Smallest Equilibrium Implementation by Global Games

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Game Theory I

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Papers

[MOT20] Morris, S., D. Oyama, and S. Takahashi (2020). "Implementation via Information Design in Binary-Action Supermodular Games."

[MOT22] Morris, S., D. Oyama, and S. Takahashi (2022). "Implementation via Information Design using Global Games."

Continuum Players, Continuous States

- Continuum of players
- Actions: $a \in \{0, 1\}$
- States: $\Theta \subset \mathbb{R}$ (closed interval)
- (Common) payoff function: $u\colon \{0, 1\} \times [0, 1] \times \Theta \to \mathbb{R}$.
  - $u(a, \ell, \theta)$: Payoff to action a when proportion ℓ of opponents play action 1 and the state is θ
  - Define $d(\ell, \theta) = u(1, \ell, \theta) - u(0, \ell, \theta)$
- $\theta \in \Theta \sim$ distribution function P

Assumptions

A1. Action monotonicity:

$d(\ell, \theta)$ is nondecreasing in ℓ.

A2. State monotonicity:

$d(\ell, \theta)$ is nondecreasing in θ.

A3. Upper dominance region:

There exists $\overline{\theta} \in \operatorname{Int} \Theta$ such that $d(0, \theta) > 0$ for all $\theta \geq \overline{\theta}$.

Laplacian Threshold

- Laplacian payoff gain:

$$\bar{\Phi}(\theta) = \int_0^1 d(\ell, \theta) d\ell$$

- Laplacian threshold:

$$\theta^\sharp = \sup\{\theta \in \Theta \mid \bar{\Phi}(\theta) \leq 0\}$$

Expected Laplacian Threshold

A4 Integrability:

$\int_\Theta \max_{\ell \in \{0,1\}} |d(\ell, \theta)| dP(\theta) < \infty$.

- Assume $\int_{\theta > \theta^*} \bar{\Phi}(\theta) dP(\theta) < 0$ and that P is continuous.
- Expected Laplacian threshold:

  Unique $\theta^* \in \operatorname{Int} \Theta$ that solves

$$\int_{\theta > \theta^*} \bar{\Phi}(\theta) dP(\theta) = 0$$

Target Outcome

- An outcome is a mapping $\nu\colon \Theta \to \Delta([0,1])$:
  - $\nu_\theta \in \Delta([0,1])$: Distribution of the proportion of action-1 players at state θ
- ν^*: Continuous analogue of the optimal outcome identified in MOT20 ("target outcome"):

$$\nu^*_\theta = \begin{cases} \delta_1 & \text{if } \theta > \theta^* \\ \delta_0 & \text{if } \theta \leq \theta^* \end{cases}$$

  - $\delta_\ell \in \Delta([0,1])$: Dirac measure on $\ell \in [0,1]$

S-Implementation of the Target Outcome

- For $\varepsilon > 0$, define ν^ε by

$$\nu^*_\theta = \begin{cases} \delta_1 & \text{if } \theta \geq \theta^* + \varepsilon \\ \delta_0 & \text{if } \theta < \theta^* + \varepsilon. \end{cases}$$

- Let $\overline{\theta}' \in \mathrm{Int}\,\Theta$ be sufficiently large that
  - $\int_{\theta^*+\varepsilon}^{\overline{\theta}'} \bar{\Phi}(\theta) dP(\theta) > 0$, and
  - $d(0, \theta) > 0$ for all $\theta \geq \overline{\theta}'$.
- Denote
  - $\underline{P} = P(\theta^* + \varepsilon)$,
  - $\overline{P} = P(\overline{\theta}')$.

- Construction:

  Conditional on the realization of θ, a signal x_i is sent to each player i according to the following law:

  - If $\theta < \theta^* + \varepsilon$, then $x_i = -\infty$ for all players i.
  - If $\theta \geq \theta^* + \varepsilon$, then $x_i = \omega + \kappa\zeta_i$ for each player i, where

$$\omega \sim \begin{cases} \text{Unif}[\underline{P}, \overline{P}] & \text{if } \theta^* + \varepsilon \leq \theta < \overline{\theta}' \\ \text{Unif}[\overline{P}, 1] & \text{otherwise;} \end{cases}$$

$$\zeta_i \sim \text{any } F \text{ on } [-\tfrac{1}{2}, \tfrac{1}{2}] \text{ (iid) with log-concave density.}$$

- If $\theta < \theta^* + \varepsilon$, then all players play action 0.

- If $\theta \geq \theta^* + \varepsilon$, then the game is equivalent to the global game with uniform prior and

$$\hat{d}(\ell, \omega) = \begin{cases} \dfrac{1}{\overline{P} - \underline{P}} \displaystyle\int_{\theta^*+\varepsilon}^{\bar{\theta}'} d(\ell, \theta) dP(\theta) & \text{if } \omega < \overline{P} \\ \dfrac{1}{1 - \overline{P}} \displaystyle\int_{\theta \geq \bar{\theta}'} d(\ell, \theta) dP(\theta) & \text{if } \omega \geq \overline{P}. \end{cases}$$

- Laplacian payoff gain in this game:

$$\hat{\bar{\Phi}}(\omega) = \int_0^1 \hat{d}(\ell, \omega) d\ell = \begin{cases} \dfrac{1}{\overline{P} - \underline{P}} \displaystyle\int_{\theta^*+\varepsilon}^{\bar{\theta}'} \bar{\Phi}(\theta) dP(\theta) & \text{if } \omega < \overline{P} \\ \dfrac{1}{1 - \overline{P}} \displaystyle\int_{\theta \geq \bar{\theta}'} \bar{\Phi}(\theta) dP(\theta) & \text{if } \omega \geq \overline{P} \end{cases}$$

$$> 0 \quad \text{for all } \omega \in \Omega.$$

- In this global game, there is a unique equilibrium, in which all players play action 1:
  - The payoff gain function $\hat{d}(\ell, \omega)$ is not continuous, but
  - the expected payoff gain when the player observes signal x and others play the k-threshold strategy

    $$D_\kappa(x, k) = \int_\Theta \hat{d}\left(1 - F\left(\frac{k - \theta}{\kappa}\right), \omega\right) dF_\kappa(\omega|x)$$

    turns out to be continuous in (x, k),

    by the monotonicity assumption (monotone likelihood ratio) on the noise distribution F

    (Lemma 1 in MOT22, Appendix A).

Optimality of the Target Outcome

- Objective function $V\colon [0,1] \times \Theta \to \mathbb{R}$

A7. Objective Action Monotonicity:

For each θ, $V(\ell, \theta)$ is nondecreasing in ℓ.

A8. Restricted Convexity:

$V(\ell, \theta) \leq \ell V(1, \theta)$ whenever $\Phi(\ell, \theta) > \Phi(1, \theta)$.

A9. Objective State Monotonicity:

For each ℓ, $V(\ell, \theta)$ is nondecreasing in θ.

1. Limit of optimal outcomes of finite approximations (N players, N states)

   For each N, apply MOT20.

   (MOT22, Appendix B)

2. Solution of an optimal information design problem with continuum players and continuous states

   (with a heuristic "law of large numbers" assumption)

   (MOT22, Appendix C)

Optimal Information Design with Continuum Players

A15. Action Continuity:

For each θ, $d(\ell, \theta)$ is lower semi-continuous in ℓ.

- An information structure is a pair $(X, (\pi_\theta)_{\theta \in \Theta})$ such that
  - X is a Polish space of signals;
  - $\pi_\theta \in \Delta(\Delta(X))$ for each $\theta \in \Theta$;
  - for each $Q \in \mathcal{B}(\Delta(X))$, $\pi_\theta(Q)$ is measurable in θ.

(If Z is Polish, i.e., separable and completely metrizable, then $\Delta(Z)$ is again Polish with respect to the weak topology.)

- Interpretation:
  1. Designer commits to $(X, (\pi_\theta)_{\theta\in\Theta})$.
  2. $\theta \in \Theta$
  3. $q \in \Delta(X) \sim \pi_\theta$
  4. Signals sent according to q "independently" across players

     Empirical distribution of signal realizations $= q$
     ("law of large numbers")
  5. Interim belief conditional on signal $x \in X$:
     $\pi(\cdot|x) \in \Delta(\Delta(X) \times \Theta)$

- Example:
  - $X = \{-\infty\} \cup \left(\Omega + \kappa\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \quad (\Omega = [\underline{P}, 1])$
  - $\pi_\theta \in \Delta(\Delta(X))$:
    Dirac measure on $\bar{q}_\theta \in \Delta(X)$ defined as follows:
    - If $\theta < \theta^* + \varepsilon$, then $\bar{q}_\theta = \delta_{-\infty}$.
    - If $\theta^* + \varepsilon \leq \theta < \bar{\theta}'$ (resp. $\theta \geq \bar{\theta}'$), then $\bar{q}_\theta$ is the distribution of $x = \omega + \kappa\zeta$ where ω follows the uniform distribution over $[\underline{P}, \overline{P}]$ (resp. $[\overline{P}, 1]$) and ζ follows log-concave F over $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

- Strategy: measurable function $s\colon X \to \{0,1\}$
  (also symmetric strategy profile)
- Identify strategy s with set $S = \{x \in X \mid s(x) = 1\}$

  Σ: set of all strategies ($= \mathcal{B}(X)$)
- Outcome: $\nu\colon \Theta \to \Delta([0,1])$
  - For any $E \in \mathcal{B}([0,1])$, $\nu_\theta(E)$ is measurable in θ.
  - ν_θ: probability distribution of the proportion of action-1 players
  - O: set of all outcomes
- $S \in \Sigma$ induces $\nu \in O$ by

$$\nu_\theta(E) = \pi_\theta(\{q \in \Delta(X) \mid q(S) \in E\}).$$

- Expected payoff again against $S \in \Sigma$ conditional on $x \in X$:

$$D(S|x) = \int_{\Delta(X)\times\Theta} d(q(S), \theta) d\pi(q, \theta|x)$$

- $S \in \Sigma$ is an equilibrium if $D(S|x) \geq 0$ for all $x \in S$ and $D(S|x) \leq 0$ for all $x \in X \setminus S$.
- By A1 (Action Monotonicity) and A15 (Action Continuity), there is a smallest equilibrium $\underline{S}$, and sequential best response from the smallest strategy $\emptyset \in \Sigma$ converges to $\underline{S}$.
- $\nu \in O$ is S-implementable if there exists an information structure whose smallest equilibrium induces ν.
- $SI \subset O$: set of S-implementable outcomes