6$	
5				$\varepsilon(1-\varepsilon)^7$	$\ddots$
$\vdots$					$\ddots$

$0 < \varepsilon < 1$

- Payoffs:

$$u_i((a_i, a_j), (t_i, t_j)) = \begin{cases} 0 & \text{if } i = 1,\ t_i = 1, \text{ and } a_i = NI \\ 1 & \text{if } i = 1,\ t_i = 1, \text{ and } a_i = I \\ g_i(a_i, a_j) & \text{otherwise} \end{cases}$$

  - $t_1 = 1$: "crazy" type
  - Other types: "normal" type

Contagion

$t_1 \backslash t_2$	1	2	3	4	$\cdots$
1	ε				
2	$\varepsilon(1-\varepsilon)$	$\varepsilon(1-\varepsilon)^2$			
3		$\varepsilon(1-\varepsilon)^3$	$\varepsilon(1-\varepsilon)^4$		
4			$\varepsilon(1-\varepsilon)^5$	$\varepsilon(1-\varepsilon)^6$	
5				$\varepsilon(1-\varepsilon)^7$	$\ddots$
$\vdots$					$\ddots$

- By construction, $t_1 = 1$ plays I as a dominant action.
- For $t_2 = 1$,
  $(P(t_1 = 1|t_2 = 1), P(t_1 = 2|t_2 = 1)) = \left(\frac{1}{2-\varepsilon}, \frac{1-\varepsilon}{2-\varepsilon}\right)$.

  $t_2 = 1$ believes that player 1 plays I with probability at least $\frac{1}{2-\varepsilon} > \frac{1}{2}$.

  $\Rightarrow t_2 = 1$ plays I as a unique rationalizable action.

$t_1 \backslash t_2$	1	2	3	4	$\cdots$
1	ε				
2	$\varepsilon(1-\varepsilon)$	$\varepsilon(1-\varepsilon)^2$			
3		$\varepsilon(1-\varepsilon)^3$	$\varepsilon(1-\varepsilon)^4$		
4			$\varepsilon(1-\varepsilon)^5$	$\varepsilon(1-\varepsilon)^6$	
5				$\varepsilon(1-\varepsilon)^7$	$\ddots$
$\vdots$					$\ddots$

- For $\tau \geq 1$, assume that both players of all types $t_i \leq \tau$ play I.
- For $t_1 = \tau + 1$,

  $$(P(t_2 = \tau | t_1 = \tau+1), P(t_2 = \tau+1 | t_1 = \tau+1)) = \left(\frac{1}{2-\varepsilon}, \frac{1-\varepsilon}{2-\varepsilon}\right).$$

  By the induction hypothesis, $t_1 = \tau + 1$ believes that player 2 plays I with probability at least $\frac{1}{2-\varepsilon} > \frac{1}{2}$.

  $\Rightarrow t_1 = \tau + 1$ plays I as a unique rationalizable action.
- Similarly, $t_2 = \tau + 1$ plays I as a unique rationalizable action.

- By induction, both players of all types play I as a unique rationalizable action.

Symmetric Version

$t_1 \backslash t_2$	1	2	3	4	5	$\cdots$
1		$\varepsilon \frac{1}{2}$				
2	$\varepsilon \frac{1}{2}$		$\varepsilon(1-\varepsilon)\frac{1}{2}$			
3		$\varepsilon(1-\varepsilon)\frac{1}{2}$		$\varepsilon(1-\varepsilon)^2\frac{1}{2}$		
4			$\varepsilon(1-\varepsilon)^2\frac{1}{2}$		$\varepsilon(1-\varepsilon)^3\frac{1}{2}$	
5				$\varepsilon(1-\varepsilon)^3\frac{1}{2}$		$\ddots$
$\vdots$					$\ddots$	

- $t_1 = 1$, $t_2 = 1$: Crazy types
- Generated by the following signal structure:
  - m drawn from $\mathbb{Z}_+$ according to the distribution $\varepsilon(1-\varepsilon)^m$;
  - noise $(\xi_1, \xi_2) = (1, 2), (2, 1)$ with probability $\frac{1}{2}$ each;
  - player i receives signal $t_i = m + \xi_i$.

Discussion

1. Discontinuity of strategic behavior at complete information limit

- In the email game incomplete information perturbation,

$$P(\text{every player } i \text{ knows that his payoffs are given by the original game}) = 1 - \varepsilon.$$

$\cdots$ "ε-elaboration" of $\mathbf{g}$

- For $\varepsilon > 0$, the game has a unique equilibrium, which plays I everywhere.
- For $\varepsilon = 0$, the game (of complete information) has two strict equilibria (and one totally mixed equilibrium).
- Failure of lower semi-continuity of the equilibrium correspondence at $\varepsilon = 0$.

2. Full implementation with payoff perturbations

- Say that an action profile a^* in a complete information game $\mathbf{g}$ is *fully implementable with payoff perturbations* if for any sufficiently small $\varepsilon > 0$, there exists an ε-elaboration of $\mathbf{g}$ such that playing a^* everywhere is a unique Bayes Nash equilibrium.
- The Email game demonstrates that a risk-dominant equilibrium in 2×2 coordination games is fully implementable with payoff perturbations.
- Extensions:
  - Strict $\mathbf{p}$-dominant equilibrium with $\sum_{i \in I} p_i \leq 1$ (Kajii and Morris 1997)
  - Monotone potential maximizer in supermodular monotone potential games (Frankel, Morris, and Pauzner 2003)
  - Oyama and Takahashi (2020)

3. Robustness to incomplete information (Kajii and Morris 1997)

- The risk-dominated equilibrium of $\mathbf{g}$ is not robust to incomplete information,

  i.e., for any $\varepsilon > 0$, there is an ε-elaboration of $\mathbf{g}$ such that the risk-dominated equilibrium is never played.

- $\nu \in \Delta(A)$ is *robust* to incomplete information if

  for any $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that any ε-elaboration with $\varepsilon \leq \bar{\varepsilon}$ has an equilibrium σ such that $\|\nu_\sigma - \nu\| \leq \delta$

  (where $\nu_\sigma \in \Delta(A)$ is the outcome induced by σ).

- By definition, if an outcome is fully implementable, then no other outcome is robust;

  if an outcome is robust, then no other outcome is fully implementable.

- A risk-dominant equilibrium in 2×2 coordination games is robust.

  More generally, a $\mathbf{p}$-dominant equilibrium with $\sum_{i \in I} p_i < 1$ is robust (Kajii and Morris 1997).

- Monotone potential maximizer in supermodular monotone potential games is robust (Morris and Ui 2005).