Global Games I

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Game Theory I

October 12, 2023

Binary Actions, Continuum of Players, Symmetric Payoffs

- Morris, S. and H. S. Shin (2003). “Global Games: Theory and Applications,” in M. Dewatripont, L. P. Hansen, and S. J. Turnovsky, eds., Advances in Economics and Econometrics: Theory and Applications: Eighth World Congress, Volume 1, Cambridge University Press: Cambridge, Section 2.2.

General Prior, "Common Values"

Global game $G(\kappa)$:

- Continuum of players
- Actions: $a \in \{0, 1\}$
- (Common) payoff function: $u \colon \{0, 1\} \times [0, 1] \times \mathbb{R} \to \mathbb{R}$.
  - $u(a, \ell, \theta)$: Payoff to action a when proportion ℓ of opponents play action 1 and the state is θ
  - Define $d(\ell, \theta) = u(1, \ell, \theta) - u(0, \ell, \theta)$
- $\theta \in \mathbb{R} \sim$ density p: continuous, interval support
- Each player i observes a private signal $x_i = \theta + \kappa\varepsilon_i$.
  - $\kappa > 0$
  - $\varepsilon_i \sim$ density f: continuous, interval support

Assumptions

1. Action monotonicity:

   $d(\ell, \theta)$ is nondecreasing in ℓ.

2. State monotonicity:

   $d(\ell, \theta)$ is nondecreasing in θ.

3. Dominance regions:

   There exist $\underline{\theta}, \overline{\theta} \in \mathbb{R}$ such that

   - if $\theta \leq \underline{\theta}$, then $d(\ell, \theta) < 0$ for all $\ell \in [0, 1]$; and
   - if $\theta \geq \overline{\theta}$, then $d(\ell, \theta) > 0$ for all $\ell \in [0, 1]$.

4. Strict Laplacian state monotonicity:

   There exists a unique θ^* solving $\int_0^1 d(\ell, \theta) d\ell = 0$.

Laplacian Actions

Let $d(\ell)$ be a complete information game with a continuum of symmetric players and binary actions.

- Action 1 is a Laplacian action if

$$\int_0^1 d(\ell)d\ell > 0.$$

- Action 0 is a Laplacian action if

$$\int_0^1 d(\ell)d\ell < 0.$$

- That is, action a is a Laplacian action if it is a strict best response to the uniform belief over the proportion of players who play a.

  $\cdots$ Generalization of risk-dominance

Potential

- The function

$$v(\ell) = \int_0^{\ell} d(\ell')d\ell'$$

is called a *potential function* of the game $d(\ell)$.

  - $v'(\ell) = d(\ell)$

- Suppose that $d(\ell)$ is nondecreasing.

$\Rightarrow v(\ell)$ is convex, and hence is maximized at $\ell = 0$ or $\ell = 1$.

- $\ell = 1$ (all playing action 1) is a (unique) potential maximizer if

$$\int_0^1 d(\ell)d\ell > 0.$$

- $\ell = 0$ (all playing action 0) is a (unique) potential maximizer if

$$\int_0^1 d(\ell)d\ell < 0.$$

Example: Linear Payoffs

- Assume $d(\ell, \theta) = \ell + \theta - 1$
- $\underline{\theta} = -\delta$ and $\overline{\theta} = 1 + \delta$ for $\delta > 0$ small
- $\int_0^1 d(\ell, \theta) d\ell = \theta - \frac{1}{2}$
- $\theta^* = \frac{1}{2}$

Example: Regime Change Game

- Assume

$$d(\ell,\theta) = \begin{cases} -c & \text{if } \ell \leq 1-\theta \\ 1-c & \text{if } \ell > 1-\theta \end{cases}$$

  where $0 < c < 1$

- $\underline{\theta} = -\delta$ and $\overline{\theta} = 1+\delta$ for $\delta > 0$ small
- $\int_0^1 d(\ell,\theta)d\ell = \theta - c$
- $\theta^* = c$

Uniform Prior, "Private Values"

- For κ small, $G(\kappa)$ is approximated by the "simplified version" $G^*(\kappa)$ where θ follows uniform prior (instead of general prior) and d depends on signal x_i (instead of state θ).
  - When κ small, x_i is close to θ, and
  - the prior does not matter.

Simplified global game $G^*(\kappa)$:

- $\theta \sim$ Uniform prior on some large interval $[a, b]$
- $d(\ell, x)$: Payoff difference for $a = 1$ over $a = 0$ when proportion ℓ of opponents play action 1 and the signal is x

Uniqueness

Proposition 1

The essentially unique strategy s *surviving iterated deletion of strictly dominated strategies in* $G^*(\kappa)$ *satisfies* $s(x) = 0$ *for all* $x < \theta^*$ *and* $s(x) = 1$ *for all* $x > \theta^*$*.*

Uniform Prior

- Suppose that θ follows a uniform distribution p on a large interval $[a, b]$: $p(\theta) = \frac{1}{b-a}$.
- The conditional density $f(\theta|x)$ of θ given signal $x = \theta + \kappa\varepsilon$ (for x away from the boundary):

$$
\begin{aligned}
f(\theta|x) &= \frac{\frac{1}{\kappa} f\left(\frac{x-\theta}{\kappa}\right) p(\theta)}{\int \frac{1}{\kappa} f\left(\frac{x-\theta'}{\kappa}\right) p(\theta') d\theta'} \\
&= \frac{f\left(\frac{x-\theta}{\kappa}\right)}{\int f\left(\frac{x-\theta'}{\kappa}\right) d\theta'} \\
&= \frac{f\left(\frac{x-\theta}{\kappa}\right)}{\int \kappa f(z) dz} \\
&= \frac{1}{\kappa} f\left(\frac{x-\theta}{\kappa}\right).
\end{aligned}
$$

Heuristic Argument—Contagion

- By Dominance regions, players observing a signal above some threshold $\overline{\xi}_1$ play 1.
- Assuming that players with signals above $\overline{\xi}_1$ play 1,

  by Action monotonicity and State monotonicity, players observing a signal above some threshold $\overline{\xi}_2$ play 1, where $\overline{\xi}_2 \leq \overline{\xi}_1$.
- $\dots$
- We have $\overline{\xi}_1 \geq \overline{\xi}_2 \geq \cdots \searrow \overline{\xi}^*$.
- Similarly, from below we have $\underline{\xi}_1 \leq \underline{\xi}_2 \leq \cdots \nearrow \underline{\xi}^*$.

- In the limit, a player with signal $\overline{\xi}^*$ when opponents play 1 above $\overline{\xi}^*$ and 0 below $\overline{\xi}^*$ must be indifferent between playing 1 and 0.
- A player with signal x when opponents play 1 above x and 0 below x has a uniform belief over the proportion of opponents playing 1.
- By Strict Laplacian state monotonicity, it must be that $\overline{\xi}^* = \theta^*$.
- The same applies to $\underline{\xi}^*$: thus $\underline{\xi}^* = \theta^*$.
- Hence, uniqueness holds with $\overline{\xi}^* = \underline{\xi}^* = \theta^*$.

Laplacian Belief

- Suppose that players play the k-threshold strategy.

  (k-threshold strategy plays action 1 iff $x = \theta + \kappa\varepsilon \geq k$ iff $\varepsilon \geq \frac{k-\theta}{\kappa}$)

- Proportion of players who play 1 given θ:

$$1 - F\left(\frac{k-\theta}{\kappa}\right)$$

- Distribution of the proportion of players who play 1 conditional on signal $x_i = k$:

$$
\begin{aligned}
&P\left(1 - F\left(\frac{k-\theta}{\kappa}\right) \leq \ell \,\middle|\, x_i = k\right) \\
&= \int_{-\infty}^{k-\kappa F^{-1}(1-\ell)} \left(1 - F\left(\frac{k-\theta}{\kappa}\right)\right) \frac{1}{\kappa} f\left(\frac{x-\theta}{\kappa}\right) d\theta \\
&= \int_{-\infty}^{F^{-1}(1-\ell)} (1 - F(z)) f(z) dz \\
&= P(1 - F(\varepsilon_i) \leq \ell) \\
&= P(\varepsilon_i \geq F^{-1}(1-\ell)) \\
&= 1 - F(F^{-1}(1-\ell)) = \ell.
\end{aligned}
$$

$\cdots$ Uniform distribution

Proof of Proposition 1 (under additional assumption)

- Write $D^*_\kappa(x,k)$ for the expected payoff gain when the player observes signal x and others play the k-threshold strategy:

$$D^*_\kappa(x,k) = \int_{-\infty}^{\infty} d\left(1 - F\left(\frac{k-\theta}{\kappa}\right), x\right) \frac{1}{\kappa} f\left(\frac{x-\theta}{\kappa}\right) d\theta$$
$$= \int_{-\infty}^{\infty} d\left(1 - F\left(z + \frac{k-x}{\kappa}\right), x\right) f(z) dz.$$

  (k-threshold strategy plays action 1 iff $x = \theta + \kappa\varepsilon \geq k$ iff $\varepsilon \geq \frac{k-\theta}{\kappa}$)

- By Action monotonicity and State monotonicity, $D^*_\kappa(x,k)$ is nondecreasing in x and nonincreasing in k.
- We assume that $D^*_\kappa(x,k)$ is continuous in (x,k).
  - Satisfied if $d(\ell, x)$ is continuous in (ℓ, x).
  - Satisfied in the regime change game.

- Define $\underline{\xi}_0, \underline{\xi}_1, \underline{\xi}_2, \ldots$ by $\underline{\xi}_0 = -\infty$ and

$$\underline{\xi}_{n+1} = \inf\{x \mid D_\kappa^*(x, \underline{\xi}_n) = 0\}.$$

  By continuity, $D_\kappa^*(\underline{\xi}_{n+1}, \underline{\xi}_n) = 0$.

- Then we have $\underline{\xi}_0 \le \underline{\xi}_1 \le \underline{\xi}_2 \le \cdots$:
  - $\underline{\xi}_0 = -\infty < \underline{\theta} \le \underline{\xi}_1$ by Dominance regions.
  - Suppose that $\underline{\xi}_{n-1} \le \underline{\xi}_n$.

    If $x < \underline{\xi}_n$, then $D_\kappa^*(x, \underline{\xi}_n) \le D_\kappa^*(x, \underline{\xi}_{n-1}) \le D_\kappa^*(\underline{\xi}_n, \underline{\xi}_{n-1}) = 0$ since $D_\kappa^*(x, k)$ is nonincreasing in k and nondecreasing in x.

    But by the definition of $\underline{\xi}_n$, we must have $D_\kappa^*(x, \underline{\xi}_n) < 0$.

    By $D_\kappa^*(\underline{\xi}_{n+1}, \underline{\xi}_n) = 0$, we have $\underline{\xi}_{n+1} \ge \underline{\xi}_n$.

- Symmetrically, define $\overline{\xi}_0, \overline{\xi}_1, \overline{\xi}_2, \ldots$ by $\overline{\xi}_0 = \infty$ and

$$\overline{\xi}_{n+1} = \sup\{x \mid D^*_\kappa(x, \overline{\xi}_n) = 0\}.$$

By continuity, $D^*_\kappa(\overline{\xi}_{n+1}, \overline{\xi}_n) = 0$.

- Then we have $\overline{\xi}_0 \geq \overline{\xi}_1 \geq \overline{\xi}_2 \geq \cdots$.
- Then a strategy s survives n rounds of iterated deletion of strictly dominated strategies if and only if

$$s(x) = \begin{cases} 0 & \text{if } x < \underline{\xi}_n, \\ 1 & \text{if } x > \overline{\xi}_n. \end{cases}$$

- Now let $n \to \infty$.

Then $\underline{\xi}_n$ converges to some $\underline{\xi}_*$ $(> \underline{\theta})$ and $\overline{\xi}_n$ converges to some $\overline{\xi}_*$ $(< \overline{\theta})$.

- By continuity, $D^*_\kappa(\underline{\xi}_*, \underline{\xi}_*) = 0$ and $D^*_\kappa(\overline{\xi}_*, \overline{\xi}_*) = 0$.

- For any x and κ, we have

$$D^*_\kappa(x,x) = \int_{-\infty}^{\infty} d\left(1 - F\left(\frac{x-\theta}{\kappa}\right), x\right)\frac{1}{\kappa} f\left(\frac{x-\theta}{\kappa}\right) d\theta$$
$$= \int_0^1 d(\ell, x) d\ell$$

(by change of variables $\ell = 1 - F\left(\frac{x-\theta}{\kappa}\right)$).

- Therefore, by Strict Laplacian state monotonicity, $D^*_\kappa(\underline{\xi}_*, \underline{\xi}_*) = 0$ and $D^*_\kappa(\overline{\xi}_*, \overline{\xi}_*) = 0$ imply that $\underline{\xi}_* = \overline{\xi}_* = \theta^*$.